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Fixed Point Theorems in G-Metric Spaces With $\mathcal{W}$ Maps

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Abstract: The concept of a $\mathcal{W}$ function was introduced very recently by Chen [1] and was used to prove several common fixed point theorems in complete generalized metric spaces. Using a a subclass of these $\mathcal{W}$ functions introduced in [1], we prove some fixed point theorems with one, and a pair of self maps on a G-metric space

Keywords: G-metric spaces, Contraction mappings, Fixed points.

1 Introduction

The $\mathcal{W}$ map was introduced recently by Chen [1] in 2012 and was used to deduce some fixed point theorems in generalized metric spaces (g.m.s.) [2]. The results generalized common fixed point theorems on g.m.s. Another generalization of a metric space, the *G*-metric space, was proposed by Mustafa and Sims [3] which refined the postulates proposed by Dhage [4]. Subsequently Mustafa and Sims [5] developed some fixed point results for maps satisfying various contractive conditions on complete *G*-metric spaces. Fixed point theorems on a *G*-metric space with a pair of self maps were also developed recently by Choudhury *et al* [6] and Manro *et al* [7].

In this article we develop some fixed point theorems on *G*-metric spaces using a subclass of the $\mathcal{W}$ function proposed in Ref. [1]. We first give the basic definitions and notions used throughout the present paper.

Definition 1.1 [3] Let X be a non empty set and $G: X \times X \times X \to \mathrm{R}^{+}$ be a function satisfying the following properties

1. $G(x,y,z)=0$ if $x=y=z$
2. $G(x,x,y)>0$ for all $x,y\in X, x\neq y,$
3. $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$,
4. $G(x,y,z)=G(x,z,y)=G(y,z,x)=\cdots$ symmetry in all three variables,
5. $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$ (rectangle inequality).

Then *G* is called a generalized metric, or more specifically a G-metric on X and the pair (X,G) is called a *G*-metric space.

Proposition 1.2 [3] Let (X,G) be a *G*-metric space, then for any $x,y,z,a\in X$ it follows that:

1. If $G(x,y,z)=0$ then $x=y=z,$
2. $G(x,y,z)\leq G(x,x,y)+G(x,x,z),$
3. $G(x,y,y)\leq 2G(y,x,x),$
4. $G(x,y,z)\leq G(x,a,z)+G(a,y,z).$

Definition 1.3 [3] Let (X,G) be a G-metric space. The sequence $\{x_n\} \subseteq X$ is G-convergent to x if for any arbitrary $\varepsilon > 0$ there is a positive integer N such that $G(x,x_n,x_n) < \varepsilon$ for $n \geq N$, that is if $\lim_{n\to\infty} G(x,x_n,x_n) = 0$.

Proposition 1.4 [3] Let (X,G) be a G-metric space, then for a sequence $\{x_n\} \subset X$ and a point $x \in X$ the following are equivalent.

1. $\{x_n\}$ is G-convergent to x.
2. $G(x_n,x_n,x) \to 0$ as $n \to \infty$.
3. $G(x,x,x_n) \to 0$ as $n \to \infty$.
4. $G(x_m,x_n,x) \to 0$ as $m,n \to \infty$.

Proposition 1.5 [3] Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all its variables.

Definition 1.6 [3] Let (X,G) be a G-metric space. Then the sequence $\{x_n\} \subseteq X$ is called G-Cauchy if for every $\varepsilon > 0$, there is a positiver integer N such that $G(x_n,x_m,x_l) < \varepsilon$ for $n,l,m > N$.

Proposition 1.7 [3] In a G-metric space (X,G) the following are equivalent.

1. The sequence $\{x_n\}$ is G-Cauchy.
2. For every $\varepsilon > 0$, there is a positive integer N such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m > N$.

Definition 1.8 [3] A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Definition 1.9 [6] Let (X,G) be a G-metric space and $T: X \to X$ be a self mapping on (X,G). T is called a contraction if $G(Tx,Ty,Tz) \leq \alpha G(x,y,z)$ for all $x,y,z \in X$ where $0 \leq \alpha < 1$.

As remarked in [6], it follows easily that every contractive self mapping on (X,G) is continuous.

Definition 1.10 [1] We call φ a $\mathcal{W}$ function if the function φ satisfies the following conditions:

1. $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$,
2. $\lim_{t_n \to t} \inf \varphi(t_n) < t$ for all $t > 0$.

We now define a restricted class of $\mathcal{W}$ functions:

Definition 1.11 We call $\varphi: \mathrm{R}^+ \to \mathrm{R}^+$ a $\mathcal{W}_\beta$ function if there is a β such that $0 < \beta < 1$, and the following conditions are satisfied:

1. $\varphi(t) \leq \beta t$ for all $t > 0$ and $\varphi(0) = 0$,
2. $\lim_{t_n \to t} \inf \varphi(t_n) \leq \beta t$ for all $t > 0$.

Example 1.12 Consider the function $\varphi: \mathrm{R}^+ \to \mathrm{R}^+$ defined by $\varphi(t) = \frac{1}{3}t$. Then φ is $\mathcal{W}_{1/2}$ since $\varphi(t) < \frac{1}{2}t$ for all $t > 0$.

2 Main Results

Our main results are presented in subsections 2.1 and 2.2.

2.1 Fixed point theorems for a single map

In this section we give some fixed point theorems for a single self mapping $T: X \to X$ on a metric space (X,G). The following is a simple consequence of Definition 1.5. [6]:

Theorem 2.1 Let (X,G) be a G-metric space, and let $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{W}_\beta$ function. Let $T: X \to X$ be a self mapping on G. If $G(Tx,Ty,Tz) = \varphi(G(x,y,z))$ then,

1. T is a contraction mapping.
2. $T: X \to X$ is continuous

Theorem 2.2 Let (X,G) be a complete G-metric space, and let $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{W}_\beta$ function. Let $T: X \to X$ be a self mapping on G. If $G(Tx,Ty,Tz) = \varphi(G(x,y,z))$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $x_n = T^n x_0$. Then,

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= \varphi(G(x_{n-1}, x_n, x_n)) \\ &\le \beta G(x_{n-1}, x_n, x_n) \\ &\le \beta^2 G(x_{n-1}, x_n, x_n) \\ &\vdots \\ &\le \beta^n G(x_0, x_1, x_1) \end{aligned} \tag{2.1}$$

So for all $m,n \in \mathbb{N}$ and $n < m$ we have,

$$\begin{aligned} G(x_n, x_m, x_m) &< (\beta^n + \beta^{n+1} + \cdots \beta^m) G(x_0, x_1, x_1) \\ &< \frac{\beta^n}{1-\beta} G(x_0, x_1, x_1) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.2}$$

Hence $\{x_n\}$ is a Cauchy sequence in (X,G) and due to G-completeness, $x_n \to u$ as $n \to \infty$. Since T is a contraction mapping, it is continuous and therefore $Tx_n \to Tu$ as $n \to \infty$. Writing $Tx_n = z_n$ we find that $z_n \to z = Tu$ as $n \to \infty$. Now by Proposition 1.3 [3],

$$\lim_{n\to\infty} G(z_n, u, u) = G(z, u, u) = G(Tu, u, u), \tag{2.3}$$

and as $x_n \to u$ as $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} G(z_n, u, u) &= \lim_{n,m\to\infty} G(z_n, x_{m+1}, x_{m+1}) \\ &= \lim_{m\to\infty} G(Tu, Tx_m, Tx_m) \\ &= \lim_{m\to\infty} \varphi(G(u, x_m, x_m) \\ &\le \beta \lim_{m\to\infty} G(x_m, x_m, u) \\ &\le 0, \text{since } x_n \to u \text{ as } n \to \infty. \end{aligned} \tag{2.4}$$

However as $G(z_n,u,u) \ge 0$, $\lim_{n\to\infty} G(z_n,u,u) \ge 0$ and by Eq.(3) we must have $\lim_{n\to\infty} G(z_n,u,u) = G(T,u,u) = 0$, which shows $Tu = u$. Hence u is a fixed point of T.

To prove the uniqueness suppose $u, v \in X$, $u \ne v$ and $Tu = u$ and $Tv = v$. If possible let $G(u,v,v) > 0$. Then,

$$G(Tu,Tv,Tv) = \varphi(G(u,v,v) \le \beta G(u,v,v) < G(u,v,v), \text{since } 0 < \beta < 1.$$

But this is a contradiction and hence $G(u,v,v) = 0$, which gives $u = v$.

Example 2.3 Let $X=[-1,1]$ and let $G: X\times X\times X \to \mathbb{R}^+$ be defined by $G(x,y,z)=|x-y|+|y-z|+|z-x|$ for all $x,y,z\in X$. Then (X,G) is a G-metric space. Define $T: X\to X$ by $Tx=\frac{x+1}{5}$ for all $x\in X$. Then $G(Tx,Ty,Tz)=\frac{1}{5}G(x,y,z)$. Let $\varphi(t)=\frac{1}{5}t$ for all $t>0$. Then clearly φ is $\mathcal{W}_{1/2}$, and all conditions of Theorem 2.2 are satisfied. $T\frac{1}{4}=\frac{1}{4}$, so $\frac{1}{4}$ is a fixed point of T and it is unique.

The following theorem was proved by Mustafa and Sims [5] on a G-metric space:

Theorem 2.4 [5] Let (X,G) be a complete G-metric space and let $T: X\to X$ be a mapping which satisfies the following conditions for all $x,y,z\in X$,

$$G(Tx,Ty,Tz)\le k\ \max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz),G(x,Ty,Ty)\\ G(y,Tz,Tz),G(z,Tx,Tx)\},$$

where $k\in[0,1/2)$. Then T has a unique fixed point (say u) and T is G-continuous at u. This result is true for $0\le k<1$ when (X,G) is a bounded metric space, as was put in Remark 2.2 of [5]. We extend these ideas to a G-metric space with a $\mathcal{W}_\beta$ map.

Theorem 2.5 Let (X,G) be a complete G-metric space and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a $\mathcal{W}_\beta$ function. Let $T: X\to X$ be a mapping which satisfies the following conditions for all $x,y,z\in X$,

$$G(Tx,Ty,Tz)\le\varphi(\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz),G(x,Ty,Ty)\\ G(y,Tz,Tz),G(z,Tx,Tx)\}),$$

Then T has a unique fixed point x and T is G-continuous at x if $0<\beta<\frac{1}{2}$. If further (X,G) is bounded then the result is true for $0<\beta<1$.

Proof. Since $G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz),G(x,Ty,Ty),G(y,Tz,Tz)$ and $G(z,Tx,Tx)$ are all non negative, the conditions of the theorem imply that there is a β , $0<\beta<1$, such that

$$G(Tx,Ty,Tz)\le\beta\ \max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz),G(x,Ty,Ty)\\ G(y,Tz,Tz),G(z,Tx,Tx)\},$$

So if $0<\beta<\frac{1}{2}$ the proof of the theorem follows immediately from Theorem 2.3. [5]. In addition if G is bounded, then the result is true for $0<\beta<1$ by Remark 2.2 of [5].

2.2 A fixed point theorem for a pair of maps

Common fixed points for a pair of, and several maps were dealt with by many authors. In particular, fixed point theorems with several maps were given by Chen [1] on a g.m.s. Choudhury *et al* [6] gave some fixed point results for a pair of maps in a $G-$metric space. Manro *et al* [7] gave a fixed point theorem for a pair of self maps on a $G-$metric space. They also introduced the notion of $R-$weak commutativity and extended the fixed point theorem with the help of this property. All of these works relied on the notion of compatibility of two mappings in some form in the respective metric spaces considered.
In this section we develop a new fixed point theorem for a pair of self maps on a $G-$metric space (X,G) . First we give the definition of compatibility as given in [6].

Definition 2.6 [6] Let f, g be two self mappings on a $G-$metric space (X,G). The mappings f and g are said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

In this section we develop a new fixed point theorem for a pair of self maps on a $G-$metric space (X,G).

Theorem 2.7 Let (X,G) be a complete $G-$metric space and let $\varphi : \mathrm{R}^+ \to \mathrm{R}^+$ be a w_β function. Suppose f and g are two self maps on X such that the following conditions are satisfied:

1. $f(X) \subseteq g(X)$.
2. f or g is continuous.
3. $G(fx, fy, fz) = \varphi(\max\{G(fx, gy, gz), G(gx, fy, gz), G(gx, gy, fz)\})$.
4. f, g are compatible.

Then f and g have a common fixed point if $0 < \beta < \frac{1}{3}$.

Proof. Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $fx_0 = gx_1$. Generally, given $x_n \in X$ one can choose $x_{n+1} \in X$ such that $y_n = fx_n = gx_{n+1}$. Then

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) &= \varphi(\max\{G(fx_n, gx_{n+1}, gx_{n+1}), G(gx_n, fx_{n+1}, gx_{n+1}), \\ &\qquad G(gx_n, gx_{n+1}, fx_{n+1})\}) \\ &= \varphi(\max\{G(fx_n, fx_n, fx_n), G(gx_n, fx_{n+1}, fx_n), \\ &\qquad G(gx_n, fx_n, fx_{n+1})\}) \\ &= \varphi(\max\{0, G(gx_n, fx_n, fx_{n+1}), G(gx_n, fx_n, fx_{n+1})\}) \\ &= \varphi(G(gx_n, fx_n, fx_{n+1})) \\ &= \varphi(G(fx_{n-1}, fx_n, fx_{n+1})) \\ &\le \beta G(fx_{n-1}, fx_n, fx_{n+1}) \end{aligned} \tag{2.5}$$

where we have used the symmetry property of G. By the rectangle inequality,

$$G(fx_{n-1}, fx_n, fx_{n+1}) \le G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1}) \tag{2.6}$$

Also by Proposition 1.1 [3] and the symmetry of G we have

$$\begin{aligned} G(fx_n, fx_n, fx_{n+1}) &= G(fx_{n+1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1}) \\ &\le 2\, G(fx_n, fx_{n+1}, fx_{n+1}). \end{aligned} \tag{2.7}$$

So (2.6) becomes,

$$G(fx_{n-1}, fx_n, fx_{n+1}) \le G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1}) \tag{2.8}$$

If we use this in (2.5) we get,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le \beta\{G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1})\}$$

which simplifies to,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le \frac{\beta}{1-2\beta} G(fx_{n-1}, fx_n, fx_n) \tag{2.9}$$

If $0 < \beta < \frac{1}{3}$, writing $q = \frac{\beta}{1-2\beta}$ we find $0 < q < 1$ and by using (9) recursively we have,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q^n G(fx_0, fx_1, fx_1). \tag{2.10}$$

Using the rectangle inequality and (2.10) we have for all $n,m \in N$ and $n < m$

$$\begin{aligned} G(y_n, y_m, y_m) &\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G(y_{m-1}, y_m, y_m) \\ &\le (q^n + q^{n+1} + \cdots + q^{m-1}) G(fx_0, fx_1, fx_1) \\ &\le \frac{q^n}{1-q} G(fx_0, fx_1, fx_1) \end{aligned} \tag{2.11}$$

Making $n,m \to \infty$ we find $G(y_n, y_m, y_m) \to 0$, so $\{y_n\}$ is a Cauchy sequence in G. Since (X,G) is complete, there is a $z \in X$ such that $z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1}$. For definiteness we assume g is continuous so that $\lim_{n\to\infty} gfx_n = g\lim_{n\to\infty} fx_n = gz$. Further since f and g are compatible $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, and hence by the continuity of G as a metric we have $\lim_{n\to\infty} G(fgx_n, gz, gz) = 0$. So $gz = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n$. Now,

$$\begin{aligned} G(fgx_n, fx_n, fx_n) &= \varphi(\max\{G(fgx_n, gx_n, gx_n), G(ggx_n, fx_n, gx_n), G(ggx_n, gx_n, fx_n)\}) \\ &= \varphi(\max\{G(fgx_n, gx_n, gx_n), G(ggx_n, gx_n, fx_n)\}), \end{aligned} \tag{2.12}$$

where we have used the symmetry property of G. Taking limits as $n \to \infty$ and using the continuity property of the metric G and that of the function g we have we have from (2.12),

$$G(gz, z, z) \le \beta G(gz, z, z) < G(gz, z, z), \tag{2.13}$$

since $0 < \beta < 1$. But this is not possible if $G(gz, z, z) > 0$. So $G(gz, z, z) = 0$ and it follows from this that $gz = z$. So z is a fixed point of g. We next prove that z is also a fixed point of f.

$$\begin{aligned} G(fx_n, fx_n, fz) &= \varphi(\max\{G(fx_n, gx_n, gz), G(gx_n, fx_n, gz), G(gx_n, gx_n, fz)\}) \\ &= \varphi(\max\{G(fx_n, gx_n, gz), G(gx_n, gx_n, fz)\}) \\ &\le \beta G(fx_n, gx_n, gz) \text{ or } \beta G(gx_n, gx_n, fz), \end{aligned} \tag{2.14}$$

where we have used the symmetry property of the metric G. Taking limit as $n \to \infty$ we find that either

$$G(z, z, fz) \le \beta G(z, z, z), \tag{2.15}$$

or,

$$G(z, z, fz) \le \beta G(z, z, fz) < G(z, z, z), \text{ since } 0 < \beta < 1. \tag{2.16}$$

To show the uniqueness of the common fixed point, let z_1 be another common fixed point of f and g. If $G(z, z_1, z_1) > 0$,

$$\begin{aligned} G(z, z_1, z_1) &= G(fz, fz_1, fz_1) \\ &= \varphi(\max\{G(fz, gz_1, gz_1), G(gz, fz_1, gz_1), G(gz, gz_1, fz_1)\}) \\ &= \varphi(\max\{G(z, z_1, z_1), G(z, z_1, z_1), G(z, z_1, z_1)\}) \\ &= \beta G(z, z_1, z_1) < G(z, z_1, z_1), \end{aligned} \tag{2.17}$$

since $0 < \beta < 1$. This is a contradiction since $G(z, z_1, z_1) > 0$ and hence $G(z, z_1, z_1) = 0$, which gives $z = z_1$.

3 Conclusion

To conclude, we have

1. introduced a subclass of $\mathcal{W}$ functions which we called a $\mathcal{W}_\beta$ function.
2. We have used $\mathcal{W}_\beta$ functions to introduce contractive conditions on a G-metric space.
3. We have developed several fixed point theorems with one, and a pair of maps on G-metric spaces with the help of $\mathcal{W}_\beta$ functions.

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