Applied Mathematics & Information Sciences
An International Journal

Note on Partial Differential Equations with Non-Constant Coefficients and Convolution Method

Adem Kılıçman[1] *and Hassan Eltayeb*[2]

[1] Department of Mathematics and Institute for Mathematical Research University Putra Malaysia, Malaysia
[2] Mathematics Department, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, KSA

Received: Received Feb. 12, 2011; Revised July 11, 2011; Accepted Aug. 02, 2011
Published online: 1 January 2012

Abstract: In this study we extend the classification of partial differential equations to the further using the convolutions products. The purpose of this study is to compute the solutions of some explicit initial-boundary value problems for one-dimensional wave equation with variable coefficients by means of Laplace transform which in general has no solution.

Keywords: Sumudu transform; differential equation

1. introduction

The topic of partial differential equations is very important subject yet there is no general method to solve all the PDEs. The behavior of the solutions very much depend essentially on the classification of PDEs therefore the problem of classification for partial differential equations is very natural and well known since the classification governs the sufficient number and the type of the conditions in order to determine whether the problem is well posed and has a unique solution.

It is also well known that some of second-order linear partial differential equations can be classified as Parabolic, Hyperbolic or Elliptic however if a PDE has coefficients which are not constant, it is rather a mixed type. In many applications of partial differential equations the coefficients are not constant in fact they are a function of two or more independent variables and possible dependent variables. Therefore the analysis to describe the solution may not be hold globally for equations with variable coefficients that we have for the equations having constant coefficients.

On the other side there are some very useful physical problems where its type can be changed. One of the best known example is for the transonic flow, where the equation is in the form of

$$\left(1-\frac{u^2}{c^2}\right)\phi_{xx}-\frac{2uv}{c^2}\phi_{xy}+\left(1-\frac{v^2}{c^2}\right)\phi_{yy}+f(\phi)=0$$

where u and v are the velocity components and c is a constant, see [3].

Similarly, partial differential equations with variable coefficients are also used in finance, for example, the arbitrage-free value C of many derivatives

$$\frac{\partial C}{\partial \tau}+s^2\frac{\sigma^2(s,\tau)}{2}\frac{\partial^2 C}{\partial s^2}+b(s,\tau)\frac{\partial C}{\partial s}-r(s,\tau)C=0$$

with three variable coefficients $\sigma(s,\tau)$, $b(s,\tau)$ and $r(s,\tau)$. In fact this partial differential equation holds whenever C is twice differentiable with respect to s and once with respect to τ, see [16].

However, in the literature there was no systematic way to generate a partial differential equations with variable coefficients by using the equations with constant coefficients, the most of the partial differential equations with variable coefficients depend on nature of particular problems.

Recently, in [5], A. Kılıcman and H. Eltayeb , introduced a new method producing a partial differential equation by using the PDEs with constant coefficients and classification of partial differential equations having polynomial

* Corresponding author: e-mail: akilicman@putra.upm.edu.my, hgadain@ksu.edu.sa

coefficients. Later the same authors extended this setting in [11] to the finite product of convolution of hyperbolic and elliptic PDEs where the authors considered the positive coefficients of polynomials.

In the literature, there are several important partial differential equations and which are in the form of:

$P(D)u = f(x,y)$

and in order to solve, one might have either of the following cases, see Kılıçman [14].

Question: Now if we multiply the differential operator with a function by using the convolution then arise a problem of the classification for the new equations. That is a new classification problem of the

$(Q(x,t) * *P(D))\, u = f(x,t).$

The purpose of this study is to compute the solutions of some explicit initial-boundary value problems for one-dimensional wave equation with variable coefficients by means of Laplace transform which in general has no solution.

Definition 1.*Let $F_1(x,y)$ and $F_2(x,y)$ be integrable functions then the convolution of $F_1(x,y)$ and $F_2(x,y)$ defined as $(F_1 * *F_2)(x,y) = \int_0^y \int_0^x F_1(x-\zeta, y-\eta)F_2(\zeta,\eta)d\zeta\, d\eta$and known double convolution respect to x and y, see [17]. It can easily be extended to more general case as follows: If $F_1(x_1,x_2,...,x_n)$ and $F_2(x_1,x_2,...,x_n)$ be integrable functions then the convolution of F_1 and F_2 exists and given by $(F_1 * * * ... * F_2)(x_1,x_2,...,x_n) = \int_0^{x_1}\int_0^{x_2}...\int_0^{x_n} F_1(X)F_2(\zeta_1,\zeta_2,..\zeta_n)d\zeta$ where $X = (x_1-\zeta_1, x_2-\zeta_2,...,x_n-\zeta_n)$ and $d\zeta = d\zeta_1 d\zeta_2...d\zeta_n$, see [2].*

The properties of the convolution were discussed in [13].

Now let us consider the general linear second order partial differential equation with non-constant coefficients in the form of

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} \quad (1)$$
$$+d(x,y)u_x + e(x,y)u_y + f(x,y)u = 0$$

and almost linear equation in two variable

$$au_{xx} + bu_{xy} + cu_{yy} + F(x,y,u,u_x,u_y) = 0 \quad (2)$$

where a,b,c, are polynomials and defined by

$$a(x,y) = \sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} a_{\alpha\beta}x^{\alpha}y^{\beta},$$
$$b(x,y) = \sum_{\zeta=1}^{n}\sum_{\eta=1}^{m} b_{\zeta\eta}x^{\zeta}y^{\eta},$$
$$c(x,y) = \sum_{l=1}^{n}\sum_{k=1}^{m} c_{kl}x^{k}y^{l}$$

and $(a,b,c) \neq (0,0,0)$ where the expression $au_{xx} + 2bu_{xy} + cu_{yy}$ is called the principal part of Eq (2), since the principal part mainly determines the properties of solution. Throughout this paper we also use the following notations

$$|a_{mn}| = \sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} |a_{\alpha\beta}|, \; |b_{mn}| = \sum_{\zeta=1}^{n}\sum_{\eta=1}^{m} |b_{\zeta\eta}|,$$
$$|c_{mn}| = \sum_{l=1}^{n}\sum_{k=1}^{m} |c_{kl}|.$$

Now in order to generate a new PDEs we convolute Eq (2) by a polynomial with single convolution as $p(x)*^x$ where $p(x) = \sum_{i=1}^{m} p_i x^i$ then Eq (2) becomes

$$p(x) *^x \begin{bmatrix} a(x,y)u_{xx} + b(x,y)u_{xy} \\ +c(x,y)u_{yy} + F(x,y,u,u_x,u_y) \end{bmatrix} = 0 \quad (3)$$

where the symbol $*^x$ indicates single convolution with respect to x, and we shall classify Eq (3) instead of Eq (2) by considering and examining the function

$$D(x,y) = (p(x) *^x b(x,y))^2 - (p(x) *^x a(x,y)) \quad (4)$$
$$(p(x) *^x c(x,y))$$

From Eq (4), one can see that if D is positive then Eq (3) is called Hyperbolic, if D is negative then Eq (3) is called Elliptic, otherwise it is parabolic.
First of all, we compute and examine the coefficients of principal part of Eq (3) as follow

$$A_1(x,y) = p(x) *^x a(x,y)$$
$$= \sum_{i=1}^{m} p_i x^i *^x \sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} a_{\alpha\beta}x^{\alpha}y^{\beta}$$

by using single convolution definition and integration by parts, thus we obtain the first coefficient of Eq (3) in the form of

$$A_1(x,y = \sum_{\beta=1}^{n}\sum_{i=1}^{m}\sum_{\alpha=1}^{m} \frac{p_i a_{\alpha\beta} i! x^{\alpha+i+1}y^{\beta}}{((\alpha+1)((\alpha+2)...(\alpha+i+1))} \quad (5)$$

similarly, for the coefficients of the second part in Eq (3) we have

$$B_1(x,y) = \sum_{j=1}^{n}\sum_{\zeta=1}^{m}\sum_{i=1}^{m} \frac{p_i b_{\zeta\eta} i! x^{\zeta+i+1}y^{\eta}}{((\zeta+1)((\zeta+2)...(\zeta+i+1))} \quad (6)$$

also the last coefficient of Eq (3) given by

$$C_1(x,y) = \sum_{l=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{m} \frac{p_i c_{kl} i! x^{k+i+1}y^{l}}{((k+1)((k+2)...(k+i+1))} \quad (7)$$

then one can easily set up

$$D_1(x,y) = B_1^2(x,y) - A_1(x,y)C_1(x,y). \quad (8)$$

then there are several cases and the classification of partial differential equations with polynomials coefficients depend on very much the signs of the coefficients, see [5]. Similar analysis can be carried out with a single convolution $*^y$ as well as the double convolution.

2. An Application: Wave Equation

A number of problems in engineering give rise to the following well-known partial differential equations: wave equation, one dimensional heat flow, two dimensional heat flow, which in steady state becomes the two dimensional Laplace's equation, Poisson's equation, these equation arises in electrostatics and elasticity theory and transmission line equation.

For example, in a long electrical cable or a telephone wire both the current and voltage depend upon a position along the wire as well as the time. By using the basic laws of electrical circuit theory, the electrical current $i(x,t)$ satisfies the following PDE

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC+GL)\frac{\partial i}{\partial t} + RGi. \tag{9}$$

where the constant R, L, C and G are the resistance, inductance, capacitance and leakage conductance respectively, and the distance measured along the length of the cable represented by x. The voltage also satisfies equation (9). Several special cases of (9) arise in particular situations, for example, for a submarine cable G is negligible and frequencies are low so that inductive effects can also be neglected, thus one may place

$$G = L = 0.$$

In this case the equation (9) becomes

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}, \tag{10}$$

which is also known as submarine equation or telegraph equation, then we see that equation (10) satisfies the one dimensional heat equation. For high frequency alternating current, again with negligible leakage, then equation (9) can be approximated by

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}, \tag{11}$$

which is called the high frequency line equation, also this equation satisfies the one dimensional wave equation. Another application of wave equation, wave propagation under moving load. For example, if we consider high frequency line equation of moving load in wave equation with a non-homogenous forcing term as follows

$$\frac{\partial^2 i}{\partial t^2} - \frac{\partial^2 i}{\partial x^2} = H(x)\otimes H(t),\ t,x>0, \tag{12}$$

where the symbol $\otimes$ represents tensor product and H heaviside function, under the initial conditions

$$\begin{aligned} &i(0,t)=\delta(t),\ \ i(x,0)=\delta(x)\\ &i_x(0,t)=\delta'(t),\ i_t(x,0)=\delta'(x) \end{aligned} \tag{13}$$

where we consider inductance and capacitance are equal unit, δ dirac delta. We solve this equation, by using DLT technique as follow:

$$s^2 I(s,p) - sI(p,0) - \frac{\partial}{\partial t}I(p,0)$$

$$-(p^2 I(s,p) - pI(0,s) - \frac{\partial}{\partial x}I(0,s)) = \frac{1}{sp},$$

and single Laplace transform for initial data as

$$I(0,s)=1, I(p,0)=1,\ I_t(p,0)=pandI_x(0,s)=s(14)$$

we substitute equation (14) into equation (**??**) and we obtain

$$I(p,s) = \frac{1}{sp(s^2-p^2)}. \tag{15}$$

Now, on using double inverse Laplace transform for both sides of equation (15) we obtain the solution which is known as current in the form of

$$i(x,t) = \frac{1}{2}t^2.$$

Now, let us reconsider the equation (14), and multiply the left hand side of this equation by the polynomial $\Psi(x,t)$ by using the double convolution in order to have non-constant coefficients

$$\Psi(x,t) ** [i_{tt} - i_{xx}] = f(x,t) \tag{16}$$

under the same initial conditions. Now by taking double Laplace transform for the equation (16) and single Laplace transform for initial conditions then the solution of equation (16) given by

$$\begin{aligned} i(x,t) &= L_s^{-1}L_p^{-1} \\ &\times \begin{bmatrix} \frac{sH_1(p)}{(s^2-p^2)} - \frac{pG_1(s)}{(s^2-p^2)} + \frac{pH_1(p)-H_1(0)}{(s^2-p^2)} \\ -\frac{sG_1(s)-G_1(0)}{(s^2-p^2)} + \frac{F(p,s)}{(s^2-p^2)\Psi(p,s)} \end{bmatrix} \end{aligned} \tag{17}$$

provided that the inverse double Laplace transform exist. In [5] and [15] the authors consider the convolution terms are polynomials.

Since the Sumudu transform is a convenient tool for solving differential equations in the time domain, without the need for performing an inverse Sumudu transform, see [8], we apply this transform in the next section to solve the wave equation.

*Example 1.*Let us consider the non-homogenous wave equation in the form

$$\begin{aligned} i_{tt} - i_{xx} &= \frac{1}{2}e^{x+t} - \frac{1}{2}\cos(x)e^t - \frac{1}{2}e^x\cos(t) \\ &\quad + \frac{1}{2}\cos(x+t), \\ i(0,t) &= \delta(t),\ i_t(0,t)=\delta'(t), \\ i(x,0) &= \delta(x),\ i_x(x,0)=\delta'(x). \end{aligned} \tag{18}$$

We note that all the initial conditions have a singularity at $x=t=0$, $(t,x)\in R_+^2$. It is easy to see that the non-homogenous term of the equation (18) can be written in the form of the double convolution as follows:

$$\sin(x+t) ** e^{x+t} = \frac{1}{2}e^{x+t} - \frac{1}{2}\cos(x)e^t$$

$$-\frac{1}{2}e^x\cos(t) + \frac{1}{2}\cos(x+t).$$

If we now apply the double Sumudu transform with respect to x for i_{xx}, we get the double Sumudu transform in the form

$$S_2\left[i_{xx}(x,t);(u,v)\right]=\frac{1}{u^2}I(u,v) \tag{19}$$
$$-\frac{1}{u^2}I(0,v)-\frac{1}{u}\frac{\partial}{\partial x}I(0,v).$$

Similarly, the double Sumudu transform of f_{tt} is given by

$$S_2\left[i_{tt}(x,t);(u,v)\right]=\frac{1}{v^2}I(u,v) \tag{20}$$
$$-\frac{1}{v^2}I(u,0)-\frac{1}{v}\frac{\partial}{\partial t}I(u,0)$$

and the single Sumudu transform for the initial conditions is given by

$$I(0,v)=\frac{1}{v},I(u,0)=\frac{1}{u},I_x(0,v) \tag{21}$$
$$=\frac{1}{v^2}and\ I_t(u,0)=\frac{1}{u^2},$$

where $u;v$ are the transform variables for $x;t$ respectively. On using equations (19), (20) and (21) and after some rearrangements, we obtain

$$I(u,v)=\frac{u^3v^3(u+v)}{(u^2-v^2)(u^2+1)(v^2+1)(1-u)(1-v)}. \tag{22}$$

In order to obtain the inverse of equation (22), we use the following formula, see [13],

$$S^{-1}\left(I\left(u,v\right)\right)$$
$$=-\frac{1}{4\pi^2}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{(s+p)t}I\left(\frac{1}{p},\frac{1}{s}\right)\frac{ds\,dp}{sp}$$
$$=\sum residues\left[e^{(s+t)t}\frac{I\left(\frac{1}{p},\frac{1}{s}\right)}{sp}\right].$$

We then obtain the symbolic solution of equation (18) as follows

$$\begin{aligned}i(x,t)&=\frac{1}{4}e^x\cos(t)-\frac{1}{4}e^t\cos(x)+\frac{1}{8}\sin(t-x)\\&+\frac{1}{4}t\sin(x+t)+\frac{1}{4}\cos(x+t)\\&+\frac{1}{4}e^{x+t}+\frac{1}{4}e^{x+t}t+\frac{1}{8}\sin(x+t).\end{aligned}$$

We note that in the literature there is no systematic way to generate a partial differential equation with variable coefficients from the PDE with constant coefficients, however the most of the partial differential equations with variable coefficients depend on nature of particular problems, see [5] and [15].

Concluding Remarks: Now in order to make the classification of the new equation

$$\left(Q(x,t)**P(D)\right)u=f(x,t)$$

then we might have some certain remarks.

(i)If the function $Q(x,t)$ is a polynomial.
(ii)If the function $Q(x,t)$ is non-polynomial function.

However if the function $Q(x,t)$ is a continuous then it is possible to approximate by a polynomial thus still is an open problem whether the classification is an invariant property by the convolution product. Further one can also look at the solution of the new equations and study the relationship between original equations and the recent one.

Acknowledgement: The authors gratefully acknowledge that this research was partially supported by University Putra Malaysia under the Research University Grant Scheme(RUGS) 05-01-09-0720RU.

References

[1] F. B. M. Belgacem, A. A. Karaballi and L. S. Kalla, *Analytical investigations of the Sumudu transform and applications to iIntegral production equations*. Math. Probl. Engr. 3(2003), pp. 103–118.

[2] Y. A. Brychkov, H. J. Glaeske, A. P. Prudnikov and V. K. Tuan, Multidimensional Integral Transformations, Gordon and Breach Science Publishers 1992.

[3] G. Q. Chen and M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, J. Amer. Math. Soc., 16 (2003), pp. 461–494.

[4] H. Eltayeb and A. Kılıçman, *On some applications and new integral transform*, International Journal of Mathematical Analysis, 4(3)(2010), pp. 123–132.

[5] H. Eltayeb and A. Kılıçman, *A Note on Solutions of Wave, Laplace's and Heat Equations with Convolution Terms by Using Double Laplace Transform*, Appl. Math. Lett., 21(2008), pp. 1324–1329.

[6] R. Estrada and S. A. Fulling, *How singular functions define distributions*, Journal of Physics A: Mathematical and General, 35(13)(2002), pp. 3079–3089.

[7] B. Fisher, B. Jolevsaka-Tuneska and A. Kılıçman, *On Defining the Incomplete Gamma Function*, Integral Transform and Special Functions, **14**(4)(2003), pp. 293–299.

[8] M. G. M. Hussain and F. B. M. Belgacem, *Transient solutions of Maxwell's equations based on sumudu transform*, Progress In Electromagnetics Research, PIER 74(2007), pp. 273–289.

[9] R. P. Kanwal, Generalized Functions Theory and Applications Birkhauser, Boston, 2007.

[10] A. Kılıçman and H. Eltayeb, *A note on defining singular integral as distribution and partial differential equations with convolution terms*. Mathematics and Computer Modeling, 49(1-2)(2009), pp. 327–336.

[11] A. Kılıçman and H. Eltayeb, *On finite product of convolutions and classifications of hyperbolic and elliptic equations*, Mathematical and Computer Modeling (2011), doi:10.1016/j.mcm.2011.05.031.

[12] A. Kılıçman, *A remark on the products of distributions and semigroups*, Journal of Mathematical Sciences: Advances and Applications, 1(2)(2008), pp. 423–430.

[13] A. Kılıçman and H. E. Gadain, *An application of double Laplace transform and double Sumudu transform*, Lobachevskii Journal of Mathematics, 30(3)(2009), pp. 214–223.
[14] A. Kılıçman and H. Eltayeb, *A note on classification of hyperbolic and elliptic equations with polynomial coefficients*, Applied Mathematics Letters 21(11) (2008), pp. 1124–1128.
[15] A. Kılıçman and H. Eltayeb, *A note on integral transforms and partial differential equations*, Applied Mathematical Sciences, 4(3)(2010), pp. 109–118.
[16] R. C. Merton, Theory of rational option pricing, Bell Journal of Economics and Management Science 4(Spring) (1973), pp. 141–183.
[17] N. Sneddon, S. Ulam and M. Stark, Operational Calculus in Two Variables and its Applications, Pergamon Press Ltd 1962.
[18] J. M. Tchuenche and N. S. Mbare, *An application of the double Sumudu transform*, App. Math. Sci. 1(1)(2007), pp. 31–39.
[19] G. K. Watugala, *Sumudu transforma new integral transform to solve differential equations and control engineering problems*, Mathematical Engineering in Industry, 6(4)(1998), pp. 319–329.
[20] G. K. Watugala, *The Sumudu transform for functions of two variables*, Mathematical Engineering in Industry, 8(4)(2002), pp. 293–302.
[21] A. I. Zayed, Functions and Generalized Functions Transformations, CRC press, 1996.
[22] H. Eltayeb, A. Kılıçman & B. Fisher, *A New Integral Transform and Associated Distributions*, Integral Transforms and Special Functions, 21(5)(2010), pp. 367—379.
[23] J. Zhang, *A Sumudu based algorithm for solving differential equations*, Computer Science Journal of Moldova, vol.15, no.3(45)(2007), pp. 303–313.

Adem Kilicman is full Professor in the Department of Mathematics at University Putra Malaysia. He received his Bachelor and Master degrees from Hacettepe University in 1989 and 1991 respectively, Turkey. He obtained his PhD from University of Leicester in 1995, UK. He has been actively involved several academic activities in the Faculty of Science and Institute of Mathematical Research (INSPEM). Adem Kilicman is also member of some Associations; PERSAMA, SIAM, IAENG, AMS. His research areas include Differential Equations, Functional Analysis and Topology.

Hassan Eltayeb: serves as a Assistant Professor at the Department of Mathematics, King Saud University, Saudi Arabia. He received his BSc. and MSc. degrees from Department of Mathematics, University of Khartoum; University of Sudan, Sudan and Ph.D. degree from University Malaysia Terengganu, Malaysia. He has worked in University Putra Malaysia one year as Postdoctoral researcher. Currently he is a full tenure in King Saud University, Saudi Arabia. His research areas includes partial differential equations.