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Improved Accuracy of Linear Multistep Methods

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Abstract: We present a technique for improving the accuracy of a given multistep method. We first propose a new formulation of the θ-method providing a general framework for studying stability and allowing to select the appropriate values of the parameter θ that increase the order of accuracy. The idea is followed through to generate optimal linear multistep methods.

Keywords: Linear multistep methods, θ-method, optimal method, absolute stability.

1. Introduction

Consider the linear k-step method

$$\rho(E)U^n = \Delta t\sigma(E)f^n \tag{1}$$

for solving the first-order initial-value problem

$$\frac{du}{dt} = f(u,t), \qquad u(0) = u^0, \tag{2}$$

where $\rho(\xi)$ and $\sigma(\xi)$ are the characteristic polynomials

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \qquad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j,$$

and E is the shift operator defined by

$$EU^n = U^{n+1}.$$

In (1), U^n is the numerical solution at time level $t_n = n\Delta t$, Δt is the time step, and $f^n = f(U^n, t_n)$. The coefficients α_n and β_n are real constants with $\alpha_k \neq 0$, and not both α_0 and β_0 are zero. It is conventional to normalize (1) by letting $\alpha_k = 1$. When $\beta_k = 0$, the method is said to be explicit; otherwise it is implicit. The local truncation error of the method at time t_{n+k} is given by

$$\epsilon(t_{n+k}) = \frac{1}{\Delta t}\left[\sum_{j=0}^{k} \alpha_j u(t_{n+j}) - \Delta t \sum_{j=0}^{k} \beta_j u'(t_{n+j})\right]. \tag{3}$$

If u is sufficiently smooth, then Taylor series expansions about $t = t_n$ yield after collecting terms

$$\epsilon(t_{n+k}) = \frac{1}{\Delta t}\left[C_0 u(t_n) + C_1 \Delta t u'(t_n) + C_2 \Delta t^2 u''(t_n) + \cdots + C_p \Delta t^p u^{(p)}(t_n) + \cdots\right], \tag{4}$$

where C_p are constants given by

$$\begin{cases} C_0 = \sum_{j=0}^{k} \alpha_j, \\ C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j), \\ C_2 = \sum_{j=0}^{k} \left(\frac{1}{2} j^2 \alpha_j - j\beta_j\right), \\ \quad \vdots \\ C_p = \sum_{j=0}^{k} \left(\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} j^{p-1} \beta_j\right), \quad p = 3, 4, \cdots. \end{cases} \tag{5}$$

We verify that $C_0 = \rho(1)$ and $C_1 = \rho'(1) - \sigma(1)$, so consistency is expressed by the relations:

$$\rho(1) = 0 \qquad \text{and} \qquad \rho'(1) = \sigma(1). \tag{6}$$

We say that (1) has order p and error constant C_{p+1} if, in (4), $C_0 = C_1 = \cdots = C_p = 0$, $C_{p+1} \neq 0$.

A linear multistep method is said to be zero-stable if the roots of the characteristic polynomial ρ satisfy the root

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condition; that is, they reside in the closed unit disk and any of unit modulus is simple. The absolute stability of a linear multistep method is analyzed by applying it to the linear test equation

$$\frac{du}{dt} = \lambda u, \tag{7}$$

where λ is a complex number. The region of absolute stability of (1) is the set of all $\bar{h} = \lambda \Delta t$ in the complex plane for which the roots of the characteristic equation

$$\rho(\xi) - \bar{h}\sigma(\xi) = 0$$

satisfy the root condition.

The simplest numerical methods for solving the initial-value problem (2) are the forward and backward Euler methods given by

$$U^{n+1} - U^n = \Delta t f^n \quad \text{and} \quad U^{n+1} - U^n = \Delta t f^{n+1}, \tag{8}$$

respectively. Both methods are first-order accurate. The so-called θ-method for solving (2) is regarded as a weighted average method obtained by taking the weighted average of the two formulas [9],

$$U^{n+1} - U^n = \Delta t(\theta f^{n+1} + (1-\theta) f^n), \tag{9}$$

where $0 \le \theta \le 1$. Formula (9) can be interpreted geometrically: the slope of the solution is assumed to be piecewise constant and is provided by a linear combination of derivatives at the endpoints of each time interval [6].

The local truncation error of (9) is given by

$$\epsilon(t_{n+1}) = \Delta t \left(\frac{1}{2} - \theta\right) u''(t_n) + \Delta t^2 \left(\frac{1}{6} - \frac{\theta}{2}\right) u'''(t_n) + O(\Delta t^3). \tag{10}$$

Forward and backward Euler methods correspond to the choices of $\theta = 0$ and $\theta = 1$ in (9), respectively, while the choice $\theta = 1/2$ yields the trapezoidal rule which is second-order accurate.

Based on a new interpretation of the θ-method, we show that, for a given linear multistep method of order p, it is possible to construct another linear multistep method of order $p+1$, using a parameter θ. The main result is stated in Theorem 1.

The rest of the paper is organized as follows. In Section 2, we investigate a new formulation of the θ-method providing a general framework for studying stability and allowing to select the appropriate values of the parameter θ that increase the order of accuracy. This idea is followed through in Section 3 to gererate modified linear multistep methods with improved properties. The case of second-order methods is treated similarly in Section 4.

2. General formulation of the θ-method

We now introduce the parameterized multistep method

$$\rho(E)U^n = \Delta t \sigma(E) f^n + \theta \Delta t \rho(E) f^n, \tag{11}$$

where the parameter θ is nonnegative. The method is obtained by adding the term $\theta \Delta t \rho(E) f^n$ to the right-hand side of (1). Formula (9) can be rearranged in the following form

$$U^{n+1} - U^n = \Delta t f^n + \theta \Delta t (f^{n+1} - f^n),$$

and hence it fits the general pattern (11) with $\rho(\xi) = \xi - 1$ and $\sigma(\xi) = 1$. The simple geometric intuition is now replaced by a general and formal approach.

Properties of the new method (11) are given in the following propositions. We shall assume throughout the paper that the original method (1) is consistent and zero-stable.

Proposition 1 *There holds*

(i) The method (11) *preserves consistency and zero-stability properties for all values of* $\theta \ge 0$.
(ii) There is a unique value θ^* *of* θ *for which the method* (11) *is at least second-order accurate. The method is first-order accurate otherwise.*

Proof. (i) We have $\rho'(1) = \sigma(1) + \theta\rho(1)$ since $\rho(1) = 0$. Hence, the consistency conditions (6) are satisfied. The zero-stability of the method is clear.
(ii) The method (11) is at least second-order accurate if the parameter θ is selected so that the coefficient C_2 in the local truncation error of (11) is zero; that is, if

$$\frac{1}{2}[\rho''(1) + \rho'(1)] - \sigma'(1) - \theta\rho'(1) = 0.$$

Since $\rho'(1)$ cannot vanish ($\xi = 1$ cannot be a repeated root of ρ), solving the previous equation for θ yields

$$\theta^* = \frac{\frac{1}{2}[\rho''(1) + \rho'(1)] - \sigma'(1)}{\rho'(1)}.$$

If $\theta \ne \theta^*$, then $C_2 \ne 0$ and the method is only first-order accurate.

Next, we investigate the stability of (11). A numerical method is said to be A-stable if its region of absolute stability contains the left half-plane $Re(\lambda \Delta t) \le 0$. The A-stability property is often desirable for stiff ODE problems. Dahlquist [4] has proved that any A-stable linear multistep method has order of accuracy $p \le 2$. On the other hand, Cryer [3] has proved there exist $A(0)$-stable linear multistep methods of arbitrary high order. Since the eigenvalues associated with parabolic problems are real and negative, the application of an $A(0)$-stable method yields an unconditionally stable scheme, that is, no stability restriction on the size of Δt can result (see, e.g., Chapter 5 in [5]). We have the following stability result.

Proposition 2 *There holds*

(i) If the interval of absolute stability of (1) *is the interval* $[-w,0]$*, where* $w>0$ *is a real number, then the method* (11) *is* $A(0)$*-stable if* $\theta \geq w^{-1}$*, and its interval of absolute stability reduces to* $[-w_\theta,0]$ *if* $\theta < w^{-1}$*, where*

$$w_\theta = \frac{w}{1-\theta w}.$$

(ii) If the region of absolute stability of (1) *contains the disk* $\{\bar{h}\in\mathbb{C},\ |\bar{h}+a|\leq a\}$*, where* $a>0$ *is a real number, then* (11) *is A-stable if* $2\theta \geq a^{-1}$.

Proof. The region of absolute stability of (11) is the set of all $\bar{h}=\lambda\Delta t$ in the complex plane for which all the roots ξ of the characteristic equation

$$\rho(\xi)-\bar{h}(\sigma(\xi)+\theta\rho(\xi))=0$$

satisfy the root condition. We rearrange this equation in the form

$$\rho(\xi)-\Psi(\bar{h})\sigma(\xi)=0,$$

where Ψ is the one-to-one mapping defined by

$$\Psi(\bar{h})=\frac{\bar{h}}{1-\theta\bar{h}}. \qquad (12)$$

We denote by $\mathbb{C}^-$ the left half complex plane $\mathrm{Re}(\bar{h})\leq 0$. To prove the proposition, we shall first determine the images of $(-\infty,0]$ and $\mathbb{C}^-$ by the mapping Ψ. We first notice that Ψ maps the interval $(-\infty,0]$ into $(-1/\theta,0]$. The method is then $A(0)$-stable if $(-1/\theta,0]$ is a subset of $[-w,0]$. This holds when $\theta\geq w^{-1}$. If $\theta<w^{-1}$, then one can verify that Ψ maps the interval $[-w_\theta,0]$ into $[-w,0]$, where the value of w_θ is given in the proposition.

For the second part of the proposition, one can verify that Ψ maps the whole set $\mathbb{C}^-$ into the set

$$S^*=\left\{\bar{h}\in\mathbb{C},\ \left|\bar{h}+\frac{1}{2\theta}\right|\leq\frac{1}{2\theta}\right\}\setminus\left\{-\frac{1}{\theta}\right\}.$$

The set S^* is the closed disk of center $-1/(2\theta)$ and radius $1/(2\theta)$ from which the point $-1/\theta$ is excluded. Indeed, we have

$$\left|\frac{\bar{h}}{1-\theta\bar{h}}+\frac{1}{2\theta}\right|=\left|\frac{1+\theta\bar{h}}{1-\theta\bar{h}}\,\frac{1}{2\theta}\right|\leq\frac{1}{2\theta}$$

since

$$\left|\frac{1+\theta\bar{h}}{1-\theta\bar{h}}\right|\leq 1$$

if $\mathrm{Re}(\bar{h})\leq 0$. The method (11) is A-stable if S^* is a subset of the region of absolute stability of (1). If this latter contains the disk mentioned in the proposition, then (11) is A-stable when $2\theta\geq a^{-1}$.

The stability region of the forward Euler method consists of the closed disk of center -1 and radius 1. By the proposition, the method (9) is A-stable when $\theta\geq 1/2$, and its interval of absolute stability reduces to the interval $[-2(1-2\theta)^{-1},0]$ when $\theta<1/2$. The case $\theta=1/2$ is particularly interesting because it allows the method to be both second-order accurate and A-stable.

We finally notice that if (1) is implicit, then we can rearrange it in the following form

$$\rho(E)U^n=\Delta t[\sigma(E)-\beta_k\rho(E)]f^n+\beta_k\Delta t\rho(E)f^n, \qquad (13)$$

where the polynomial $\sigma-\beta_k\rho$ is at least one degree lower than ρ. This new form of (1) is useful for the approximate factorization of some parabolic problems, see [2,7].

3. Highly accurate methods

The results in Proposition 1 are not affected when $\rho(E)$ in (11) is replaced by $(E-1)$, since we have only used the properties that $\rho(1)=0$ and $\rho'(1)\neq 0$. The disadvantage of choosing $\rho(E)$ in (11) is that if (1) is second-order accurate, then (11) is first-order accurate for any $\theta\neq 0$. We notice that if $\rho'(1)=0$, then second-order accuracy is preserved with any value of θ, and there is a unique value of θ that makes $C_3=0$, thus producing a third-order accurate method. Assuming $\rho'(1)=0$ is of course not possible for stability reason. An alternative would be to add a term of the form $\theta(E-1)^2$ to the left-hand side of (1). This has the advantage of preserving both second-order accuracy and zero-stability. Furthermore, there is a specific value of θ for which the resulting method is third-order accurate. The following theorem presents this idea is a more general form. It provides a technique for generating higher-order methods. The novelty in this theorem is the relationship between θ and the coefficient C_{p+1}. In the theorem ϕ_p is a polynomial of degree p given by $\phi_p(\xi)=(\xi-1)^p$.

Theorem 1 *Assume that the linear* k*-step method* (1) *has order* p *and error constant* C_{p+1}*. Consider the following linear multistep method*

$$E^\nu\rho(E)U^n=\Delta tE^\nu\sigma(E)f^n+\theta\Delta t\phi_p(E)f^n, \qquad (14)$$

where $\nu=\max\{0,p-k\}$*. Then, there is a unique value of* θ*,* $\theta^*=C_{p+1}$*, for which the method* (14) *is at least of order* $p+1$*. The method is of order* p *otherwise.*

Proof. The local truncation error of the pth order method (1) at time t_{n+k} is of the form

$$\epsilon(t_{n+k})=C_{p+1}\Delta t^{p+1}u^{(p+1)}(t_n)+O(\Delta t^{p+2}). \qquad (15)$$

The local truncation error of the $(k+\nu)$-step method

$$E^\nu\rho(E)U^n=\Delta tE^\nu\sigma(E)f^n,$$

can be written in the same form at time $t_{n+k+\nu}$. On the other hand, we notice that since $\phi_p^{(m)}(1)=0$ for $m=0,\ldots,p-1$, we have that

$$\begin{aligned}\phi_p(E)u'(t_n)&=\frac{1}{p!}\phi_p^{(p)}(1)\Delta t^p u^{(p+1)}(t_n)+O(\Delta t^{p+1})\\&=\Delta t^p u^{(p+1)}(t_n)+O(\Delta t^{p+1}).\end{aligned}$$

As a result, the method (14) is at least of order $p+1$ if $\theta=C_{p+1}$, and of order p otherwise.

We remark that if $p \leq k$ and k is odd, a recursive application of (14) with optimal values of θ leads to an optimal multistep method.

Other methods for generating higher-order linear multistep methods, based on damping the leading error, can be found in the literature, see for instance [1,5]. One of these methods is the so-called deferred correction. The idea of this method is to start out with a computed low-order solution, and then raise the accuracy one or two levels by using this one for estimating the leading error. This procedure can then be continued to any order of accuracy. The implementation of the method is quite easy, and has the advantage that the properties of the basic low-order method, to a large extend, characterizes the whole algorithm, see [5] for further details.

As a first application of Theorem 1, consider the midpoint method

$$U^{n+2} - U^n = 2\Delta t f^{n+1},$$

for which $C_3 = 1/3$. According to the thereom, the method

$$U^{n+2} - U^n = 2\Delta t f^{n+1} + \frac{\Delta t}{3}(f^{n+2} - 2f^{n+1} + f^n)$$

is at least third-order accurate. This formula is indeed the Simpson rule. The method is symmetric and so fourth-order accurate. Many other examples can be given. For instance, the 2-step Adams-Bashforth method

$$U^{n+2} - U^{n+1} = \frac{\Delta t}{2}(3f^{n+1} - f^n)$$

is second-order accurate and has the error constant $C_3 = 5/12$. By Theorem 1, the method

$$\begin{aligned} U^{n+2} - U^{n+1} &= \frac{\Delta t}{2}(3f^{n+1} - f^n) \\ &\quad + \frac{5}{12}\Delta t(f^{n+2} - 2f^{n+1} + f^n) \\ &= \frac{\Delta t}{12}(5f^{n+2} + 8f^{n+1} - f^n) \end{aligned}$$

is at least third-order accurate. This is the 2-step Adams-Moulton method. The 3-step Adams-Bashforth method

$$U^{n+3} - U^{n+2} = \frac{\Delta t}{12}(23f^{n+2} - 16f^{n+1} + 5f^n)$$

is third-order accurate with $C_4 = 3/8$. Hence, the improved method

$$\begin{aligned} U^{n+3} - U^{n+2} &= \frac{\Delta t}{12}(23f^{n+2} - 16f^{n+1} + 5f^n) \\ &\quad + \frac{3}{8}\Delta t(f^{n+3} - 3f^{n+2} + 3f^{n+1} - f^n) \\ &= \frac{\Delta t}{24}(9f^{n+3} + 19f^{n+2} - 5f^{n+1} + f^n) \end{aligned}$$

is at least fourth-order accurate. This is the 3-step Adams-Moulton method. It can shown that all the Adams-Moulton methods can be obtained from the Adams-Bashforth methods using (14). The relationship between Adams methods has also another interpretation based on the original derivation of the methods using interpolating polynomials.

The next example concerns the trapezoidal rule for which $\nu = 1$ in Theorem 1. Substituting $\theta = 1/2$ in (10) yields $C_3 = -1/12$. As a result, the method

$$\begin{aligned} U^{n+2} - U^{n+1} &= \frac{\Delta t}{2}(f^{n+2} + f^{n+1}) \\ &\quad - \frac{1}{12}\Delta t(f^{n+2} - 2f^{n+1} + f^n) \end{aligned}$$

is at least third-order accurate. One can verify that this is exactly the third-order Adams-Moulton method found previously. Repeating this process generates higher-order Adams-Moulton methods.

The last example concerns the BDF2 formula

$$U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n = \frac{2}{3}\Delta t f^{n+2}$$

having the error constant $C_3 = -2/9$. If we consider the parametrized multistep method

$$\begin{aligned} U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n \\ = \frac{2}{3}\Delta t f^{n+2} + \theta\Delta t(f^{n+2} - 2f^{n+1} + f^n), \end{aligned}$$

then, by Theorem 1, the choice $\theta = -2/9$ yields a method which is at least third-order accurate.

The previous examples show that (14) provides an efficient tool for the derivation of multistep methods. Starting from a simple and low-order method one can use (14) to generate a higher-order method. Formula (14) is not restricted to linear multistep methods. It can be applied to any numerical method having a local truncation error of the form given by (15).

Consider now the implicit k-step method

$$\rho(E)U^n = \Delta t\tilde{\sigma}(E)f^n \tag{16}$$

of order p and assuming for simplicity that $p \leq k+1$. We can rearrange (16) in the following form

$$\rho(E)U^n = \Delta t\sigma(E)f^n + \beta_k\Delta t\phi_{p+1}(E)f^n,$$

where $\sigma(\xi) = \tilde{\sigma}(\xi) - \beta_k\phi_p(\xi)$. Then, we can verify that the method

$$\rho(E)U^n = \Delta t\sigma(E)f^n \tag{17}$$

is explicit and of order $p-1$. Indeed, β_k is the error contant of (17). Formulas (16) and (17) can be combined to produce a predictor-corrector method, where (17) is the predictor and (16) the corrector. Let $U^{n+k,*}$ be the predicted approximation from (17) and U^{n+k} the corrected approximation from (16). Then it is easy to verify that the corrector method can be written in the following form

$$\begin{aligned} U^{n+k} = U^{n+k,*} + \beta_k\ \Delta t\ f(U^{n+k,*}) \\ + \beta_k\Delta t\bar{\phi}_{p-1}(E)f^n, \end{aligned} \tag{18}$$

where $\bar{\phi}_{p-1}(\xi) = (\xi-1)^{p-1} - \xi^{p-1}$. From Milne's device, we deduce that the predictor-corrector method is of order p; that is, of the same order than the corrector. It is interesting to notice that the corrector method (18) does not involve any characteristic polynomial. For instance, we will have this form when the predictor is an Adams-Bashforth method of order $p-1$, and the corrector is an Adams-Moulton method of order p.

We now extend some of the previous results to second-order ordinary differential equations.

4. Methods for second-order ODEs

A linear k-step method for solving the following second-order initial value problem

$$\frac{d^2u}{dt^2} = f(u,t), \qquad u(0) = u^0, \qquad \frac{du}{dt}(0) = v^0, \quad (19)$$

is defined by the difference equation

$$\rho(E)U^n = \Delta t^2 \sigma(E) f^n, \quad (20)$$

where the polynomials ρ and σ are previously defined. We normalize (20) by setting $\alpha_k = 1$. Consistency is expressed by the relations

$$\rho(1) = \rho'(1) = 0, \qquad \rho''(1) = 2\sigma(1).$$

Explicit methods are still popular for solving second-order ODEs. The method

$$U^{n+2} - 2U^{n+1} + U^n = \Delta t^2 f^{n+1} \quad (21)$$

is the most popular and is known as the leapfrog method. A more general method is given by the formula

$$U^{n+2} - 2\,U^{n+1} + U^n = \Delta t^2(\theta f^{n+2} + (1-2\theta) f^{n+1} + \theta f^n),$$

where θ is a parameter to be selected in $[0, 1/2]$. The method has been considered for instance in [8]. It reduces to (21) when $\theta = 0$, and can be rearranged in the form

$$U^{n+2} - 2\,U^{n+1} + U^n = \Delta t^2 f^{n+1} + \theta \Delta t^2 (f^{n+2} - 2f^{n+1} + f^n).$$

Thus, it can be viewed as a special case of the generalized θ-method

$$\rho(E)U^n = \Delta t^2 \sigma(E) f^n + \theta \Delta t^2 \rho(E) f^n. \quad (22)$$

Properties of (22) are summarized in the following proposition whose proof is similar to that of Proposition 1.

Proposition 3 *There holds*

(i) The method (22) *preserves consistency and zero-stability properties for all values of* θ.

(ii) There is a unique value θ^* *of* θ *for which* (22) *is at least third-order accurate. The method is second-order accurate otherwise.*

(iii) If the interval of absolute stability of (20) *is the interval* $[-w, 0]$ $(w > 0)$, *then* (22) *is* $A(0)$*-stable if* $\theta \geq w^{-1}$, *and its interval of absolute stability reduces to* $[-w_\theta, 0]$ *if* $\theta < w^{-1}$, *where*

$$w_\theta = \frac{w}{1 - \theta w}.$$

The interval of absolute stability of the leapfrog method (21) is the interval $[-4, 0]$. We deduce from proposition that (22) is $A(0)$-stable if $\theta \in [1/4, 1/2]$, and its interval of absolute stability reduces to $[-4(1-4\theta)^{-1}, 0]$ if $0 \leq \theta < 1/4$. The optimal value $\theta^* = C_3 = 1/12$ generates a fourth-order accurate method known as the Numerov method, having the interval of absolute stability $[-6, 0]$.

A general class of time discretization methods well-known in the engineering literature is given by the so-called Newmark method [10]. When applied to (19), it reads

$$U^{n+2} - 2\,U^{n+1} + U^n = \Delta t^2 \left[\theta f^{n+2} + \left(\frac{1}{2} - 2\theta + \gamma\right) f^{n+1} + \left(\frac{1}{2} + \theta - \gamma\right) f^n\right], \quad (23)$$

where $\theta \geq 0$ and $\gamma \geq 0$ are free parameters. Based on our previous analysis, we can easily determine its order of accuracy. Indeed, we first rearrange the method in the form

$$U^{n+2} - 2\,U^{n+1} + U^n = \Delta t^2 f^{n+1} + \theta \Delta t^2 (f^{n+2} - 2f^{n+1} + f^n) + \left(\gamma - \frac{1}{2}\right) \Delta t^2 (f^{n+1} - f^n),$$

or,

$$\rho(E)U^n = \Delta t^2 \sigma(E) f^n + \theta \Delta t^2 \rho(E) f^n + \left(\gamma - \frac{1}{2}\right) \Delta t^2 \eta(E) f^n,$$

where $\rho(\xi) = (\xi-1)^2$, $\sigma(\xi) = \xi$ and $\eta(\xi) = (\xi - 1)$. Therefore, we conclude that the Newmark method (23) is of order $p = 1$ if $\gamma = 1/2$, $p = 4$ if $\gamma \neq 1/2$ and $\theta = 1/12$, and $p = 2$ otherwise.

To study the stability of the method, we rearrange it in the form

$$\rho(E)U^n = \Delta t^2 \kappa(E) f^n + \theta \Delta t^2 \rho(E) f^n,$$

where

$$\kappa(\xi) = \left(\gamma + \frac{1}{2}\right)\xi - \left(\gamma - \frac{1}{2}\right).$$

A simple test of stability shows that the method

$$\rho(E)U^n = \Delta t^2 \kappa(E) f^n \quad (24)$$

has the interval of absolute stability $[-2/\gamma, 0]$ when $\gamma \geq 1/2$, but has no interval of absolutely stable when $\gamma < 1/2$. Now, by applying the results in Proposition 3 to (24) using $w = 2/\gamma$, we obtain

Proposition 4 *The Newmark method* (23) *is* $A(0)$*-stable if* $2\theta \geq \gamma \geq 1/2$ *and its interval of absolute stability reduces to* $[-2/(\gamma - 2\theta), 0]$ *when* $2\theta < \gamma$ *and* $\gamma \geq 1/2$.

5. Conclusion

We presented a procedure for improving the accuracy of a given multistep method. Improving accuracy may in general reduce stability. It is then desirable to adjust the procedure in order to construct methods that combine a high order with certain specific stability requirements.

References

[1] U. M. Ascher and L. R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, SIAM, 1998.
[2] M. Beam and R. F. Warming, Alternating direction implicit methods for parabolic equations with a mixed derivative, SIAM J. Sci. Statist. Comput., 1 (1980), pp. 131–159.
[3] C. W. Cryer, A new class of highly-stable methods: A_0-stable methods, BIT, 13 (1973), 153–159.
[4] G. Dahlquist, A special stability problem for linear multistep methods, Nordisk Tidskr. Informations-Behandling, 3 (1963), pp. 27–43.
[5] B. Gustafsson, High Order Difference Methods for the Time Dependent PDE, Springer-Verlag, Berlin, 2008.
[6] A. Iserles, Numerical Analysis of Differential Equations, Cambridge University Press, Cambridge, UK, 1996.
[7] S. Karaa, Improved accuracy for the approximate factorization of parabolic equations, Computing, 86 (2009), pp. 23–36.
[8] S. Kim and H. Lim, High-order schemes for acoustic waveform simulation, Appl. Numer. Math., 57 (2007), pp. 402–414.
[9] J. D. Lambert, Numerical Methods for Ordinary Differential Systems, John Wiley and Sons, New York, 1991.
[10] M. N. Newmark, A method of computation for structural dynamics, Proc. ASCE, J. Engrg. Mech., 85 (1959), pp. 67-94.

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