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Spacelike Curves in the Lightlike Cone

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Abstract: In this paper, we study curves in the lightlike cone. First, we show that any curves in the lightlike cone are spacelike or lightlike, and then we characterize some curves with special cone curvature function in the 4, 5, and 6-dimensional lightlike cone. Finally, we consider the relationship between Frenet curvature functions and cone curvature functions for a spacelike curve on the lightlike cone.

Keywords: Asymptotic frame, cone curvature, lightlike cone, spacelike curve

1 Introduction

In Euclidean space, we can consider the behavior of a curve by Frenet orthonormal frame and its Frenet curvatures. For example, if all of the Frenet curvatures of a curve in $\mathbb{E}^n$ are constant, then this curve is

$$\begin{aligned}\alpha(s) = &(a_1\sin(\alpha_1 s), a_1\cos(\alpha_1 s), \dots,\\ &\dots, \alpha_m\sin(\alpha_m s), \alpha_m\cos(\alpha_m s), bs)\end{aligned}$$

for $n = 2m+1$ and

$$\begin{aligned}\alpha(s) = &(a_1\sin(\alpha_1 s), a_1\cos(\alpha_1 s), \dots,\\ &\dots, \alpha_m\sin(\alpha_m s), \alpha_m\cos(\alpha_m s))\end{aligned}$$

for $n = 2m$, see [3]. Also, S. Yılmaz and M. Turgut in [10] have defined vector products in Minkowski space-time, and by this way they calculate Frenet frames of all spacelike curves.

In the Lorentzian manifold, there are three type of curves, namely spacelike, timelike, and lightlike curves, and their Frenet equations are different, see [1, 7].

Besides the Frenet orthonormal frame along a curve on a lightlike cone, an asymptotic orthonormal frame is very useful. Asymptotic orthonormal frames are applied in order to consider curves, surfaces, and hypersurfaces in the lightlike cones.

H. Liu in [4, 5] has considered curves in the lightlike cone $\mathbb{Q}^{n+1}$. For this consideration, he defined the asymptotic orthonormal frame along a curve and cone curvature functions for such curve in $\mathbb{Q}^{n+1}$. Then he obtained some conformal invariants, and he classified curves with constant cone curvatures in $\mathbb{Q}^2$ and $\mathbb{Q}^3$. Also, in $\mathbb{Q}^2$, Liu established the relation between Frenet curvature and cone curvature, and he characterized the cone curvature function for a helix. We also remark that M. Bektaş and M. Külahci in [2] have obtained a characterization of spacelike curves in the 3-dimensional lightlike cone in terms of some differential equations.

In this article, we develop and generalize the results by H. Liu [4, 5]. The setup of this paper is as follows. After giving some preliminaries in Section 2, we show in Section 3 that any nonstraight line curve in $\mathbb{Q}^{n+1}$ is a spacelike curve. In Section 4, we characterize curves with constant cone curvature functions in $\mathbb{Q}^4$, $\mathbb{Q}^5$, and $\mathbb{Q}^6$. In Section 5, we give some relation between Frenet curvatures and cone curvature functions for a curve in $\mathbb{Q}^3$, and also we obtain cone curvature functions for a curve in $\mathbb{Q}^3$ such that the vectors α_1 and α_2 have constant angle with a constant vector b.

2 Preliminaries

Let $\mathbb{E}^n$ be n-dimensional Euclidean space. For two vectors $v = (v^1, \dots, v^n)$, $w = (w^1, \dots, w^n)$ and an integer $q \in [0, n]$,

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we define the bilinear form

$$\langle v,w\rangle := \sum_{i=1}^{n-q} v^i w^i - \sum_{i=n-q+1}^{n} v^i w^i.$$

The resulting semi-Riemannian space is called Minkowski n-space, and $n=4$ is the simplest example of a relativistic space-time, see [7].

Definition 1. *A vector $v \neq 0$ in $\mathbb{E}_1^n$ is called spacelike, timelike, or lightlike if $\langle v,v\rangle > 0, \langle v,v\rangle < 0$, or $\langle v,v\rangle = 0$, respectively, and $v=0$ is spacelike.*

Definition 2. *The set of all lightlike vectors in $\mathbb{E}_1^n$ is called the lightlike cone and denoted by $\mathbb{Q}^{n-1}$.*

In the lightlike cone $\mathbb{Q}^{n+1} \subset \mathbb{E}_1^{n+2}$, there are two orthonormal frame fields. One of them is a pseudo-Frenet orthonormal frame field and the other is an asymptotic orthonormal frame field, see [4].

Definition 3. *A frame field $\{e_1,\ldots,e_n,e_{n+1},e_{n+2}\}$ on $\mathbb{E}_1^{n+2}$ is called an asymptotic orthonormal frame field provided*

$$\langle e_{n+1},e_{n+1}\rangle = \langle e_{n+2},e_{n+2}\rangle = 0,\ \langle e_{n+1},e_{n+2}\rangle = 1,$$
$$\langle e_{n+1},e_i\rangle = \langle e_{n+2},e_i\rangle = 0,\ \langle e_i,e_j\rangle = \delta_{ij},\ i,j=1,\ldots,n.$$

Definition 4. *A frame field $\{e_1,\ldots,e_n,e_{n+1},e_{n+2}\}$ on $\mathbb{E}_1^{n+2}$ is called a pseudo-Frenet orthonormal frame field provided*

$$\langle e_{n+1},e_{n+1}\rangle = -\langle e_{n+2},e_{n+2}\rangle = 1,\ \langle e_{n+1},e_{n+2}\rangle = 0,$$
$$\langle e_{n+1},e_i\rangle = \langle e_{n+2},e_i\rangle = 0,\ \langle e_i,e_j\rangle = \delta_{ij},\ i,j=1,\ldots,n.$$

Definition 5. *A curve x in $\mathbb{E}_1^{n+2}$ is called a Frenet curve provided for all $t\in I$, the vector fields*

$$x(t),\dot{x}(t),\ddot{x}(t),\ldots,x^{(n)}(t),x^{(n+1)}(t)$$

are linearly independent and the vector fields

$$x(t),\dot{x}(t),\ddot{x}(t),\ldots,x^{(n+1)}(t),x^{(n+2)}(t)$$

are linearly dependent, where $x^{(n)}(t) = \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$.

Definition 6. *A curve $x: I \to \mathbb{E}_1^{n+2}$ is called spacelike, timelike, or lightlike provided $\dot{x}(t)$ is spacelike, timelike, or lightlike, respectively for all $t\in I$.*

Definition 7. *A spacelike or timelike curve $x: I\to\mathbb{E}_1^{n+2}$ is said to be parameterized by arc length provided*

$$\langle \dot{x}(s),\dot{x}(s)\rangle = 1 \text{ or } \langle \dot{x}(s),\dot{x}(s)\rangle = -1,$$

respectively.

Remark. In this article, all of the spacelike or timelike curves are parameterized by arclength denoted by s, and $x'(s) := \frac{\mathrm{d}x(s)}{\mathrm{d}s}$.

In [4], H. Liu defined an asymptotic orthonormal frame field for a given curve in the $\mathbb{Q}^{n+1}$ as follows. Let $x: I\to \mathbb{Q}^{n+1}\subset\mathbb{E}_1^{n+2}$ be a curve. We choose the null vector field $y(s)$ and the spacelike normal space V^{n-1} of the curve x such that they satisfy

$$\langle x(s),y(s)\rangle = 1,$$
$$\langle x(s),x(s)\rangle = \langle y(s),y(s)\rangle = \langle x'(s),y(s)\rangle = 0,$$
$$V^{n-1} = \left(\mathrm{span}_{\mathbb{R}}\{x,y,x'\}\right)^{\perp},$$
$$\mathrm{span}_{\mathbb{R}}\{x,y,x',V^{n-1}\} = \mathbb{E}_1^{n+2}.$$

Therefore, by choosing suitable orthonormal vector fields $\alpha_2(s),\alpha_3(s),\ldots,\alpha_n(s)\in V^{n-1}$, we have the Frenet formulas

$$\begin{aligned}
x'(s) &= \alpha_1(s)\\
\alpha_1'(s) &= \kappa_1(s)x(s) - y(s) + \tau_1(s)\alpha_2(s)\\
\alpha_2'(s) &= \kappa_2(s)x(s) - \tau_1(s)\alpha_1(s) + \tau_2(s)\alpha_3(s)\\
&\ \vdots\\
\alpha_i'(s) &= \kappa_i(s)x(s) - \tau_{i-1}(s)\alpha_{i-1}(s) + \tau_i(s)\alpha_{i+1}(s) \qquad (1)\\
&\ \vdots\\
\alpha_n'(s) &= \kappa_n(s)x(s) - \tau_{n-1}(s)\alpha_{n-1}(s)\\
y'(s) &= -\sum_{i=1}^{n}\kappa_i(s)\alpha_i(s),
\end{aligned}$$

where $\{x(s),y(s),x'(s),\alpha_2(s),\alpha_3(s),\ldots,\alpha_n(s)\}$ is an asymptotic orthonormal frame field, called the *asymptotic orthonormal frame* on $\mathbb{E}_1^{n+2}$ along the curve x in $\mathbb{Q}^{n+1}$. The functions $\kappa_i = \langle\alpha_i',y\rangle$, $i=1,\ldots,n$ and $\tau_i = \langle\alpha_i',\alpha_{i+1}\rangle$, $i=1,\ldots,n-1$, are called *cone curvature functions* of the curve x.

Proposition 1(see [5]). *Let $x: I\to\mathbb{Q}^{n+1}\subset\mathbb{E}_1^{n+2}$ be a spacelike curve and put*

$$y(s) := -x''(s) - \frac{1}{2}\langle x''(s),x''(s)\rangle x(s). \qquad (2)$$

Then $\tau_i = 0$, $i=1,\ldots,n$.

3 Nonstraight Line Curves in the Lightlike Cone $\mathbb{Q}^{n+1}$

In Euclidean space, a regular curve is a curve which has nonzero velocity vector. In the Minkowski space $\mathbb{E}_1^3$, any timelike (lightlike) curve is regular. Also, if a curve $x: I\to\mathbb{E}_1^3$ is regular in s_0, then, by continuity, x is also regular in a neighborhood of s_0, see [6]. Similarly to this, we can prove the following.

Proposition 2. *Any timelike (lightlike) curve $x: I\to\mathbb{E}_1^{n+2}$ (with arbitrary parameter) is regular.*

Proof. Assume that the curve is timelike. We write

$$x(t) = (x_1(t), \ldots, x_{n+1}(t), x_{n+2}(t)),$$

where x_i are differentiable functions on I. In this case, we have

$$\langle \dot{x}(t), \dot{x}(t)\rangle = \dot{x}_1^2(t) + \ldots + \dot{x}_{n+1}^2(t) - \dot{x}_{n+2}^2(t) < 0.$$

In particular, $\dot{x}_{n+2}(t) \neq 0$, i.e., x is regular. On the other hand, if the curve is lightlike, we have $\dot{x}_{n+2}(t) \neq 0$ again since, on the contrary, $\dot{x}_i(t) = 0$ and $\dot{x}(t) = 0$. But this means that the curve is spacelike. □

Lemma 1. *Let $x : I \to \mathbb{Q}^{n+1} \subset \mathbb{E}_1^{n+2}$ be a curve. Then x is lightlike if and only if x is a straight line.*

Proof. Let $\langle x, x\rangle = 0$ and $\langle \dot{x}, \dot{x}\rangle = 0$, so

$$\begin{aligned} x_{n+2}^2 &= x_1^2 + \ldots + x_{n+1}^2, \\ \dot{x}_{n+2}^2 &= \dot{x}_1^2 + \ldots + \dot{x}_{n+1}^2. \end{aligned} \tag{3}$$

Differentiation of the first equation in (3) yields that

$$\begin{aligned} x_{n+2}\dot{x}_{n+2} &= x_1\dot{x}_1 + \ldots + x_{n+1}\dot{x}_{n+1}, \\ (x_{n+2})^2(\dot{x}_{n+2})^2 &= (x_1\dot{x}_1 + \ldots + x_{n+1}\dot{x}_{n+1})^2. \end{aligned} \tag{4}$$

By substituting (3) into (4) and after some calculations, we conclude that

$$\sum_{i,j=1}^{n+1} (x_i\dot{x}_j - x_j\dot{x}_i)^2 = 0.$$

Thus,

$$\frac{\dot{x}_i}{x_i} = \frac{\dot{x}_1}{x_1},\ i = 1, \ldots, n+1.$$

Finally, we have

$$\begin{aligned} x_i(s) &= A_i x_1(s),\ i = 1, \ldots, n+1, \\ x_{n+2}(s) &= \pm x_1(s)\sqrt{1 + A_1^2 + \ldots + A_{n+1}^2}, \end{aligned}$$

where A_i is some real constant. Thus, $x(s) = \overrightarrow{A} x_1(s)$ is a straight line with the real differentiable function x_1 and constant lightlike velocity vector $\overrightarrow{A}$. On the other hand, the converse statement is trivial. □

Lemma 2. *Let $x : I \to \mathbb{E}_1^{n+2}$ be a timelike curve. Then x is not lying in $\mathbb{Q}^{n+1}$.*

Proof. Assume that x is in $\mathbb{Q}^{n+1}$. Then

$$x_{n+2}^2 = x_1^2 + \ldots + x_{n+1}^2, \tag{5}$$

so that

$$\sum_{i=1}^{n+1} x_i x_i' = x_{n+2}x_{n+2}'. \tag{6}$$

Since x is timelike, we get

$$\begin{aligned} (x_1^2 + \ldots + x_{n+1}^2)({x_1'}^2 + \ldots + {x_{n+1}'}^2) - x_{n+2}^2 {x_{n+2}'}^2 \\ = -x_{n+2}^2. \end{aligned} \tag{7}$$

If we replace (6) in (7), then we obtain

$$\sum_{i,j=1}^{n+1} (x_i x_j' - x_j x_i')^2 = -x_{n+2}^2.$$

Hence $x_{n+2} = 0$, and (5) yields

$$x_i(s) = 0,\ i = 1, \ldots, n+1.$$

So $x(s) = 0$, which is a contradiction. □

Lemmas 1 and 2 yield the following theorem.

Theorem 1. *If $x : I \to \mathbb{Q}^{n+1} \subset \mathbb{E}_1^{n+2}$ is a regular curve, then x is a nonstraight line if and only if x is a spacelike curve.*

Proof. Let x be a nonstraight line curve in $\mathbb{Q}^{n+1}$. Then, by Lemma 1, this curve is not a lightlike curve, and, by Lemma 2, this curve is not a timelike curve. Therefore, it is a spacelike curve. Conversely, if the curve is spacelike and a straight line, then $x(s) = \overrightarrow{A}\tilde{x}(s)$ such that $\tilde{x}(s)$ is a real differentiable function and $\overrightarrow{A}$ is a lightlike vector as x is lightlike, a contradiction. □

Remark. For the rest of this article, we assume that the curve x is a spacelike curve parameterized by arc length.

4 Curves in the Lightlike Cones $\mathbb{Q}^4$, $\mathbb{Q}^5$, and $\mathbb{Q}^6$

H. Liu in [4, Theorems 2.3 and 3.1] has classified all curves with constant cone curvature functions on $\mathbb{Q}^2$ and $\mathbb{Q}^3$. These curves are solutions of special differential equations. Similarly to these two theorems, we obtain constant cone curvature curves in the lightlike cones $\mathbb{Q}^4$, $\mathbb{Q}^5$, and $\mathbb{Q}^6$.

Theorem 2. *Let $x : I \to \mathbb{Q}^4 \subset \mathbb{E}_1^5$ be a curve in the lightlike cone $\mathbb{Q}^4$. If the cone curvature functions of the curve are constant, then the curve satisfies the differential equation*

$$\begin{aligned} x^{(5)} &+ (\tau_1^2 + \tau_2^2 - 2\kappa_1)x''' \\ &- (\kappa_2^2 + \kappa_3^2 + 2\kappa_3\tau_1\tau_2 + 2\kappa_1\tau_2^2)x' = 0. \end{aligned} \tag{8}$$

Proof. The Frenet formulas (1) for this curve are

$$\begin{aligned} x'(s) &= \alpha_1(s) \\ \alpha_1'(s) &= \kappa_1(s)x(s) - y(s) + \tau_1(s)\alpha_2(s) \\ \alpha_2'(s) &= \kappa_2(s)x(s) - \tau_1(s)\alpha_1(s) + \tau_2(s)\alpha_3(s) \\ \alpha_3'(s) &= \kappa_3(s)x(s) - \tau_2(s)\alpha_2(s) \\ y'(s) &= -\kappa_1(s)\alpha_1(s) - \kappa_2(s)\alpha_2(s) - \kappa_3(s)\alpha_3(s). \end{aligned} \tag{9}$$

From (9), we obtain the other derivatives of x as

$$\begin{aligned} x''' =& \tau_1\kappa_2 x + (2\kappa_1 - \tau_1^2)x' + \kappa_2\alpha_2 \\ &+ (\kappa_3 + \tau_1\tau_2)\alpha_3, \end{aligned} \tag{10}$$

$$\begin{aligned} x^{(4)} =& (\kappa_2^2 + \kappa_3^2 + \kappa_3\tau_1\tau_2)x \\ &+ (2\kappa_1 - \tau_1^2)x'' - (\kappa_3\tau_2 + \tau_1\tau_2^2)\alpha_2 + \kappa_2\tau_2\alpha_3, \\ x^{(5)} =& -\kappa_2\tau_1\tau_2^2 x + (\kappa_2^2 + \kappa_3^2 + 2\kappa_3\tau_1\tau_2 + \tau_1^2\tau_2^2)x' \\ &+ (2\kappa_1 - \tau_1^2)x''' - \kappa_2\tau_2^2\alpha_2 \\ &- \tau_2^2(\kappa_3 + \tau_1\tau_2)\alpha_3. \end{aligned} \tag{11}$$

If we multiply (10) by τ_2^2 and add the resulting equation to (11), then we obtain (8). □

Corollary 1. *If $x : I \to \mathbb{Q}^4 \subset \mathbb{E}_1^5$ is a curve with constant cone curvature in the lightlike cone $\mathbb{Q}^4$, then the following cases hold.*

i) Assume $\kappa_3 = \tau_2 = 0$.
i-i) If $\kappa_2 = 0$, then

$$x(s) = a_1 s^2 + a_2 s + a_3$$

provided $\lambda = 0$,

$$a_1 \sinh(\sqrt{2\lambda})s + a_2 \cosh(\sqrt{2\lambda})s + a_3$$

provided $\lambda > 0$, and

$$a_1 \sin(\sqrt{-2\lambda})s + a_2 \cos(\sqrt{-2\lambda})s + a_3$$

provided $\lambda < 0$, where $\lambda := 2\kappa_1 - \tau_1^2$.
i-ii) If $\kappa_2 \neq 0$, then

$$\begin{aligned} x(s) =& a_1 \sinh(\mu s) + a_2 \cosh(\mu s) + a_3 \sin(\nu s) \\ &+ a_4 \cos(\nu s) + a_5, \end{aligned}$$

where $\pm\mu$ and $\pm i\nu$ are the real and imaginary roots of the equation

$$t^4 - (2\kappa_1 - \tau_1^2)t^2 - \kappa_2^2 = 0.$$

ii) If $\kappa_3 = 0$ and $\tau_2 \neq 0$, then

$$\begin{aligned} x(s) =& a_1 \sinh(\mu s) + a_2 \cosh(\mu s) + a_3 \sin(\nu s) \\ &+ a_4 \cos(\nu s) + a_5, \end{aligned}$$

where $\pm\mu$ and $\pm i\nu$ are the real and imaginary roots of the equation

$$t^4 + (\tau_1^2 + \tau_2^2 - 2\kappa_1)t^2 - (\kappa_2^2 + 2\kappa_1\tau_2^2) = 0.$$

iii) Assume $\kappa_3 \neq 0$. Consider the equation

$$\begin{aligned} &t^4 + (\tau_1^2 + \tau_2^2 - 2\kappa_1)t^2 \\ &\quad - (\kappa_2^2 + \kappa_3^2 + 2\kappa_3\tau_1\tau_2 + 2\kappa_1\tau_2^2) = 0. \end{aligned} \tag{12}$$

iii-i) If the roots of (12) are $\pm\mu$ and $\pm i\nu$, then

$$\begin{aligned} x(s) =& a_1 \sinh(\mu s) + a_2 \cosh(\mu s) \\ &+ a_3 \sin(\nu s) + a_4 \cos(\nu s) + a_5. \end{aligned} \tag{13}$$

iii-ii) If the roots of (12) are $\pm\mu$ and $\pm\nu$, then

$$\begin{aligned} x(s) =& a_1 \sin(\mu s) + a_2 \cos(\mu s) + a_3 \sin(\nu s) \\ &+ a_4 \cos(\nu s) + a_5. \end{aligned} \tag{14}$$

iii-iii) If the roots of (12) are $\pm i\mu$ and $\pm i\nu$, then

$$\begin{aligned} x(s) =& a_1 \sinh(\mu s) + a_2 \cosh(\mu s) \\ &+ a_3 \sinh(\nu s) + a_4 \cosh(\nu s) + a_5. \end{aligned} \tag{15}$$

iii-vi) If the roots of (12) are $\pm\mu \pm i\nu$, then

$$\begin{aligned} x(s) =& a_1 \sinh(\mu s)\sin(\nu s) \\ &+ a_2 \cosh(\mu s)\sin(\nu s) \\ &+ a_3 \sinh(\mu s)\cos(\nu s) \\ &+ a_4 \cosh(\mu s)\cos(\nu s) + a_5, \end{aligned} \tag{16}$$

Here, $a_i \in \mathbb{E}_1^4$, $i \in \{1,2,3,4,5\}$, are suitable constant vectors.

Proof. First, we prove i). If we set $\kappa_3 = \tau_2 = 0$ in (9), then α_3 is constant, and thus [4, Theorem 3.1] yields the statement. Next, we prove ii). The discriminant of the quadratic equation

$$t^2 + (\tau_1^2 + \tau_2^2 - 2\kappa_1)t - (\kappa_2^2 + 2\kappa_1\tau_2^2) = 0 \tag{17}$$

is

$$\Delta = \begin{cases} (2\kappa_1 - \tau_1^2)^2 + \tau_2^4 + 2\tau_1^2\tau_2^2 + 4\kappa_1\tau_2^2 + 4\kappa_2^2, & \kappa_1 \geq 0, \\ (2\kappa_1 - \tau_2^2)^2 + \tau_1^4 + 2\tau_1^2\tau_2^2 - 4\kappa_1\tau_1^2 + 4\kappa_2^2, & \kappa_1 < 0. \end{cases}$$

Since $\tau_2 \neq 0$, we have $\Delta > 0$, and thus (17) has two real roots with different signs. If we set $\tilde{x} := x'$ in (8), then we obtain a differential equation of order 4 with constant coefficients. After an integration, the statement is proved. Finally, we prove iii). If we set $\tilde{x} := x'$ in (8), then we obtain a differential equation of order 4 with constant coefficients. In general, the solution of such differential equations are $\exp(\lambda s)$, where λ is a root of (12). If λ is real, then the solution is

$$a_1 \sinh(\lambda s) + a_2 \cosh(\lambda s),$$

while if λ is imaginary, then the solution is

$$a_3 \sin(i\lambda s) + a_4 \cos(i\lambda s).$$

If $\lambda = \mu + i\nu$ is a complex number, then

$$\begin{aligned} \tilde{x}(s) =& a_1 \sinh(\mu s)\sin(\nu s) + a_2 \cosh(\mu s)\sin(\nu s) \\ &+ a_3 \sinh(\mu s)\cos(\nu s) + a_4 \cosh(\mu s)\cos(\nu s). \end{aligned}$$

An integration yields that the curve is in the forms (13), (14), (15), or (16). □

Theorem 3. *Let $x : I \to \mathbb{Q}^5 \subset \mathbb{E}_1^6$ be a curve in the lightlike cone $\mathbb{Q}^5$. If the cone curvature functions of the curve are constant, then the curve satisfies the differential equation*

$$\begin{aligned} &x^{(6)} + (\tau_1^2 + \tau_2^2 + \tau_3^2 - 2\kappa_1)x^{(4)} \\ &- (\kappa_2^2 + \kappa_4^2 + (2\kappa_1 - \tau_1^2)(\tau_2^2 + \tau_3^2) + (\kappa_3 + \tau_1\tau_2)^2)x'' \\ &= (\kappa_4\tau_2 + \kappa_2\tau_3)^2 x. \end{aligned} \quad (18)$$

Proof. The Frenet formulas (1) for this curve are

$$\begin{aligned} x'(s) =& \alpha_1(s), \\ \alpha_1'(s) =& \kappa_1(s)x(s) - y(s) + \tau_1(s)\alpha_2(s), \\ \alpha_2'(s) =& \kappa_2(s)x(s) - \tau_1(s)\alpha_1(s) + \tau_2(s)\alpha_3(s), \\ \alpha_3'(s) =& \kappa_3(s)x(s) - \tau_2(s)\alpha_2(s) + \tau_3(s)\alpha_4(s), \\ \alpha_4'(s) =& \kappa_4(s)x(s) - \tau_3(s)\alpha_3(s), \\ y'(s) =& -\kappa_1(s)\alpha_1(s) - \kappa_2(s)\alpha_2(s) \\ & - \kappa_3(s)\alpha_3(s) - \kappa_4(s)\alpha_4(s). \end{aligned} \quad (19)$$

From (19), we obtain the other derivatives of x as

$$\begin{aligned} x''' =& \tau_1\kappa_2 x + (2\kappa_1 - \tau_1^2)x' + \kappa_2\alpha_2 \\ & + (\kappa_3 + \tau_1\tau_2)\alpha_3 + \kappa_4\alpha_4, \end{aligned} \quad (20)$$

$$\begin{aligned} x^{(4)} =& (\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \kappa_3\tau_1\tau_2)x \\ & + (2\kappa_1 - \tau_1^2)x'' - (\kappa_3\tau_2 + \tau_1\tau_2^2)\alpha_2 \\ & + (\kappa_2\tau_2 - \kappa_4\tau_3)\alpha_3 + (\kappa_3\tau_3 + \tau_1\tau_2\tau_3)\alpha_4, \end{aligned} \quad (21)$$

$$\begin{aligned} x^{(5)} =& (\kappa_4\tau_1\tau_2\tau_3 - \kappa_2\tau_1\tau_2^2)x \\ & + (\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \tau_1^2\tau_2^2 + 2\kappa_3\tau_1\tau_2)x' \\ & + (2\kappa_1 - \tau_1^2)x''' + (\kappa_4\tau_2\tau_3 - \kappa_2\tau_2^2)\alpha_2 \\ & - (\kappa_3\tau_2^2 + \tau_1\tau_2^3 + \kappa_3\tau_3^2 + \tau_1\tau_2\tau_3^2)\alpha_3 \\ & + (\kappa_2\tau_2\tau_3 - \kappa_4\tau_3^2)\alpha_4, \\ x^{(6)} =& \big(2\kappa_2\kappa_4\tau_2\tau_3 - \tau_2^2(\kappa_2^2 + \kappa_3^2 + \kappa_3\tau_1\tau_2) \\ & - \tau_3^2(\kappa_3^2 + \kappa_4^2 + \kappa_3\tau_1\tau_2)\big)x \\ & + (\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \tau_1^2\tau_2^2 + 2\kappa_3\tau_1\tau_2)x'' \\ & + (2\kappa_1 - \tau_1^2)x^{(4)} + (\tau_2^2 + \tau_3^2)(\kappa_3\tau_2 + \tau_1\tau_2^2)\alpha_2 \\ & + (\tau_2^2 + \tau_3^2)(\kappa_4\tau_3 - \kappa_2\tau_2)\alpha_3 \\ & - (\tau_2^2 + \tau_3^2)(\kappa_3\tau_3 + \tau_1\tau_2\tau_3)\alpha_4. \end{aligned} \quad (22)$$

If we multiply (21) by $(\tau_2^2 + \tau_3^2)$ and add the resulting equation to (22), then we obtain (18). □

Corollary 2. *Let $x : I \to \mathbb{Q}^5 \subset \mathbb{E}_1^6$ be a curve with constant cone curvature in the lightlike cone $\mathbb{Q}^5$. Assume*

$$\kappa_4\tau_2 + \kappa_2\tau_3 \neq 0.$$

Consider the equation

$$\begin{aligned} &t^3 + (\tau_1^2 + \tau_2^2 + \tau_3^2 - 2\kappa_1)t^2 \\ &- \left(\kappa_2^2 + \kappa_4^2 + (2\kappa_1 - \tau_1^2)(\tau_2^2 + \tau_3^2) + (\kappa_3 + \tau_1\tau_2)^2\right)t \\ &- (\kappa_4\tau_2 + \kappa_2\tau_3)^2 = 0. \end{aligned} \quad (23)$$

i) If the roots of (23) are λ^2 and $(\mu \pm \mathrm{i}\nu)^2$, then

$$\begin{aligned} x(s) =& a_1\sinh(\mu s)\sin(\nu s) + a_2\cosh(\mu s)\sin(\nu s) \\ & + a_3\sinh(\mu s)\cos(\nu s) + a_4\cosh(\mu s)\cos(\nu s) \\ & + a_5\sinh(\lambda s) + a_6\cosh(\lambda s). \end{aligned}$$

ii) If the roots of (23) are λ^2, μ^2, and ν^2, then

$$\begin{aligned} x(s) =& a_1\sinh(\mu s) + a_2\cosh(\mu s) + a_3\sinh(\nu s) \\ & + a_4\cosh(\nu s) + a_5\sinh(\lambda s) + a_6\cosh(\lambda s). \end{aligned}$$

iii) If the roots of (23) are λ^2, $(\mathrm{i}\mu)^2$, and $(\mathrm{i}\nu)^2$, then

$$\begin{aligned} x(s) =& a_1\sin(\mu s) + a_2\cos(\mu s) + a_3\sin(\nu s) \\ & + a_4\cos(\nu s) + a_5\sinh(\lambda s) + a_6\cosh(\lambda s). \end{aligned}$$

Here, $a_i \in \mathbb{E}_1^6$, $i \in \{1,2,3,4,5,6\}$, are suitable constant vectors.

Proof. Denote the left-hand side of (23) by $f(t)$. Then $f(0) < 0$ and

$$\begin{aligned} f'(t) =& 3t^2 + 2(\tau_1^2 + \tau_2^2 + \tau_3^2 - 2\kappa_1)t \\ & - \left(\kappa_2^2 + \kappa_4^2 + (2\kappa_1 - \tau_1^2)(\tau_2^2 + \tau_3^2) \right. \\ & \left. + (\kappa_3 + \tau_1\tau_2)^2\right), \\ \Delta =& 3(\tau_2^2 + \tau_3^2)^2 + \left(\tau_2^2 + \tau_3^2 + 2(2\kappa_1 - \tau_1^2)\right)^2 \\ & + 12\left(\kappa_2^2 + \kappa_4^2 + (\kappa_3 + \tau_1\tau_2)^2\right) > 0. \end{aligned}$$

Thus, (23) has at least one positive root. The other roots are one of the following. In i) both are $(\mu \pm \mathrm{i}\nu)^2$. In ii), both of them are positive, i.e., they are μ^2 and ν^2. In iii), both of them are negative, i.e., they $(\mathrm{i}\mu)^2$ and $(\mathrm{i}\nu)^2$. Thus, the curves in i), ii), and iii) are satisfying (18). □

Corollary 3. *Let $x : I \to \mathbb{Q}^5 \subset \mathbb{E}_1^6$ be a curve with constant cone curvature in the lightlike cone $\mathbb{Q}^5$. Assume*

$$\kappa_4\tau_2 + \kappa_2\tau_3 = 0.$$

Consider the equation

$$\begin{aligned} &t^2 + (\tau_1^2 + \tau_2^2 + \tau_3^2 - 2\kappa_1)t \\ &- \left(\kappa_2^2 + \kappa_4^2 + (2\kappa_1 - \tau_1^2)(\tau_2^2 + \tau_3^2) + (\kappa_3 + \tau_1\tau_2)^2\right) = 0. \end{aligned} \quad (24)$$

i) If the roots of (24) are $\mu > 0$ and $\nu < 0$, then

$$\begin{aligned} x(s) =& a_1\sinh(s\sqrt{\mu}) + a_2\cosh(s\sqrt{\mu}) + a_3\sin\left(s\sqrt{-\nu}\right) \\ & + a_4\cos\left(\sqrt{-\nu}s\right) + a_5 s + a_6. \end{aligned}$$

ii) If at least one of the roots of (24) is zero, then

$$x(s) = a_1 s^2 + a_2 s + a_3 \quad (25)$$

provided $\lambda = 0$,

$$x(s) = a_1\sinh\left(s\sqrt{2\lambda}\right) + a_2\cosh\left(s\sqrt{2\lambda}\right) + a_3 \quad (26)$$

provided $\lambda > 0$, *and*

$$x(s) = a_1 \sin\left(s\sqrt{-2\lambda}\right) + a_2 \cos\left(s\sqrt{-2\lambda}\right) + a_3 \tag{27}$$

provided $\lambda < 0$, *where* $\lambda := 2\kappa_1 - \tau_1^2$.

Here, $a_i \in \mathbb{E}_1^6$, $i \in \{1,2,3,4,5,6\}$, *are suitable constant vectors.*

Proof. The discriminant of (24) is

$$\Delta = (\tau_2^2 + \tau_3^2 + 2\kappa_1 - \tau_1^2)^2 + 4\kappa_2^2 + 4\kappa_4^2 + 4(\kappa_3 + \tau_1\tau_2)^2 \geq 0.$$

First, assume $\Delta > 0$. If the roots of (24) are $\mu > 0$ and $\nu < 0$, then, as in Corollary 1, we may prove i). If one of the roots of (24) is zero, then (24) yields

$$\kappa_2^2 + \kappa_4^2 + (2\kappa_1 - \tau_1^2)(\tau_2^2 + \tau_3^2) + (\kappa_3 + \tau_1\tau_2)^2 = 0,$$
$$\kappa_2 = \kappa_4 = \kappa_3 + \tau_1\tau_2 = (2\kappa_1 - \tau_1^2)(\tau_2^2 + \tau_3^2) = 0,$$

from which together with (20), we have

$$x''' = (2\kappa_1 - \tau_1^2)x'.$$

The solution of this equation in the different cases is (25), (26), and (27). In the case $\Delta = 0$, we conclude that any two roots of (24) are zero, and the curve is then (25). $\square$

Theorem 4. *Let* $x : I \to \mathbb{Q}^6 \subset \mathbb{E}_1^7$ *be a curve in the lightlike cone* $\mathbb{Q}^6$. *If the cone curvature functions of the curve are constant, then the curve satisfies the differential equation*

$$x^{(7)} + Ax^{(5)} + Bx^{(3)} + Cx' = 0, \tag{28}$$

where A, B, and C are constant coefficients.

Proof. Proceeding as in the proofs of Theorems 2 and 3, we obtain the other derivatives of x as

$$\begin{aligned}
x''' =& \tau_1\kappa_2 x + (2\kappa_1 - \tau_1^2)x' + \kappa_2\alpha_2 \\
&+ (\kappa_3 + \tau_1\tau_2)\alpha_3 + \kappa_4\alpha_4 + \kappa_5\alpha_5, \qquad (29)\\
x^{(4)} =& (\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \kappa_5^2 + \kappa_3\tau_1\tau_2)x + (2\kappa_1 - \tau_1^2)x'' \\
&- (\kappa_3\tau_2 + \tau_1\tau_2^2)\alpha_2 + (\kappa_2\tau_2 - \kappa_4\tau_3)\alpha_3 \\
&+ (\kappa_3\tau_3 + \tau_1\tau_2\tau_3 - \kappa_5\tau_4)\alpha_4 + \kappa_4\tau_4\alpha_5, \\
x^{(5)} =& (\kappa_4\tau_1\tau_2\tau_3 - \kappa_2\tau_1\tau_2^2)x \\
&+ (\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \kappa_5^2 + \tau_1^2\tau_2^2 + 2\kappa_3\tau_1\tau_2)x' \\
&+ (2\kappa_1 - \tau_1^2)x''' + (\kappa_4\tau_2\tau_3 - \kappa_2\tau_2^2)\alpha_2 \\
&- \big((\kappa_3 + \tau_1\tau_2)(\tau_2^2 + \tau_3^2) - \kappa_5\tau_3\tau_4\big)\alpha_3 \\
&+ (\kappa_2\tau_2\tau_3 - \kappa_4\tau_3^2 - \kappa_4\tau_4^2)\alpha_4 \\
&+ (\kappa_3\tau_3\tau_4 + \tau_1\tau_2\tau_3\tau_4 - \kappa_5\tau_4^2)\alpha_5, \qquad (30)\\
x^{(6)} =& \big(2\kappa_2\kappa_4\tau_2\tau_3 + 2\kappa_3\kappa_5\tau_3\tau_4 - \tau_2^2(\kappa_2^2 + \kappa_3^2 + \kappa_3\tau_1\tau_2) \\
&- \tau_3^2(\kappa_3^2 + \kappa_4^2 + \kappa_3\tau_1\tau_2) - \tau_4^2(\kappa_4^2 + \kappa_5^2) \\
&+ \kappa_5\tau_1\tau_2\tau_3\tau_4\big)x \\
&+ (\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \kappa_5^2 + \tau_1^2\tau_2^2 + 2\kappa_3\tau_1\tau_2)x'' \\
&+ (2\kappa_1 - \tau_1^2)x^{(4)} \\
&+ \big((\tau_2^2 + \tau_3^2)(\kappa_3\tau_2 + \tau_1\tau_2^2) - \kappa_5\tau_2\tau_3\tau_4\big)\alpha_2 \\
&+ \big(\tau_2^2 + \tau_3^2)(\kappa_4\tau_3 - \kappa_2\tau_2) + \kappa_4\tau_3\tau_4^2\big)\alpha_3 \\
&- \big(\tau_2^2 + \tau_3^2)(\kappa_3\tau_3 + \tau_1\tau_2\tau_3) \\
&\quad + \tau_4^2(\kappa_3\tau_3 - \kappa_5\tau_4 + \tau_1\tau_2\tau_3)\big)\alpha_4 \\
&+ (\kappa_2\tau_2\tau_3\tau_4 - \kappa_4\tau_3^2\tau_4 - \kappa_4\tau_4^3)\alpha_5, \\
x^{(7)} =& \big((\tau_2^2 + \tau_3^2 + \tau_4^2)(\kappa_2\tau_1\tau_2^2 - \kappa_4\tau_1\tau_2\tau_3) - \kappa_4\tau_1\tau_2^2\tau_4^2\big)x \\
&+ \big(2\kappa_2\kappa_4\tau_2\tau_3 + 2\kappa_3\kappa_5\tau_3\tau_4 - 2\kappa_3\tau_1\tau_2(\tau_2^2 + \tau_3^2) \\
&\quad - \kappa_2^2\tau_2^2 - \kappa_3^2(\tau_2^2 + \tau_3^2) - \kappa_4^2(\tau_3^2 + \tau_4^2) \\
&\quad + \kappa_5\tau_1\tau_2\tau_3\tau_4 - \kappa_5^2\tau_4^2 \\
&\quad - \tau_1^2\tau_2^2(\tau_2^2 + \tau_3^2) + \kappa_5\tau_1^2\tau_3\tau_4\big)x' \\
&+ \big(\kappa_2^2 + \kappa_3^2 + \kappa_4^2 + \kappa_5^2 + \tau_1^2\tau_2^2 \\
&\quad - 2\kappa_3\tau_1\tau_2(\tau_2^2 + \tau_3^2) + 2\kappa_3\tau_1\tau_2\big)x''' \\
&+ (2\kappa_1 - \tau_1^2)x^{(5)} \\
&+ \big((\tau_2^2 + \tau_3^2 + \tau_4^2)(\kappa_2\tau_2^2 - \kappa_4\tau_2\tau_3) - \kappa_2\tau_2^2\tau_4^2\big)\alpha_2 \\
&+ \big((\tau_2^2 + \tau_3^2)(\kappa_3\tau_2^2 + \tau_1\tau_2^3) - \kappa_5\tau_3\tau_4((\tau_2^2 + \tau_3^2 + \tau_4^2) \\
&\quad + (\kappa_3 + \tau_1\tau_2)\tau_3^2\tau_4^2\big)\alpha_3 \\
&+ \big((\kappa_4\tau_3^2 + \kappa_4\tau_4^2 - \kappa_2\tau_2\tau_3)(\tau_2^2 + \tau_3^2 + \tau_4^2) \\
&\quad - \kappa_4\tau_2^2\tau_4^2\big)\alpha_4 \\
&- \big((\tau_2^2 + \tau_3^2 + \tau_4^2)(\kappa_3\tau_3\tau_4 + \tau_1\tau_2\tau_3\tau_4 - \kappa_5\tau_4^2) \\
&\quad + \kappa_5\tau_2^2\tau_4^2\big)\alpha_5. \qquad (31)
\end{aligned}$$

If we multiply (30) by $(\tau_2^2 + \tau_3^2 + \tau_4^2)$, multiply (29) by $\tau_2^2\tau_4^2$, and then add the resulting equations to (31), then we obtain (28), where A, B, and C are suitable constant coefficients. $\square$

Theorem 5. *Let* $x : I \to \mathbb{Q}^{n+1}$ *be a curve with constant cone curvature in* $\mathbb{Q}^{n+1}$. *Assume that the lightlike vector field* $y(s)$ *is as in* (2).

i) If $\kappa_i = 0$ *for all* $i \in \{2,\ldots,n\}$, *then*

$$x(s) = a_1 s^2 + a_2 s + a_3$$

provided $\kappa_1 = 0$,

$$x(s) = a_1 \sinh\left(s\sqrt{2\kappa_1}\right) + a_2 \cosh\left(s\sqrt{2\kappa_1}\right) + a_3$$

provided $\kappa_1 > 0$, *and*

$$x(s) = a_1 \sin\left(s\sqrt{-2\kappa_1}\right) + a_2 \cos\left(s\sqrt{-2\kappa_1}\right) + a_3$$

provided $\kappa_1 < 0$.

ii) If one of the $\kappa_i \neq 0$ for $i \in \{2,\dots,n\}$, then

$$x(s) = a_1 \sinh(\mu s) + a_2 \cosh(\mu s) + a_3 \sin(\nu s) + a_4 \cos(\nu s), \tag{32}$$

where $a_i \in \mathbb{E}_1^{n+2}$, $i \in \{1,2,3\}$, are suitable vectors and $\pm\mu$ and $\pm i\nu$ are the real and imaginary roots of the equation

$$t^4 - 2\kappa_1 t^2 - \sum_{i=2}^{n} \kappa_i^2 = 0.$$

Proof. First, we prove i). By Proposition 1, all of the $\tau_i = 0$, and because of $\kappa_i = 0$ for all $i \in \{2,\dots,n\}$, by the Frenet formulas (1), we conclude that

$$x'''(s) - 2\kappa_1 x'(s) = 0,$$

which has its solutions in the stated forms. Finally, we prove ii). By Proposition 1, all of the $\tau_i = 0$, thus, by the Frenet formulas (1), we conclude that

$$x^{(4)}(s) - 2\kappa_1 x''(s) - \sum_{i=2}^{n} \kappa_i^2 x(s) = 0,$$

which has its solutions in the form (32). $\square$

5 Some Results for Spacelike Curves in $\mathbb{Q}^3$

In this section, we follow the relationship between Frenet curvatures and cone curvature functions on $\mathbb{Q}^3$ in special cases. J. Walrave in [9] has classified all spacelike curves in $\mathbb{E}_1^4$, and we consider such constant cone curvature curves. By Corollary 1, the only nonstraightline curves in $\mathbb{Q}^{n+1}$ are spacelike curves, thus we will not study the other types.

M. P. Torgašev and E. Šućurović in [8, Remarks 3.2 and 3.3] have proved that if $x : I \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ is a curve in the lightlike cone $\mathbb{Q}^3$ with constant Frenet curvatures and the principal vector N or the binormal vector B_1 is timelike, then $\overline{\tau}^2 = \overline{\sigma}^2$. In the following theorems, in the cases when B_2 or N is timelike, we independently prove this, and next we will show that if B_1 is timelike or B_1, B_2 are lightlike, then $\kappa_2 = \overline{\tau} = \overline{\sigma} = 0$.

Remark. In the sequel, we will write κ_1, κ_2, and τ for cone curvature functions, and $\overline{\kappa}$, $\overline{\tau}$, and $\overline{\sigma}$ for Frenet curvatures functions.

Theorem 6. *Let $x : I \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a curve in the lightlike cone $\mathbb{Q}^3$. If the cone curvature functions of the curve are constant, then its Frenet curvatures are constant.*

Proof. Case 1. Let N and B_1 be spacelike. Thus, B_2 is timelike. In this case, we have the Frenet formulas

$$\begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}' = \begin{pmatrix} 0 & \overline{\kappa} & 0 & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} & 0 \\ 0 & -\overline{\tau} & 0 & \overline{\sigma} \\ 0 & 0 & \overline{\sigma} & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}. \tag{33}$$

By (1), for $\mathbb{Q}^3$, we have

$$N = \frac{1}{\overline{\kappa}} \alpha_1' = \frac{1}{\overline{\kappa}} (\kappa_1 x - y + \tau \alpha_2), \qquad \overline{\kappa}^2 = \tau^2 - 2\kappa_1. \tag{34}$$

Thus, the first Frenet curvature of the curve is constant. By (33) and (34), we have

$$N' = -\overline{\kappa} T + \overline{\tau} B_1 = -\overline{\kappa} T + \frac{\tau \kappa_2}{\overline{\kappa}} x + \frac{\kappa_2}{\overline{\kappa}} \alpha_2, \qquad \overline{\tau}^2 = \left(\frac{\kappa_2}{\overline{\kappa}}\right)^2, \; B_1 = \varepsilon(\tau x + \alpha_2), \tag{35}$$

where $\varepsilon = \pm 1$. Thus, $\overline{\tau}$ is constant. Now by (35) and the third equation of (33), we have

$$B_1' = -\overline{\tau} N + \overline{\sigma} B_2 = \varepsilon \kappa_2 x,$$

so that B_1' is lightlike, and as a result of orthonormality, we have

$$\overline{\tau}^2 - \overline{\sigma}^2 = 0, \; \overline{\tau}^2 = \overline{\sigma}^2,$$

and hence $\overline{\sigma}$ is constant.

Case 2. Let B_1, B_2 be spacelike and N be timelike. In this case, we have the Frenet formulas

$$\begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}' = \begin{pmatrix} 0 & \overline{\kappa} & 0 & 0 \\ \overline{\kappa} & 0 & \overline{\tau} & 0 \\ 0 & \overline{\tau} & 0 & \overline{\sigma} \\ 0 & 0 & -\overline{\sigma} & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}.$$

Similarly as in Case 1, this results in the Frenet curvatures being constant. In this case, we have $\overline{\kappa}^2 = 2\kappa_1 - \tau^2$.

Case 3. Let N, B_2 be spacelike and B_1 be timelike. We have the Frenet equations

$$\begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}' = \begin{pmatrix} 0 & \overline{\kappa} & 0 & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} & 0 \\ 0 & \overline{\tau} & 0 & \overline{\sigma} \\ 0 & 0 & \overline{\sigma} & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}. \tag{36}$$

This case is similar to Cases 1 and 2 as well, and by an elementary calculation, we conclude that $\overline{\kappa}^2 = \tau^2 - 2\kappa_1$ and $\kappa_2 = \overline{\tau} = \overline{\sigma} = 0$ are constant (see [8, Remark 3.2 and Theorem 3.7]).

Case 4. Let N be spacelike and B_1, B_2 be lightlike. In this case, we have

$$B_1 = N' + \overline{\tau} T, \; \langle B_1, B_2 \rangle = 1, \; \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 0,$$

and the Frenet formulas are

$$\begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}' = \begin{pmatrix} 0 & \overline{\kappa} & 0 & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} & 0 \\ 0 & 0 & \overline{\sigma} & 0 \\ 0 & -\overline{\tau} & 0 & -\overline{\sigma} \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}.$$

This case is similar to Case 3. We conclude that $\overline{\kappa}^2 = \tau^2 - 2\kappa_1$ and $\kappa_2 = \overline{\tau} = \overline{\sigma} = 0$ are constant. $\square$

Corollary 4. *Let $x : I \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be the curve in Cases 3 or 4 of Theorem 6. If its Frenet curvatures are constant, then it is a planar curve.*

Proof. By Theorem 6, in Cases 3 or 4, we have $\kappa_2 = 0$. Thus, by [8, Theorem 3.1], the curve is planar. □

Theorem 7. *Let $x : I \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a curve in the lightlike cone $\mathbb{Q}^3$ with constant Frenet curvatures. Then its cone curvature functions satisfy the relations*

$$\overline{\kappa}^2 = \tau^2 - 2\kappa_1, \ \kappa_2 + \tau' = \pm\overline{\tau\kappa}.$$

Moreover, in Cases 3 and 4 of Theorem 6, if $\overline{\kappa} \neq 0$, then $\overline{\tau} = \overline{\sigma} = 0$.

Proof. In all of the cases, by the second equation in (1), we have

$$\alpha_1' = \kappa_1 x - y + \tau\alpha_2, \ \overline{\kappa}^2 = \tau^2 - 2\kappa_1.$$

In Case 1, by the Frenet equation (33), we have

$$N' = -\overline{\kappa}T + \overline{\tau}B_1 = \frac{1}{\overline{\kappa}}(-\overline{\kappa}^2 x' + \kappa_1' x + \tau'\alpha_2 + \tau\kappa_2 x + \kappa_2\alpha_2).$$

Since B_1 is spacelike, we get

$$(\overline{\tau\kappa})^2 = (\kappa_2 + \tau')^2.$$

Again, by the Frenet equation (33), we have

$$\begin{aligned} B_1' &= -\overline{\tau}N + \overline{\sigma}B_2 \\ &= \frac{1}{\overline{\tau\kappa}}\{(\kappa_2^2 + 2\tau'\kappa_2 + \tau\kappa_2' + \kappa_1'')x \\ &\quad + (\tau\kappa_2 + \kappa_1' - \tau\kappa_2 - \tau\tau')x' + (\kappa_2' + \tau'')\alpha_2\}, \end{aligned}$$

and since $\tau\tau' - \kappa_1' = \kappa_2' + \tau'' = 0$ and $\kappa_1'' = \tau\tau'' + \tau'^2$, we conclude that

$$-\overline{\tau}N + \overline{\sigma}B_2 = (\overline{\tau\kappa})x.$$

Since x is lightlike, N is spacelike and B_2 is timelike, thus $\overline{\tau}^2 = \overline{\sigma}^2$.

Case 2 is similar.

In Case 3, by the Frenet equation (36), we have

$$N' = -\overline{\kappa}T + \overline{\tau}B_1 = \frac{1}{\overline{\kappa}}(-\overline{\kappa}x' + \kappa_1' x + \tau'\alpha_2 + \tau\kappa_2 x + \kappa_2\alpha_2).$$

Since B_1 is timelike, we have

$$-(\overline{\tau\kappa})^2 = (\kappa_2 + \tau')^2.$$

Since $\overline{\kappa} \neq 0$, we obtain $\overline{\tau} = 0$. □

Corollary 5. *Let $x : I \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a curve in the lightlike cone $\mathbb{Q}^3$. Assume that the lightlike vector field y is defined in (2). Then the cone curvature functions of the curve are constant if and only if its Frenet curvatures are constant.*

Proof. In this case, from $\langle N, N\rangle = 1$, we have

$$\overline{\kappa}^2 = -2\kappa_1.$$

Thus the first cone curvature function is constant if and only if the first Frenet curvature function is constant. In Case 1 of Theorem 6, we conclude

$$\overline{\tau}^2 = \left(\frac{\kappa_2}{\overline{\kappa}}\right)^2.$$

Hence the second cone curvature function is constant if and only if the second (third) Frenet curvature is constant. □

H. Liu in [4] proved that if $x : I \to \mathbb{Q}^2$ is a helix such that its velocity vector field has constant angle with a constant vector, then its cone curvature function satisfies

$$\kappa(s) = c_1(s + c_2)^{-2},$$

where $c_1 \neq 0$ and c_2 are constant. Now, we prove a similar result as follows.

Theorem 8. *Let $x : I \to \mathbb{Q}^3$ be a curve such that α_1 and α_2 have constant angle with the constant vector b. Then its cone curvature function κ_1 and κ_2 satisfy*

$$\kappa_1 l + \kappa_2\tilde{l} = c_1(s + c_2)^{-2},$$

where $\langle\alpha_1, b\rangle = l$, $\langle\alpha_2, b\rangle = \tilde{l}$, and c_1, c_2 are real constants.

Proof. From $\langle\alpha_1, b\rangle = l$, we have

$$\langle x, b\rangle = ls + l_0$$

and

$$0 = \langle\alpha_1', b\rangle = \langle\kappa_1 x - y - \tau\alpha_2, b\rangle = \kappa_1\langle x, b\rangle - \langle y, b\rangle + \tau\tilde{l}.$$

Thus,

$$\kappa_1(ls + l_0) + 2\kappa_1 l + (\kappa_2 + \tau')\tilde{l} = 0. \tag{37}$$

Similarly, $\langle\alpha_2, b\rangle = \tilde{l}$ yields

$$\langle\alpha_2', b\rangle = \kappa_2(ls + l_0) - \tau l = 0.$$

By differentiation, we have

$$\tau' = \kappa_2' s + \kappa_2 + \kappa_2'\frac{l_0}{l}. \tag{38}$$

From (38) and (37), we get

$$(\kappa_1' l + \kappa_2'\tilde{l})(ls + l_0) + 2(\kappa_1 l + \kappa_2'\tilde{l}) = 0,$$

so that

$$\frac{\kappa_1' l + \kappa_2'\tilde{l}}{\kappa_1 l + \kappa_2\tilde{l}} = -2\frac{l}{ls + l_0}.$$

Hence the solutions of this differential equation are

$$\kappa_1 l + \kappa_2\tilde{l} = c(ls + l_0)^{-2}.$$

Thus,

$$\kappa_1 l + \kappa_2\tilde{l} = c_1(s + c_2)^{-2},$$

completing the proof. □

Theorem 9. *Let $x : I \to \mathbb{Q}^3$ be a spacelike curve such that the position vector $x(s)$ has constant inner product with a constant vector b. If the lightlike vector field $y(s)$ is defined as in (2), then the cone curvature functions κ_1 and κ_2 satisfy*

$$\kappa_2 = -\left(\frac{\kappa_1'}{\kappa_2}\right)'. \tag{39}$$

Proof. Let $\langle x, b\rangle = l$ such that l is a constant. Then

$$\langle x', b\rangle = 0.$$

By the Frenet equations (1), we have

$$\langle y, b\rangle = \kappa_1 l.$$

The vector b, by the frame $\{x, x', y, \alpha_2\}$, can be expressed as

$$b = l\kappa_1(s)x(s) + ly(s) + \lambda(s)\alpha_2(s).$$

Thus, $b' = 0$ and

$$(l\kappa_1'(s) + \lambda(s)\kappa_2(s))x(s) + (\lambda'(s) - l\kappa_2(s))\alpha_2(s) = 0.$$

Therefore,

$$\lambda'(s) - l\kappa_2(s) = 0,\ l\kappa_1'(s) + \lambda(s)\kappa_2(s) = 0,$$

which yields (39). □

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