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The Effect of Immigration on The Persistence and Ergodicity of a Stochastic SIS Model for Transmission of Disease

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Abstract: This paper aims to study the dynamic behavior of a stochastic *SIS* (Susceptible-Infected-Susceptible) epidemic model with varying population size and constant flow of new members of whom a specified fraction is infective. First, we show that the solution is stochastically ultimately bounded and permanent. Then, we investigate the persistence in the mean of the variable $I(t)$ (the number of infected members). Next, we use the Markov semigroups theory to investigate the ergodicity of the solution. Mainly, we show that a stationary distribution for the solution always exists for all values of the parameters in this model. Finally, Numerical simulations are carried out to illustrate the theoretical results.

Keywords: SIS model, Immigration, persistence in the mean, Markov semigroups, Stationary distribution

1 Introduction

Mathematical epidemiology is the science that formulates the spread of diseases with the aim of identifying factors that are responsible for their existence. Compartmental models have longly been a very useful tool for modeling a communicable disease. To explain the evolution in the number of infected patients observed in a population infected by some epidemics like the plague, Kermack and McKendrick [1] proposed and investigated a classical *SIR* model. Thenceforth, many other authors proposed and studied various more realistic epidemic models and developed a mathematically-backed results and techniques that are still used till now (See, [2,3]).

A *SIS* model is an appropriate tool which serves to modulate a communicable disease, especially a bacterial disease (e.g. meningitis and pneumococcus) or a sexually-transmitted disease such as gonorrhea. To study the effect of the arrival of infected members from the outside of a population to the dynamics of a disease, F. Brauer and P. Van Den Driessche [4] have proposed the following deterministic *SIS* model that include some immigration and demographic effects

$$\begin{cases} dS = ((1-p)A - \beta SI - dS + \gamma I)\,dt, \\ dI = (pA + \beta SI - (d+\gamma+\alpha)I)\,dt, \end{cases} \tag{1}$$

Here, $S(t)$ describes the number of members of the population who are susceptible to an infection at time t, and $I(t)$ is the number of members infected at time t. Let $N(t)$ be the total population size at time t. In the *SIS* model the disease confers no immunity, which means that $N(t) = S(t) + I(t)$. The different positive parameters in the model are described by giving the following assumptions:

The demographic assumptions:

1. There exist a constant flow A of new individuals into the population in unit time, where a fraction $p, (0 \le p \le 1)$, of A is infective.
2. There is a constant per capita natural death rate constant $d > 0$ in each class.

The epidemiological assumptions:

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3.There exist a fraction $\gamma \geq 0$ of infectives that is recovered and a fraction $\alpha \geq 0$ of infectives that die from the infection in unit time.
4.β is the contact rate. Each infective produces βN contacts sufficient to transmit the infection in unit time.

The infection in this model cannot be eliminated from the population because of the constant flow of new infectives pA. Due to this, the free equilibrium state does not exist. Furthermore, since the number of recovered infectives γI returns to the class of susceptible, the disease confers no immunity. The rate of new members infected considered is βSI, the rate β may likewise represent the successful contact rate between infected and susceptible individuals. For $p > 0$, Brauer et al. [4] has proved in the case where $\beta > 0$, the existence of a unique positive endemic equilibrium (I^*, N^*), which is globally and locally asymptotically stable. As a result, for $p > 0$, the disease remains endemic. However, the existence of a threshold $\mathscr{R}_0$ (basic reproduction number) requires p to tend to zero. In this case we have $\mathscr{R}_0 = \frac{\beta A}{d(d+\gamma+\alpha)}$, as well if $\mathscr{R}_0 < 1$, then I^* tends to zero.

In the real world, the parameters of a mathematical model are permanently subject to random variability. There are many types of noise that serve to represent these variabilities mathematically. However, in our case (environmental random variability in terrestrial systems) the white noise is the most appropriate (see, Steele 1985 [5], Vasseur and Yodzis 2004 [6]). Accordingly, and by using the technique of parameter perturbation (e.g. Zhang et al. [7], El Ansari et al. [8]), we introduce a Gaussian white noise disturbance into model (1) by considering the case where the per capita natural death rate constant $d > 0$ is subject to random fluctuations, We replace d by $d + \sigma dB$ in both equations of (1), where $B(t)$ is a standard one-dimensional Brownian motions and σ is its intensity. We obtain the following stochastic system

$$\begin{cases} dS = ((1-p)A - \beta SI - dS + \gamma I)dt - \sigma SdB, \\ dI = (pA + \beta SI - (d+\gamma+\alpha)I)dt - \sigma IdB, \end{cases} \tag{2}$$

The main interest of this paper is to present a stochastic study of a *SIS* model (2) with a degenerate diffusion matrix on one hand. On the other hand, it takes into consideration a demographic case, which is the possibility of new arrivals of infected individuals from the outside of a population. In fact, this is very interesting because it considers the phenomenon of immigration (which is constantly increasing across the world) in the study of the spread of a disease. To validate model (2), we use the same method as in [9,10] to prove the existence and uniqueness of the positive solution $(S(t), I(t))$ with probability one, if we start from any positive initial value $(S(0), I(0))$.

In Section 2 of this paper, we give the necessary mathematical background that we use to prove the different results concerning the integral Markov semigroups. In Section 3, we show the stochastic ultimate boundedness and the permanence of the solution of model (2) for any values of its parameters. In Section 4, interesting results concerning the persistence in the mean of the disease are showed. In section 5, we investigate the ergodicity of the solution of model (2) depending on the semigroup theory. Finally, in Section 6 we give some numerical simulations to illustrate our results.

2 Markov semigroups

Throughout the rest of this paper, we let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathbb{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathbb{F}_0$ contains all $\mathbb{P}$-null sets), we also denote .

$$\mathbb{R}^2_+ = \{(x_1, x_2) |\ x_i > 0,\ i = 1, 2\}$$

In general, consider the n dimensional stochastic differential equation

$$dX(t) = f(X(t))dt + g(X(t))dB(t), \text{ for } t \geq 0 \tag{3}$$

with initial value $X(0) = x_0 \in \mathbb{R}^n$. $B(t)$ denotes an d-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$. We denote by $C^{2,1}(\mathbb{R}^n \times [t_0, \infty], \mathbb{R}_+)$ the family of all nonnegative functions $V(X,t)$ defined on $\mathbb{R}^n \times [t_0, \infty]$ such that, they are continuously twice differentiable in X and once in t. The differential operator $\mathscr{L}$ is defined in [16]. The diffusion matrix is defined as follows
$A(X) = g(X)^T g(X)$.
The following theorem gives a criterion for positive recurrence in terms of Lyapunov function [17]

Theorem 1. *The system (2) is positive recurrent if there is a bounded open subset Δ of $\mathbb{R}^n$ with a regular boundary, and the following holds*
(i) there exist some $\delta \in (0,1]$ such that, for all $x \in \Delta$,

$$\delta |\xi|^2 \leq \xi^T \Sigma(x) \xi \leq \delta^{-1} |\xi|^2, \quad \textit{for any } \xi \in \mathbb{R}^n,$$

(ii) there exist a nonnegative function $\mathscr{V} : \Delta^c \to \mathbb{R}$ such that $\mathscr{V}$ is twice continuously differentiable and that for some $\theta > 0$

$$\mathscr{L}\mathscr{V} \leq -\theta, \quad \textit{for any } x \in \Delta^c$$

Moreover, the positive Markov process $X(t)$ has a unique ergodic stationary distribution π. That is, if h is a function integrable with respect to the measure π, then

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{1}{t} \int_0^t h(X(s))ds = \int_{\mathbb{R}^n} h(x)\pi(dx) \right) = 1.$$

Many authors use Theorem 2.1 in order to check the existence of a unique ergodic stationary distribution for various models (See, [18,19]). In our model, it is easy to see that $A(S,I)$ is degenerate, then $A(S,I)$ does not satisfy the uniform ellipticity condition i.e. the condition (i) cannot be satisfied. In contrast we can use the theory of integral Markov semigroups to analyze the asymptotic properties. For the convenience of readers, we present some definitions and results concerning the Markov semigroups (more details in [11,13]). In general, let $(E,\mathscr{E},m)$ be a σ-finite measure space. define D the subset of $L^1(m)$ which contains all densities

$$D=\left\{f\in L^1 : f\geq 0, \|f\|_{L^1}=1\right\}.$$

A linear mapping $P: L^1\to L^1$ is called a Markov operator if $P(D)\subset D$. The Markov operator P is called an integral or kernel operator if there exist a measurable function $\mathscr{K}$: $E\times E\to[0,\infty)$ such that

$$\int_E \mathscr{K}(x,y)m(dx)=1 \quad \forall y\in E,$$

$$Pf(x)=\int_E \mathscr{K}(x,y)f(y)m(dx) \quad \forall f\in D.$$

A family $\{P(t)\}_{t\geq 0}$ of Markov operator is called a Markov semigroup if it satisfies the conditions

(i)$P(0)=Id$,
(ii)$P(t+s)=P(t)P(s)\ \ for\ s,t\geq 0$
(iii)for each $f\in L^1$ the function $t\to P(t)f$ is continuous with respect to the L^1 norm.

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called integral, if for each $t>0$ the operator $P(t)$ is an integral Markov semigroup. A density f_* is called invariant under the semigroup $\{P(t)\}_{t\geq 0}$ if $P(t)f_*=f_*\ \forall t\geq 0$. The markov semigroup $\{P(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t\to\infty}\|P(t)f-f_*\|_{L^1}=0 \quad \forall f\in D.$$

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A\in\mathscr{E}$ if for every $g\in D$.

$$\lim_{t\to\infty}\int_A P(t)g(x)m(dx)=0.$$

The following theorem which can be found in [12,13] gives a sufficient condition for an integral Markov semigroup to be either asymptotically stable or sweeping.

Theorem 2.*let E be a metric space with σ-finite measure and $\mathscr{E}$ the σ-algebra of Borel sets. $\{P(t)\}_{t\geq 0}$ be a an integral Markov semigroup with a continuous kernel $\mathscr{K}(t,x,y)$ for $t\geq 0$. We assume that for every $f\in D$, we have*

$$\int_0^\infty P(t)f dt>0 \quad \textit{a.e. (almost everywhere)}$$

then this semigroup is asymptotically stable or sweeping with respects to compact sets .

The property that a Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping for a sufficiently large family of sets is called the Foguel alternative.

In this section, we discuss how the solution varies in $\mathbb{R}^2_+$. We give first of all the definition of stochastic ultimate boundedness and stochastic permanence.

3 Stochastic ultimate boundedness and permanence

Definition 1. Stochastically ultimate boundedness [20]. *The solution $X(t)$ of model (2) is said to be stochastically ultimately bounded, if for any $\varepsilon\in(0,1)$, there is a positive constant $\rho=\rho(\varepsilon)$, such that for any initial value $X(0)=X_0\in\mathbb{R}^2_+$, the solution $X(t)$ of model (2) has the property that*

$$\limsup_{t\to+\infty}\mathbb{P}\{|X(t)|>\rho\}<\varepsilon$$

Definition 2. Stochastic permanence [21,22]. *The model (2) is said to be stochastically permanent if for any $\varepsilon\in(0,1)$, there exist a pair of positive constants $\rho=\rho(\varepsilon)$ and $\chi=\chi(\varepsilon)$, such that for any initial value $X_0\in\mathbb{R}+^2$, the solution $X(t)$ of model (2) has the property that*

$$\liminf_{t\to+\infty}\mathbb{P}\{|X(t)|\leq\rho\}\geq 1-\varepsilon$$

and

$$\liminf_{t\to+\infty}\mathbb{P}\{|X(t)|\geq\chi\}\geq 1-\varepsilon.$$

Theorem 3.*The solution of model (2) is stochastically ultimate bounded and permanent for any initial value $(S_0,I_0)\in\mathbb{R}^2_+$.*

*Proof.*Let $X(t)=(S(t),I(t))$ and $N=S+I$. We define $V(X(t))=N(t)+\dfrac{1}{N(t)}$. Applying Itô formula we have

$$\begin{aligned}\mathscr{L}V(X(t))=&(A-dN-\alpha I)-\frac{A-dN-\alpha I}{N^2}+\frac{\sigma^2}{N}\\ \leq&-d(N+\frac{1}{N})+\frac{2d}{N}+A-\frac{A}{N^2}+\frac{\alpha I}{N^2}+\frac{\sigma^2}{N}\\ \leq&-d(N+\frac{1}{N})+\frac{2d}{N}+A-\frac{A}{N^2}+\frac{\alpha+\sigma^2}{N}\\ \leq&-dV(S,I)+C,\end{aligned}\tag{4}$$

where

$$\begin{aligned}C=&\sup_{N\in\mathbb{R}_+}\left\{A-\frac{A}{N^2}+\frac{2d+\alpha+\sigma^2}{N}\right\}\\ =&\frac{4A^2+(2d+\alpha+\sigma^2)^2}{4A}\end{aligned}$$

On the other hand, by application of Itô formula and (4), we get

$$\begin{aligned} E(e^{dt}V(t)) &= E(V(0)) + E\left[\int_0^t e^{ds}\big(dV(s)+LV(s)\big)ds\right] \\ &\le E(V(0)) + CE\left[\int_0^t e^{ds}ds\right] \\ &\le E(V(0)) + \frac{C}{d}\left(e^{dt}-1\right), \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned} E(V(t)) &\le e^{-dt}E(V(0)) + \frac{C}{d}\left(1-e^{-dt}\right) \\ &\le E(V(0)) + \frac{C}{d}H, \end{aligned} \tag{6}$$

let φ be a positive constant sufficiently large such that, $\frac{H}{\varphi}<1$. By Chebyshev's inequality we get

$$P\left\{N+\frac{1}{N}>\varphi\right\} \le \frac{1}{\varphi}E\left(N+\frac{1}{N}\right) \le \frac{H}{\varphi}\varepsilon$$

This implies

$$1-\varepsilon \le P\left\{N+\frac{1}{N}\le\varphi\right\} \le P\left\{\frac{1}{\varphi}\le N\le\varphi\right\}$$

noting that $N^2 \le 2|X|^2 \le 2N^2$, then we have

$$P\left\{\frac{1}{\sqrt{2}\varphi} \le \frac{N(t)}{\sqrt{2}} \le |X(t)| \le N(t) \le \varphi\right\} \ge 1-\varepsilon, \tag{7}$$

by Definition 1 and Definition 2, the solutions of model (2) are stochastically ultimately bounded and permanent.□

4 Persistence in the mean

The purpose of this section is to investigate the persistence in the mean of the solution of model (2). To this end, we first recall the definition of the persistence in the mean, then we give two important lemmas that we are using to prove Theorem 4.

For convenience and simplicity, in the next analysis we define the following notations

$$\mathscr{R}_\sigma = \frac{\beta A}{d(d+\gamma+\frac{\sigma^2}{2})}$$

and

$$\langle f\rangle_t = \frac{1}{t}\int_0^t f(s)ds$$

Definition 3.[23] *The variable I in (2) is said to be persistent in the mean if*

$$\liminf_{t\to+\infty}\langle I\rangle_t > 0$$

Lemma 1.*Let $(S(t),I(t))$ be the solution of system (2). With any initial value $(S(0),I(0))\in\mathbb{R}^2_+$, we have*

$$\lim_{t\to\infty}\frac{S(t)+I(t)}{t} = 0 \ \ \textit{a.s. (almost surely)}$$

*Proof.*Let $u(t)=S(t)+I(t)$, and $w(u)=(1+u)^\theta$, where $\theta>0$ is chosen later. Using Itô's formula, we get

$$dw(u(t)) = \mathscr{L}w(u)dt + \theta(1+u)^{\theta-1}(-\sigma u dB), \tag{8}$$

where

$$\begin{aligned} Lw(u) =& \theta(1+u)^{\theta-1}[A-du-\alpha I] \\ &+ \frac{\theta(\theta-1)}{2}(1+u)^{\theta-2}\sigma^2u^2 \\ =& \theta(1+u)^{\theta-2}\Big\{(1+u)(A-du-\alpha I) \\ &+ \frac{(\theta-1)}{2}\sigma^2u^2\Big\} \\ &\le \theta(1+u)^{\theta-2}\Big\{(1+u)(A-du) + \frac{\theta-1}{2}\sigma^2u^2\Big\} \\ =& \theta(1+u)^{\theta-2}\Big\{-\Big[d-\frac{(\theta-1)}{2}\sigma^2\Big]u^2 \\ &+ (A-d)u + A\Big\} \end{aligned}$$

we choose $\theta>0$ such that $d-\dfrac{\theta-1}{2}\sigma^2 := \lambda > 0$, so

$$\mathscr{L}w(u) \le \theta(1+u)^{\theta-2}\Big\{-\lambda u^2 + (A-d)u + A\Big\}$$

and

$$\begin{aligned} dw(u) \le& \theta(1+u)^{\theta-2}\Big\{-\lambda u^2 + (A-d)u + A\Big\}dt \\ &- \theta\sigma u(1+u)^{\theta-1}dB \end{aligned}$$

for $0<k<\theta\lambda$, we have

$$d\left(e^{kt}w(u(t))\right) = \mathscr{L}\left(e^{kt}w(u)\right)dt - \theta\sigma e^{kt}u(1+u)^{\theta-1}dB$$

thus

$$E\left(e^{kt}w(u(t))\right) = w(u(0)) + E\int_0^t \mathscr{L}\left(e^{ks}w(u(s))\right)ds \tag{9}$$

where

$$\begin{aligned} \mathscr{L}\left(e^{kt}w(u(t))\right) =& ke^{kt}w\big(u(t)\big) + e^{kt}\mathscr{L}w\big(u(t)\big) \\ \le& \theta e^{kt}(1+u)^{\theta-2}\Big\{\frac{k}{\theta}(1+u)^2 \\ &- \lambda u^2 + (A-d)u + A\Big\} \\ =& \theta e^{kt}(1+u)^{\theta-2}\Big\{-\Big(\lambda-\frac{k}{\theta}\Big)u^2 \\ &+ \Big(A-d+\frac{2k}{\theta}\Big)u + A + \frac{k}{\theta}\Big\} \\ \le& \theta e^{kt}H \end{aligned}$$

where

$$H = \sup_{u\in\mathbb{R}_+}\Big\{(1+u)^{\theta-2}\Big(-(\lambda-\frac{k}{\theta})u^2+(A-d+\frac{2k}{\theta})u + A + \frac{k}{\theta}\Big)\Big\}$$

therefore, it follows from (9) that

$$E(e^{kt}(1+u)^{\theta}) \le (1+u(0))^{\theta} + \frac{\theta H}{k}e^{kt}$$

consequently

$$\limsup_{t\to\infty} E((1+u(t))^{\theta}) \le \frac{\theta H}{k} =: H_0$$

which together with the continuity of u implies that there exist a constant $M>0$, such that

$$E((1+u(t))^{\theta}) \le M, \qquad t\ge 0 \tag{10}$$

with (10), we can proceed as in [24] to complete the proof. □

Lemma 2. *For any initial value $(S(0),I(0)) \in \mathbb{R}^2_+$, the solution $(S(t),I(t))$ of system (2) verifies*

$$\lim_{t\to\infty}\frac{\int_0^t S(s)dB(s)}{t} = 0, \textit{ and } \lim_{t\to\infty}\frac{\int_0^t I(s)dB(s)}{t} = 0 \quad a.s. \tag{11}$$

Proof. We proceed as in Lemma 7 of [8]. □

Theorem 4. *For any positive initial value $(S(0),I(0))$, the variable $I(t)$ of model (2) is persistent in the mean a.s., more precisely,*

$$\liminf_{t\to+\infty}\langle I\rangle_t \ge \frac{pA}{d+\gamma+\alpha}$$

Proof. We have

$$dI = (pA+\beta SI-(d+\gamma+\alpha)I)dt - \sigma IdB$$

then

$$\begin{aligned}\frac{I(t)-I(0)}{t} =& pA+\beta\langle SI\rangle_t-(d+\alpha+\gamma)\langle I\rangle_t \\ &- \frac{\sigma}{t}\int_0^t I(r)dB(r) \\ \ge& pA-(d+\alpha+\gamma)\langle I\rangle_t - \frac{\sigma}{t}\int_0^t I(r)dB(r)\end{aligned}$$

thus

$$\begin{aligned}\liminf_{t\to+\infty}\frac{I(t)-I(0)}{t} \ge& pA-(d+\alpha+\gamma)\liminf_{t\to+\infty}\langle I\rangle_t \\ &- \liminf_{t\to+\infty}\frac{\sigma}{t}\int_0^t I(r)dB(r)\end{aligned} \tag{12}$$

then by Lemma 1 and Lemma 2 we get

$$0 \ge pA-(d+\alpha+\gamma)\liminf_{t\to+\infty}\langle I\rangle_t$$

therefore

$$\liminf_{t\to+\infty}\langle I\rangle_t > \frac{pA}{d+\alpha+\gamma} \tag{13}$$

This completes the proof. □

Remark. Theorem 4 tells us that the number of infected individuals $I(t)$ tends to a point in time average that is considered stable in time average.

5 Stochastically asymptotic stability and stationary distribution

The main result of this section is the following theorem which gives a sufficient condition for the existence of a stationary distribution of system (2) and its asymptotic stability.

Theorem 5. *Let $(S(t),I(t))$ be the solution of system (2), then for every $t>0$ the distribution of $(S(t),I(t))$ has a density $u(t,x,y)$, and there exist a unique density $u_*(x,y)$ such that*

$$\lim_{t\to\infty}\iint_{\mathbb{R}^2}|u(t,x,y)-u_*(x,y)|\,dxdy = 0.$$

First, we introduce an integral Markov semigroup connected with system (2). Consider the space $(\mathbb{R}^2_+,\mathscr{B}(\mathbb{R}^2_+),m)$ where $\mathscr{B}(\mathbb{R}^2_+)$ be the σ-algebra of Borel subsets of $\mathbb{R}^2_+$, and m be the Lebesgue measure. By $\mathscr{P}(t,x_0,y_0,A)$ we denote the transition probability function for the diffusion process (S_t,I_t), i.e. $\mathscr{P}(t,x_0,y_0,A) = \mathbb{P}\big((S_t,I_t)\in A|S_0=x_0,I_0=y_0\big)$. In Lemma 3 we prove that for each point $(x_0,y_0)\in\mathbb{R}^2$ and $t>0$ the measure $\mathscr{P}(t,x_0,y_0,.)$ is absolutely continuous with respect to the Lebesgue measure.
Denote by $\mathscr{K}(t,x,y;x_0,y_0)$ the density of $\mathscr{P}(t,x_0,y_0,.)$. For any $t\ge 0$ we define the operator $P(t)$

$$P(t)f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathscr{K}(t,x,y;u,v)f(u,v)dudv, \quad \textit{for any } f\in D$$

Consequently $\{P(t)\}_{t\ge 0}$ is an integral Markov semigroup. Thus asymptotic stability of the semigroup $\{P(t)\}_{t\ge 0}$ implies the convergence in L^1 of the densities of the process (S_t,I_t) to the invariant density.

The proof of Theorem 5 is based on the following strategy described in [12].

– In Lemma 3 , by using the Hörmander condition, we show that the transition function of the process (S_t,I_t) is absolutely continuous.

–In Lemma 4 , we use support theorems to prove that the density of the transition function is positive on a set Γ.
–In Lemma 5 , we demonstrate that the set Γ is an *attractor*.
–In Lemma 6 , we show that the Markov semigroup satisfies the Foguel alternative, according to Theorem 2.
–In Lemma 7 , we exclude sweeping by showing that there exist a Khasminskii function.

Lemma 3.*For every point $(x_0,y_0) \in \mathbb{R}^2_+$, the transition probability function $\mathscr{P}(t,x,y;x_0,y_0)$ has a continuous density $\mathscr{K}(t,x,y;x_0,y_0) \in \mathscr{C}^\infty(\mathbb{R}_+,\mathbb{R}^2_+,\mathbb{R}^2_+)$.*

*Proof.*Let $a(x)$ and $b(x)$ two vectors fields in $\mathbb{R}^n$, then the Lie bracket $[a,b]$ is a vector field given by

$$[a,b]_j(x) = \sum_{i=1}^{n}\left(a_i\frac{\partial b_j}{\partial x_i} - b_i\frac{\partial a_j}{\partial x_i}\right), \quad j = 1,\dots,n$$

let

$$a(x,y) = \begin{bmatrix}(1-p)A-\beta xy - dx+\gamma y\\ pA+\beta xy-(d+\gamma+\alpha)y\end{bmatrix}$$

and

$$b(x,y) = \begin{bmatrix}-\sigma x\\ -\sigma y\end{bmatrix}$$

we have

$$[a,b] = \begin{bmatrix}-\sigma(1-p)A-\sigma\beta xy\\ -\sigma pA+\sigma\beta xy\end{bmatrix}$$

and

$$\big[[a,b],b\big] = \begin{bmatrix}\sigma^2(1-p)A-\sigma^2\beta xy\\ \sigma^2 pA+\sigma^2\beta xy\end{bmatrix}$$

Put

$$\delta_1 = \begin{vmatrix}-\sigma(1-p)A-\sigma\beta xy & -\sigma x\\ -\sigma pA+\sigma\beta xy & -\sigma y\end{vmatrix}$$

and

$$\delta_2 = \begin{vmatrix}\sigma^2(1-p)A-\sigma^2\beta xy & -\sigma x\\ \sigma^2 pA+\sigma^2\beta xy & -\sigma y\end{vmatrix}$$

As

$$\delta_1 = \sigma^2\left((1-p)Ay+\beta xy^2+\beta x^2y-pAx\right)$$

and

$$\delta_2 = \sigma^3\left(-(1-p)Ay+\beta xy^2+\beta x^2y+pAx\right)$$

We deduce that the vectors $b(x,y)$, $[a,b](x,y)$ and $\big[[a,b],b\big](x,y)$ span the space $\mathbb{R}^2$ for any $(x,y)\in\mathbb{R}^2_+$. Based on the Hörmander theorem (See Theorem 8 in [11]), the measure $\mathscr{P}(t,x_0,y_0,.)$ has a density $\mathscr{K}(t,x,y;x_0,y_0) \in \mathscr{C}^\infty(\mathbb{R}_+,\mathbb{R}^2_+,\mathbb{R}^2_+)$. □

In order to check the positivity of $\mathscr{K}$, we describe a method based on support theorem [25, ?].
Denoting $X(t) = \ln S(t)$ and $Y(t) = \ln I(t)$, by Itô's formula system (2) becomes

$$\begin{cases} dX(t) = f_1(X,Y)dt-\sigma dB,\\ dY(t) = f_2(X,Y)dt-\sigma dB,\end{cases} \tag{14}$$

where

$$f_1(x,y) = -(d+\frac{1}{2}\sigma^2)+(1-p)Ae^{-x}-\beta e^y+\gamma e^{y-x}$$

and

$$f_2(x,y) = pAe^{-y}-(d+\gamma+\alpha+\frac{1}{2}\sigma^2)+\beta e^x.$$

Fix a point $(x_0,y_0)\in\mathbb{R}^2_+$ and a function $\phi\in L^2([0,T],\mathbb{R})$ and consider the following system

$$\begin{cases} dx_\phi(t) = -\sigma\phi(t)+f_1(x_\phi(t),y_\phi(t))\\ dy_\phi(t) = -\sigma\phi(t)+f_2(x_\phi(t),y_\phi(t))\\ x_\phi(0)=x_0\\ y_\phi(0)=y_0\end{cases} \tag{15}$$

Let $D_{x_0,y_0;\phi}$ be the Fréchet derivative of the function $h\mapsto x_{\phi+h}(T)$ from $L^2([0,T];\mathbb{R})$ to $\mathbb{R}^2$. If for some $\phi\in L^2([0,T];\mathbb{R})$ the derivative $D_{x_0,y_0;\phi}$ has rank 2, then $\mathscr{K}(T,x,y;x_0,y_0)>0$ for $x=x_\phi(T)$ and $y=y_\phi(T)$. The derivative $D_{x_0,y_0;\phi}$ can be found by means of perturbation method for ordinary differential equations. Namely, let $\Lambda(t) = f'(x_\phi(t),y_\phi(t))$ where f' is the Jacobien of $f=\begin{bmatrix}f_1\\ f_2\end{bmatrix}$ and let $Q(t,t_0)$ for $T\geq t\geq t_0\geq 0$ be a matrix function such that $Q(t_0,t_0)=I_2$, $\frac{\partial Q(t,t_0)}{\partial t} = \Lambda(t)Q(t,t_0)$ and let $V=\begin{bmatrix}-\sigma\\ -\sigma\end{bmatrix}$, then

$$D_{x_0,y_0;\phi}h = \int_0^T Q(T,s)Vh(s)ds. \tag{16}$$

Finally, we note that

$$f_2(x,y)-f_1(x,y) = -(\frac{\gamma+\alpha}{2})+\frac{a(z)(e^x)^2-b(z)e^x+c(z)}{e^x},$$

where

$$\begin{aligned} z &= y - x, \\ a(z) &= \beta e^z + \beta > 0, \\ b(z) &= \frac{\gamma+\alpha}{2} + \gamma e^z > 0, \\ c(z) &= A\Big(pe^{-z} - (1-p)\Big). \end{aligned}$$

If $c(z) < 0$ ($i.e.$ $x + \log(\frac{p}{1-p}) < y$), there exist x^* such that

$$f_2(x^*,y) - f_1(x^*,y) \leq -(\frac{\gamma+\alpha}{2}) \tag{17}$$

So, let

$$\Gamma(C) = \{(x,y) \in \mathbb{R}^2_+, x + C < y\} \text{ and } \Gamma = \Gamma(C_0),$$

where C_0 is the largest number that verifies $f_2(x,y) - f_1(x,y) \geq 0$ for all $(x,y) \notin \Gamma(C)$.

Lemma 4. *For each $(x_0,y_0) \in \Gamma$ and for almost every $(x,y) \in \Gamma$, there exist $T > 0$ such that $\mathscr{K}(T,x,y;x_0,y_0) > 0$.*

Proof. **Step 1**
First, we check that the rank of $D_{x_0,y_0;\phi}$ is 2. Let $\varepsilon \in (0,T)$ and $h = \mathbf{1}_{[T-\varepsilon,T]}$.
Since

$$Q(T,s) = I_2 - \Lambda(T)(T-s) + o(T-s)$$

from (16) we obtain

$$D_{x_0,y_0;\phi} h = \varepsilon V - \frac{1}{2}\varepsilon^2 \Lambda(T)V + o(\varepsilon^2)$$

Since

$$V = \begin{bmatrix} -\sigma \\ -\sigma \end{bmatrix} \text{ and } \Lambda(T)V = \begin{bmatrix} \sigma((1-p)A + \gamma e^{y-x}) \\ -\sigma\beta e^x + \sigma pAe^{-y} \end{bmatrix}$$

then V and $\Lambda(T)V$ are linearly independent for any $(x,y) \in \mathbb{R}^2 \backslash U$, where

$$U = \{(x,y) \in \mathbb{R}^2, \quad \det\big(V, \Lambda(T)V\big) = 0\}.$$

Note that $m(U) = 0$, and $D_{x_0,y_0;\phi}$ has rank 2 for almost every $(x,y) \in \mathbb{R}^2$.

The next steps are to prove that for any two points $(x_0,y_0) \in \Gamma$ and $(x_1,y_1) \in \Gamma$ there exist a control function ϕ and $T > 0$ such that the solution of system (15) satisfies $x_\phi(T) = x_1$ and $y_\phi(T) = y_1$.
Using the substitution $z_\phi = y_\phi - x_\phi$, then (15) becomes

$$\begin{cases} x'_\phi = -\sigma\phi(t) + g_1(x_{\phi(t)}, z_{\phi(t)}), \\ z'_\phi = g_2(x_{\phi(t)}, z_{\phi(t)}), \end{cases} \tag{18}$$

where

$$\begin{aligned} g_2(x,z) &= f_2(x,x+z) - f_1(x,x+z), \\ &= -(\gamma+\alpha) + pAe^{-(x+z)} + \beta e^x - (1-p)Ae^{-x} \\ &\quad + \beta e^{x+z} - \gamma e^z, \end{aligned}$$

and

$$\begin{aligned} g_1(x,z) &= f_1(x,x+z), \\ &= -(d + \frac{1}{2}\sigma^2) + (1-p)Ae^{-x} - \beta e^{x+z} + \gamma e^z. \end{aligned}$$

Step 2
Fix $z_0 \neq z_1$ such that $\min(z_0,z_1) > C_0$. We prove that there exist ϕ, T and a solution (x_ϕ, z_ϕ) of system (18) and a constant x^* such that $z_\phi(0) = z_0, z_\phi(T) = z_1$ and $x_\phi(t) = x^*$.
Case 1: $z_0 < z_1$
We have, for all $z \in [z_0,z_1]$ and $x \in \mathbb{R}$

$$g_2(x,z) > -(\gamma+\alpha) + \beta e^x - (1-p)Ae^{-x} - \gamma e^{z_1} \tag{19}$$

Let x^* be sufficiently large such that

$$-(\gamma+\alpha) + \beta e^{x^*} - (1-p)Ae^{-x^*} - \gamma e^{z_1} > 0 \tag{20}$$

and let $z(t)$ solution of

$$\begin{cases} \dot{z}(t) = g_2(x^*, z(t)), \\ z(0) = z_0, \end{cases}$$

in the maximal interval $[0,\tau)$. Let $\phi(t) = \dfrac{1}{\sigma} g_1(x^*, z(t))$, then $(x_{\phi(t)}, z_{\phi(t)})$ is solution of (18).
There exist $\tau_1 > 0$ such that

$$z(\tau_1) > z_1 \text{ or } z(\tau_1) < z_0. \tag{21}$$

If not, $z(t)$ is bouneded and $\tau = \infty$, so by (19) and (20) we have

$$\dot{z}(t) \geq -(\gamma+\alpha) + \beta e^{x*} - (1-p)Ae^{-x^*} - \gamma e^{z_1} > 0$$

then $\lim\limits_{t\to\infty} z(t) = \infty$ which contradict $z(t)$ is bounded.
Since (21), then there exist $T \in (0,\tau)$ such that $z_\phi(T) = z_1$.
Case 2: if $z_1 < z_0$
Since (17), then there exist x^* such that

$$g_2(x^*,z) \leq \frac{-(\gamma+\alpha)}{2}, \text{ for any } z \in [z_1,z_0]$$

Using a similar argument in case 1, we get the desired conclusion.
Note that, in this step, there exist an infinity of values of x^*.
Step 3
Fix $x_0 \neq x_1 \in \mathbb{R}$.
Case 1: if $x_0 < x_1$

Let $L > 0$ be sufficiently large such that $x_1 < x_0 + \frac{L}{2}$ and let A_0, $A_1 > A_0$ and $\varepsilon > 0$.
Put

$$m = \max_{[x_0,x_0+L]\times[A_0,A_1]} \{|g_1(x,z)| + |g_2(x,z)|\}$$

and

$$\tau_0 = \varepsilon m^{-1}, \phi \equiv \frac{-3\sigma^{-1}\varepsilon^{-1}mL}{4},$$

choose ε such that

$$\varepsilon < \frac{L}{4} \text{ and } \varepsilon < \frac{A_1 - A_0}{4},$$

then we have

$$\frac{3\varepsilon^{-1}mL}{4} - m \leq |\sigma\phi| - |g_1(x_\phi, z_\phi)| \leq -\sigma\phi + g_1(x_\phi, z_\phi),$$

which implies that

$$\tau_0(-\sigma\phi + g_1(x_\phi, z_\phi)) \geq \frac{3L}{4} - \varepsilon \geq \frac{L}{2},$$

thus, for every $z_0 \in [A+\varepsilon, A_1-\varepsilon]$, the solution of system (18) with $x_\phi(0) = x_0$ and $z_\phi(0) = z_0$ has the following properties, $z_\phi(t) \in [z_0-\varepsilon, z_0+\varepsilon]$ for $t \leq \tau_0$ and $x_\phi(\tau_0) \in (x_0 + \frac{L}{2}, x_0 + L)$.
So, for $z_1 \in [A_0 + 2\varepsilon, A_1 - 2\varepsilon]$ there exist $z_0 \in [z_1-\varepsilon, z_1+\varepsilon]$ and $T \in (0, \tau_0)$ such that $x_\phi(T) = x_1$ and $z_\phi(T) = z_1$.
Case 2: if $x_1 < x_0$
Let $L > 0$ be sufficiently large such that $x_0 - \frac{L}{2} < x_1$.
The same proof in case 1 works for $x_1 \in (x_0 - \frac{L}{2}, x_0)$.
Step 4
Now, we claim that for any (x_0, y_0) and $(x_1, y_1) \in \Gamma$, there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0$, $y_\phi(0) = y_0$, $x_\phi(T) = x_1$, and $y_\phi(T) = y_1$.
Let $\varepsilon > 0$ sufficiently small such that

$$(x_0, y_0 - \varepsilon) \in \Gamma \text{ and } (x_1, y_1 - \varepsilon) \in \Gamma,$$

let $z_i = y_i - x_i$, $i = 1, 2$. Without loss of generality, we assume that $z_0 \leq z_1$.
From step 2, there exist x^*, $T_1 > 0$, and ϕ_1 such that

$$z_{\phi_1}(0) = z_0 - \varepsilon, z_{\phi_1}(T_1) = z_1 + \varepsilon \text{ and } x_{\phi_1}(t) = x^*. \quad (22)$$

Note that we can choose $x^* \neq x_0$ and $x^* \neq x_1$.
From step 3, there exist $z_0^* \in (z_0 - \frac{\varepsilon}{2}, z_0 + \frac{\varepsilon}{2})$, $T_2 > 0$, and ϕ_2 such that

$$x_{\phi_2}(0) = x_0, x_{\phi_2}(T_2) = x^*, z_{\phi_2}(0) = z_0 \text{ and } z_{\phi_2}(T_2) = z_0^*. \quad (23)$$

By using the similar argument in step 3, there exist $z^* \in (z_1 - \frac{\varepsilon}{2}, z_1 + \frac{\varepsilon}{2})$, T_3 and $\tilde{\phi}_3$ such that the solution of system

$$\begin{cases} x'_{\tilde{\phi}} = \sigma\tilde{\phi}(t) - g_1(x_{\tilde{\phi}}(t), z_{\tilde{\phi}}(t)), \\ z'_{\tilde{\phi}} = -g_2(x_{\tilde{\phi}(t)}, z_{\tilde{\phi}(t)}), \end{cases}$$

verifies

$$x_{\tilde{\phi}_3}(0) = x_1, x_{\tilde{\phi}_3}(T_3) = x^*, z_{\tilde{\phi}_3}(0) = z_1 \text{ and } z_{\tilde{\phi}_3}(T_3) = z^*,$$

let $x_{\phi_3}(t) = x_{\tilde{\phi}_3}(T_3 - t)$ and $z_{\phi_3}(t) = z_{\tilde{\phi}_3}(T_3 - t)$, then $(x_{\phi_3}(t), z_{\phi_3}(t))$ is the solution of (18) that verifies

$$x_{\phi_3}(0) = x^*, x_{\phi_3}(T_3) = x_1, z_{\phi_2}(0) = z^* \text{ and } z_{\phi_2}(T_3) = z_1. \quad (24)$$

In view of

$$z_0^* \in (z_0 - \frac{\varepsilon}{2}, z_0 + \frac{\varepsilon}{2}) \text{ and } z^* \in (z_1 - \frac{\varepsilon}{2}, z_1 + \frac{\varepsilon}{2})$$

we assume that $z_0^* = z_{\phi_1}(\tau_1)$ and $z^* = z_{\phi_1}(\tau_2)$, where $\tau_1, \tau_2 \in (0, T_1)$. Without loss of generality, we assume that $\tau_1 \leq \tau_2$.
Let

$$\phi(t) = \begin{cases} \phi_2(t), & 0 \leq t \leq T_2, \\ \phi_1(t - T_2 + \tau_1), & T_2 < t \leq T_2 + \tau_2 - \tau_1, \\ \phi_3(t - T_2 - \tau_2 + \tau_1), & T_2 + \tau_2 - \tau_1 < t \leq T, \end{cases}$$

where $T = T_2 + \tau_2 - \tau_1 + T_3$. By (22), (23) and (24) we have

$$x_\phi(0) = x_0, z_\phi(0) = z_0, x_\phi(T) = x_1 \text{ and } z_\phi(T) = z_1.$$

From step 1 it follows that $\mathscr{K}(T, x_0, y_0, x_1, y_1) > 0$, which completes the proof. □

The following Lemma can be proved by using the same method as in Lemma 3 of [12]. We hence omit the proof here.

Lemma 5. *For every density f we have*

$$\lim_{t\to\infty} \iint_\Gamma P(t) f(x,y)\, dx dy = 1.$$

Remark. In Theorem 5 the support of the invariant density u_* is the set Γ.

By the support of a measurable function f we simply mean the set

$$\text{supp}(f) = \{(x,y) \in \mathbb{R}_+^2 : f(x,y) \neq 0\}.$$

Lemma 6. *The semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or is sweeping with respect to compact sets.*

Proof. According to Lemma 3 , $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup with a continuous kernel $\mathscr{K}(t, x, y; x_0, y_0)$, and from Lemma 4 for every density f, we have

$$\int_0^\infty P(t) f dt > 0 \quad a.e. \quad (25)$$

From Lemma 5, we know that it is sufficient to demonstrate the restriction of the semigroup $\{P(t)\}_{t\geq 0}$ to the space $L^1(\Gamma)$. From (25), the Foguel alternative also follows. □

Lemma 7. *The semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.*

Proof. We search a positive function V such that

$$\sup_{(S,I)\in\mathbb{R}_+^2\setminus\Delta_\rho} \mathscr{L}V(S,I) \leq -1, \text{ for some } \rho > 0$$

where

$$\Delta_\rho = (\frac{1}{\rho},\rho)\times(\frac{1}{\rho},\rho).$$

Let $V = \frac{1}{S} + c(I+S) - \log I$, where $c > 1$ is chosen later. We have

$$\begin{aligned}\mathscr{L}V(S,I) = &-\frac{(1-p)A}{S^2} + \frac{\beta I}{S} + \frac{d}{S} - \frac{\gamma I}{S^2} + \frac{\sigma^2}{S}\\ &+ c(pA - (S+I)d - \alpha I) - \frac{pA}{I} - \beta S\\ &+ d + \gamma + \alpha + \frac{\sigma^2}{2}\end{aligned}$$

using the inequality $\frac{1}{S} \leq \frac{\varepsilon}{S^2} + \frac{1}{4\varepsilon}$, for some $\varepsilon > 0$, we have

$$\begin{aligned}\mathscr{L}V(S,I) \leq &-\frac{(1-p)A - \varepsilon(\sigma^2+d)}{S^2} - \frac{(\gamma-\varepsilon\beta)I}{S^2}\\ &- I(c\alpha - \frac{\beta}{4\varepsilon}) - d(S+I) - \beta S - \frac{pA}{I}\\ &+ \frac{d+\sigma^2}{4\varepsilon} + d + \gamma + \alpha + \frac{\sigma^2}{2}\end{aligned}$$

Choose $\varepsilon < \min\left(\frac{\gamma}{\beta}, \frac{(1-p)A}{\sigma^2+d}\right)$ and $c > \frac{\beta}{4\varepsilon\alpha}$, we get

$$\lim_{I\to 0^+} V(S,I) = \lim_{S\to 0^+} V(S,I)) = -\infty$$

and

$$\lim_{I\to +\infty} V(S,I) = \lim_{S\to +\infty} V(S,I) = -\infty$$

consequently, for ρ large enough we have

$$\sup_{(S,I)\in\mathbb{R}_+^2\setminus\Delta_\rho} \mathscr{L}V(S,I) \leq -1.$$

We find a positive Khasminskii function V which allows to exclude the sweeping of $\{P(t)\}_{t\geq 0}$, according to Lemma 6 the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.□

Example 1. We choose the parameters in systems (1) and (2) as follows: $p = 0.001$, $A = 1$, $\beta = 0.01$, $d = 1.5$, $\gamma = 0.9$, $\alpha = 0.5$, and $\sigma = 0.35$.

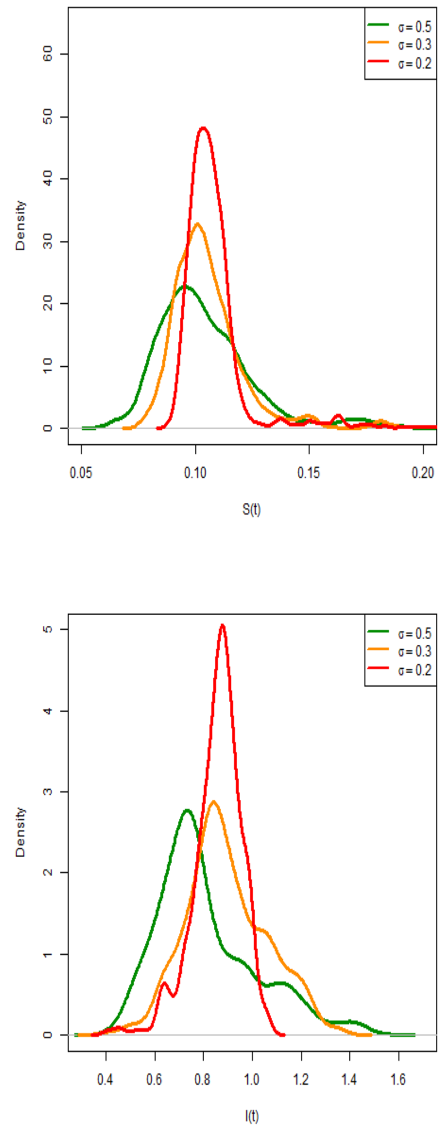
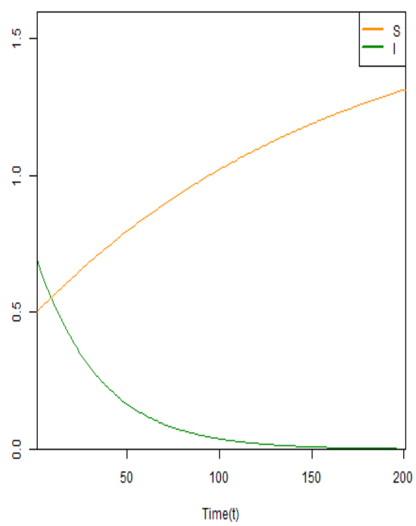


Fig. 1: Computer simulation of the density function of the invariant stationary distribution of system (2) for different values of σ, and parameters of Example 1, and $(S(0), I(0)) = (0.5, 0.4)$.

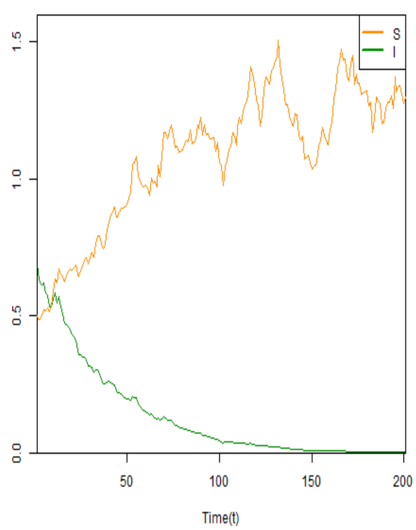
6 Discussions and Numerical Simulations

In this section, some numerical simulations are carried out to illustrate the different theoretical results obtained. We use Milstein's higher-order method [27,28] to derive the corresponding discretization equations of model (2).

Example 2. We choose the parameters in systems (1) and (2) as follows: $p = 0.8$, $A = 10$, $\beta = 3$, $d = 1.5$, $\gamma = 0.2$, $\alpha = 0.5$.



(a: the Deterministic System)



(b: the Stochastic System)

Fig. 2: Computer simulation of the path $S(t), I(t)$ for the models (1)(a) and (2)(b) , using parameters of Example 2, and $(S(0), I(0)) = (0.5, 0.7)$.

We choose the parameters of Example 1 arbitrary, numerical simulations in fig.1 affirm (as Theorem 5) the existence of a stationary distribution for model (2). Simulations are run under the different values of σ ($\sigma = 0.5$ or $\sigma = 0.3$ or $\sigma = 0.2$). The smoothed curves represented in fig.1 are the probability densities functions of $S(t)$ and $I(t)$. The distributions represented in fig.1 reflect that the stationary distribution has a big change when we increase the value σ. In other words, the distribution is closer to a standard distribution when σ becomes gradually smaller. Using the above parameters we have $\mathscr{R}_\sigma = 0.0027 << 1$, and p is very small. We notice that the level of the disease becomes gradually smaller in long time. fig.2 (b) illustrates this situation (the number of infected cases $I(t)$, represented by the green line, tends to a small value). We leave the proof of this result for a future investigation.

7 Perspective

The added value of this paper lies in the fact that it presents a stochastic study for a SIS model that describes the dynamics of a communicable disease into a population with positive flow of infectives pA. In this case, it is impossible to have a disease-free equilibrium. We have proved that regardless of the values of its parameters, the solution of model (2) is stochastically ultimately-bounded and permanent. Then we have proved that the number of infected individuals $I(t)$ is always persistent in the mean. The diffusion matrix of model (2) is degenerate, and the use of the semigroup theory described in [11,12,13,14,15] is appropriate to investigate the existence of a stationary distribution. We have proved that the semigroup $\{\mathscr{P}(t)\}_{t\geq 0}$ connected with system (2) is asymptotically stable regardless of the values of the parameters of this system, which guarantees regularly the existence of a unique stationary distribution. A numerical simulation explains how the intensity σ of fluctuations can change the deviation of this distribution.

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