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Unitary Aberrations on Pixellated Screens

*Kenan Uriostegui and Kurt Bernardo Wolf**

Instituto de Ciencias Físicas, Universidad Nacional Autónoma, de México, Av. Universidad s/n, Cuernavaca, Mor. 62210, Mexico

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Abstract: The finite oscillator based on the Lie group of spin $\mathsf{U}(2)$ provides a model for finite one-dimensional (1D) arrays of $N = 2j+1$ pixels, for j integer or half-integer which, as $j \to \infty$, deforms to the continuous 1D model of geometric optics. Translations, linear transformations and aberrations in the latter are canonical and have their $N \times N$ unitary counterparts in the former. Since in $\mathsf{U}(N)$ there are only N^2 independent transformations, we identify the finite counterparts of translations, linear transformations and aberrations within the finite model, applicable to the correction of aberrated images or signals on N-pixel linear arrays.

Keywords: Finite optical models, signal analysis, phase space

1 Introduction: The finite optical model

Starting from the 1D geometric optical model where the coordinates of the phase space of rays are their position q and momentum p at a line screen of sensors or leds, the finite oscillator model [1,2] is built as a Lie-algebraic *deformation* of these observables to two $N \times N$ (non-commuting) matrices $\mathbf{Q} = \|Q_{m,m'}\|$ and $\mathbf{P} = \|P_{m,m'}\|$, with their commutator $\mathbf{K} = -\mathrm{i}[\mathbf{Q},\mathbf{P}]$, acting on N-vectors $\mathbf{f} = \{f_m\}_{m=-j}^{j}$ that represent the N-point signals or pixellated images of that discrete model based on the $\mathsf{SU}(2)$ group of quantum angular momentum [3], so that $N := 2j+1$, for any fixed value of $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. The elements of these matrices are

$$q \mapsto Q_{m,m'} := m\,\delta_{m,m'}, \quad m, m' \in \{-j, -j+1, \ldots, j\}, \quad (1)$$

$$p \mapsto P_{m,m'} := -\mathrm{i}\tfrac{1}{2}C^j_m\,\delta_{m+1,m'} + \mathrm{i}\tfrac{1}{2}C^j_{-m}\,\delta_{m-1,m'}, \quad (2)$$

$$h \mapsto K_{m,m'} := \tfrac{1}{2}C^j_m\,\delta_{m+1,m'} + \tfrac{1}{2}C^j_{-m}\,\delta_{m-1,m'}. \quad (3)$$

where

$$C^j_m := \sqrt{(j-m)(j+m+1)}, \quad (4)$$

The three matrices are traceless and self-adjoint: the *position* matrix $\mathbf{Q}$ in (1) is diagonal, the *momentum* matrix $\mathbf{P}$ is skew-symmetric and pure imaginary, while the symmetric real *mode* matrix $\mathbf{K}$ can be associated to the classical oscillator Hamiltonian $h = \frac{1}{2}(p^2 + q^2)$ shifted by $(j+\frac{1}{2})\mathbf{1}$. The $N = 2j+1$ eigenvalues of each are equally spaced: $\{-j, -j+1, \ldots, j\}$.

For the classical model, the *oscillator* Lie algebra osc_1 has four generators: p, q, h, and 1; under Poisson brackets they close as

$$\{h,q\} = -p, \quad \{h,p\} = q, \quad \{q,p\} = 1, \quad \{1,\circ\} = 0. \quad (5)$$

Their corresponding matrices (1)–(4) close under commutation as,

$$[\mathbf{K},\mathbf{Q}] = -\mathrm{i}\mathbf{P}, \quad [\mathbf{K},\mathbf{P}] = \mathrm{i}\mathbf{Q}, \quad [\mathbf{Q},\mathbf{P}] = \mathrm{i}\mathbf{K}, \quad [\mathbf{1},\circ] = \mathbf{0}. \quad (6)$$

This set of commutators defines a basis for the Lie algebra of phase and spin, $\mathsf{u}(2) = \mathsf{u}(1) \oplus \mathsf{su}(2)$, in an $N \times N$ matrix representation, $N = 2j+1$, determined by the value of the $\mathsf{su}(2)$ Casimir invariant [4,5],

$$\mathbf{C} := \mathbf{Q}^2 + \mathbf{P}^2 + \mathbf{K}^2 = j(j+1)\mathbf{1}. \quad (7)$$

This process of finite quantization has been called the *discrete-quantization process* [6] from geometric to finite optics, roughly parallel to that from classical to quantum angular momentum.

Translations of position and momentum of phase space in the geometric *paraxial* optical model are generated by the exponentiated Poisson operators $\exp(u\{p,\circ\})$ and $\exp(-v\{q,\circ\})$. The corresponding matrices in the finite model, $\exp(\mathrm{i}u\mathbf{P})$ and $\exp(-\mathrm{i}v\mathbf{Q})$, are $N \times N$ *unitary* matrices that act on the finite image vectors $\mathbf{f}$ as analogues to those translations; $\exp(\mathrm{i}\alpha\mathbf{K})$ produces fractional Fourier-Kravchuk transformations (rotations between the position and momentum axes) [3,7], and

* Corresponding author e-mail: bwolf@icf.unam.mx

$\exp \mathrm{i}\phi\mathbf{1}$ impresses phases. They close into the four-parameter Lie *group* $\mathsf{U}(2)$ of 2×2 unitary matrices. This is a subgroup within the manifold of all N^2 independent unitary transformations $\mathsf{U}(N)$ that can be inflicted on the linear vector space of images $\mathbf{f}=\{f_m\}_{m=-j}^{j}$. In Sect. 2 we recall the classification of canonical transformations in the classical model and their corresponding unitary matrix maps in the finite-array models.

In Sect. 3 we examine the characteristic signature of phase space 'translations' on finite pixellated images, while in Sect. 4 we address the maps that correspond to classical linear canonical transformations, generated by Poisson operators of the three *quadratic* functions q^2, qp, and p^2, expliciting the action of their corresponding $N\times N$ matrices on the pixellated images. Classically, translations and linear canonical transformations constitute the inhomogeneous symplectic Lie group $\mathsf{ISp}(2,\mathsf{R})$. Higher powers $q^u p^v$ generate *aberrations* in the classical model; we consider also their finite counterparts as $\mathsf{U}(N)$ matrices that do *not* belong to the previous subset of translations [8], but can be assumed to be 'close' to the classical linear subgroup, so that power expansions can be usefully made in the finite model for small values of the departure parameters as a requisite for their correction. A closing Sect. 5 presents some conclusions on the scope of this correction and the issue of the parametrizations of the N^2-dimensional manifold of the group $\mathsf{U}(N)$.

2 Classical canonical and finite unitary maps of phase space

We transit the Royal Road by first formalizing the classical geometric model in phase space with its linear and aberration transformations, to determine their finite counterparts as $N\times N$ self-adjoint matrices that exponentiate to the unitary transformations in the finite model that will thus conserve information.

In geometric optics with 1D screens, the observable of position $q\in\mathsf{R}$ marks the intersection of a ray with a standard $z=0$ screen and ranges over the full real line. Optical momentum is related to ray inclination θ to the screen normal by $p=n\sin\theta$, with n the refractive index. To enter the metaxial régime, the range of p is extended to the full real line so that $(q,p)\in\mathsf{R}^2$ is the phase space plane whose symplectic structure is contained in the basic Poisson bracket $\{q,p\}=1$, which is skew-symmetric, bilinear, and follows the Leibniz rule for products, so that series expansions can be used [9].

2.1 Classical canonical transformations and factored-product expansions

Canonical transformations of the classical phase space plane $(q,p)\in\mathsf{R}^2$ are generated by the monomials $M_{k,m}:=p^{k+m}q^{k-m}$ in the phase space coordinates, whose Poisson operators are

$$\widehat{M}_{k,m}:=\{p^{k+m}q^{k-m},\circ\}=\frac{\partial M_{k,m}}{\partial q}\frac{\partial}{\partial p}-\frac{\partial M_{k,m}}{\partial p}\frac{\partial}{\partial q},\qquad(8)$$

with

$$\begin{array}{ll}\textit{rank} & k\in\{0,\tfrac12,1,\tfrac32,\ldots\},\\ \textit{weight} & m\in\{k,k-1,\ldots,-k\}.\end{array}\qquad(9)$$

This provides, for the operators that generate phase space maps through their exponentials $\exp(\sum_{k,m}\alpha_{k,m}\widehat{M}_{k,m})$, the classification (k,m) by rank and weight for 1D aberrations [10,11], [9, Chap. 13]. The Lie exponentials of the generators (8) act on beam density functions $\rho(q,p)$ or, for phase space display purposes on the 2-vector $\binom{q}{p}$, generating *canonical* transformations that preserve its symplectic structure [4], in particular the basic Poisson bracket $\{q,p\}=1$.

For rank $k=0$ the classical map is the identity since $\widehat{M}_{0,0}=1$ yields a null Poisson bracket. When $k=\frac12$ we have phase space *translations* by $\alpha\in\mathsf{R}$,

$$\begin{aligned}\exp(\alpha\widehat{M}_{\frac12,\frac12})\binom{q}{p}&=\binom{q-\alpha}{p},\\ \exp(\alpha\widehat{M}_{\frac12,-\frac12})\binom{q}{p}&=\binom{q}{p+\alpha},\end{aligned}\qquad(10)$$

and with rank $k=1$ we generate *linear* transformations,

$$\begin{aligned}\exp(\alpha\widehat{M}_{1,1})\binom{q}{p}&=\binom{q-2\alpha p}{p},\\ \exp(\alpha\widehat{M}_{1,0})\binom{q}{p}&=\binom{e^{-\alpha}q}{e^{\alpha}p},\\ \exp(\alpha\widehat{M}_{1,-1})\binom{q}{p}&=\binom{q}{p+2\alpha q}.\end{aligned}\qquad(11)$$

These maps close into the 5-parameter inhomogeneous symplectic group of linear canonical transformations of phase space $\mathsf{ISp}(2,\mathsf{R})$, which is a distinguished subgroup of all canonical transformations of the phase space plane.

For ranks $k>1$ the transformations are nonlinear in (q,p) and generally referred to as *aberrations*, that yield the closed expression

$$\exp(\alpha\widehat{M}_{k,m})\binom{q}{p}=\begin{pmatrix}q\left(1+\sum_{n=1}^{\infty}\frac{(-\alpha)^n}{n!}c^{-}_{k,m;n}M_{n(k-1),nm}\right)\\ p\left(1+\sum_{n=1}^{\infty}\frac{(+\alpha)^n}{n!}c^{+}_{k,m;n}M_{n(k-1),nm}\right)\end{pmatrix},\qquad(12)$$

where $c^{\sigma}_{k,m;n}:=\prod_{s=0}^{n-1}\big(k+\sigma(2s-1)m\big)$. Letting w stand for p and/or q, the action of $M_{k,m}\sim w^{2k}$ in Eq. (12) on the phase space coordinates has the leading terms $w+c\alpha w^{2k-1}+\cdots$, so one calls the exponent $A:=2k-1$ the aberration *order* of the generator $M_{k,m}$ in the geometric model of optics. All these transformations are canonical, i.e., $\{q,p\}=\{e^{\alpha\widehat{M}}q,e^{\alpha\widehat{M}}p\}$; they conserve the volume and structure of phase space; no light is neither lost nor gained.

The linear and aberration performance of optical instruments comprising several elements with individually known coefficient vectors $\alpha_k = \{\alpha_{k,m}\}_{m=-k}^{k}$ can be concatenated as elements of an infinite-parameter aberration *group* using the *factored product expansion* [10, 11]

$$\widehat{A}(\alpha) = \cdots \times \exp(\mathrm{i}\alpha_k \cdot \widehat{M}_k) \times \cdots \times \exp(\mathrm{i}\alpha_{\frac{3}{2}} \cdot \widehat{M}_{\frac{3}{2}}) \times \exp(\mathrm{i}\alpha_1 \cdot \widehat{M}_1). \tag{13}$$

For reasons of mathematical simplicity, the translation factor $\exp(\mathrm{i}\alpha_{\frac{1}{2}} \cdot \widehat{M}_{\frac{1}{2}})$ is normally excluded from the right of (14), and instead one refers to aberration expansions about a *design ray* curtailed to some upper rank k as transformations that are canonical *up to* order $A = 2k-1$. These operators can be concatenated: $\widehat{A}(\alpha_1)\widehat{A}(\alpha_2) = \widehat{A}(\alpha_3(\alpha_1,\alpha_2))$, whose product coefficients $\alpha_3(\alpha_1,\alpha_2)$ have been tabulated to aberration order 7 in [9, Chap. 14]. In the finite model however, translations are naturally included within the unitary group $\mathsf{U}(N)$ as we proceed to recall below.

2.2 Finite unitary transformations and factored-product expansions

The Royal Road to *finite* quantization of the monomial functions $p^a q^b$ to matrices, leads to the consideration of monomials with powers of the *three* generators of $\mathsf{U}(2)$ in their $N := (2j+1)$-dimensional representations, $\mathbf{Q}$, $\mathbf{P}$, *and* of the *mode* matrix $\mathbf{K}$. (The unit matrix $\mathbf{1}$ is also present but will not be addressed separately). These matrices are related by the Casimir invariant (7), so we only need to count $\mathbf{K}$ —the generator of Fourier-Kravchuk rotations of phase space [8]— with exponents 0 or 1. Hence, instead of the one pyramid of generators $\widehat{M}_{k,m}$ in (8), in the finite model we have *two* pyramids of matrix generators,

$$\mathbf{M}^0_{k,m}(\mathbf{Q},\mathbf{P}) := \{\mathbf{P}^{k+m}\mathbf{Q}^{k-m}\}_{\text{Weyl}}, \qquad m|_{-k}^{k}, \tag{14}$$

$$\mathbf{M}^1_{k,m}(\mathbf{Q},\mathbf{P},\mathbf{K}) := \{\mathbf{K}\,\mathbf{M}^0_{k-\frac{1}{2},m}(\mathbf{Q},\mathbf{P})\}_{\text{Weyl}}, \quad m|_{-k+\frac{1}{2}}^{k-\frac{1}{2}}, \tag{15}$$

where $\{\mathbf{A}^u\mathbf{B}^v\mathbf{C}^w\}_{\text{Weyl}}$ is the *Weyl ordering* of the symbols within the braces, i.e., the sum of all permutations of the $u+v+w$ objects divided by $(u+v+w)!$; when the forming matrices are self-adjoint, so is the Weyl ordering of their powers. The range of ranks k, classically unbounded in (9), is restricted in in the $N = 2j+1$ finite model to

$$\begin{aligned} &k \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, j\} && \text{in } \mathbf{M}^0_{k,m}, \\ &k \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, j-\tfrac{1}{2}\} && \text{in } \mathbf{M}^1_{k,m}, \end{aligned} \qquad m|_{-k}^{k}, \tag{16}$$

$$\text{so} \qquad \sum_{\sigma,k,m} 1 = (2j+1)^2,$$

The natural analogue of rank-k geometric transformations (12) is thus the sum of self-adjoint matrices with coefficients $\alpha_k = \{\alpha^\sigma_{k,m}\}$,

$$\mathbf{M}_k(\alpha_k) = \sum_{m=-k}^{k} \alpha^0_{k,m}\mathbf{M}^0_{k,m} + \sum_{m=-k+\frac{1}{2}}^{k-\frac{1}{2}} \alpha^1_{k,m}\mathbf{M}^1_{k,m}, \tag{17}$$

while the analogue of the factored-product expansion (14) that provides $N \times N$ complex unitary matrices with parameters $\alpha = \{\alpha_k\}$, is

$$\mathbf{A}(\alpha) = \exp(\mathrm{i}\mathbf{M}_j(\alpha_j)) \times \exp(\mathrm{i}\mathbf{M}_{j-\frac{1}{2}}(\alpha_{j-\frac{1}{2}})) \times \cdots \times \exp(\mathrm{i}\mathbf{M}_1(\alpha_1)) \times \exp(\mathrm{i}\mathbf{M}_{\frac{1}{2}}(\alpha_{\frac{1}{2}}))\, e^{\mathrm{i}\alpha^0_{0,0}}, \tag{18}$$

where we note that the product is not open-ended as (14), but has $2j+1$ factors, with the rightmost being an overall phase. The natural inner product in the vector space of signals $\mathbf{f} = \{f_m\}_{m=-j}^{j}$, where the matrices of the basis (14)–(15) are self-adjoint and their i-exponentials are *unitary* is, of course,

$$(\mathbf{f},\mathbf{g}) := \sum_{m=-j}^{j} f^*_m g_m \;\Rightarrow\; \begin{cases} (\mathbf{f},\mathbf{M}\mathbf{g}) = (\mathbf{M}^\dagger\mathbf{f},\mathbf{g}), \\ (\mathbf{A}(\alpha)\,\mathbf{f},\mathbf{A}(\alpha)\,\mathbf{g}) = (\mathbf{f},\mathbf{g}). \end{cases} \tag{19}$$

Thus one has the Lie group $\mathsf{U}(N)$ of all N^2 transformations of $2j+1 = N$-pixel 1D images, that are reversible (by $\alpha \mapsto -\alpha$) and thus conserve information.

3 Finite phase space ‘translation’ maps

To evince the action of the exponentiated matrices $\exp(\mathrm{i}\alpha\mathbf{M}^\sigma_{k,m})$ on an N-point signal $\mathbf{f} = \{f_m\}_{m=-j}^{j}$, let us consider the translation and linear maps, (10) and (11) that belong to the linear (paraxial) subgroup in the geometric model seen above.

3.1 Linear phase generated by $\mathbf{Q}$

We note first that the *position* matrix $\mathbf{M}^0_{\frac{1}{2},-\frac{1}{2}} = \mathbf{Q}$ in (1) acts as $(\mathbf{Q}\mathbf{f})_m = m f_m$, so the exponential action is

$$\begin{aligned} f_m \mapsto f^{(\alpha)}_m &= (e^{\mathrm{i}\alpha\mathbf{Q}}\mathbf{f})_m \\ &= f_m + \mathrm{i}\alpha(\mathbf{Q}\mathbf{f})_m - \tfrac{1}{2!}\alpha^2(\mathbf{Q}^2\mathbf{f})_m + \cdots \\ &= e^{\mathrm{i}\alpha m} f_m, \end{aligned} \tag{20}$$

impressing a linear phase on the pixels of $\mathbf{f}$, as originating from an inclined plane wave on the screen. This action corresponds with (10) as a translation in momentum space that here is cyclic in α with period 2π when j is integer, or modulo 4π when half-integer. However, if we regard α in (20) as a *small* parameter, we should regard only the first term after the unity, namely $(\mathbf{Q}\mathbf{f})_m = m f_m$.

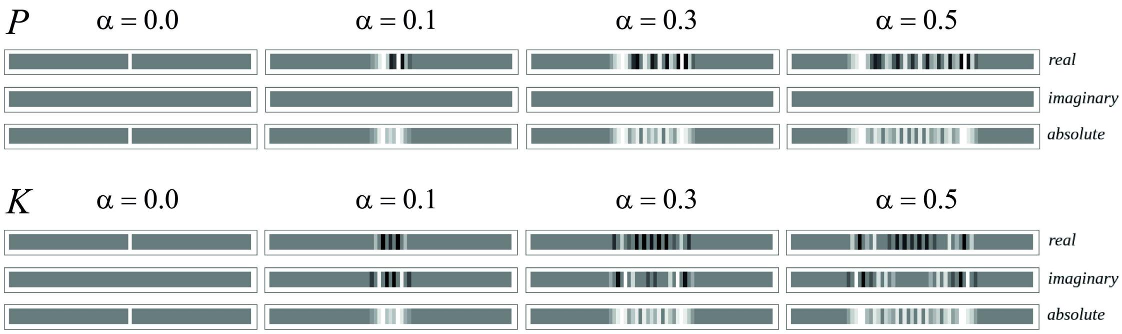

Fig. 1: Transformation of a 65-point real signal $f_m^{(0)} := \delta_{0,m}$ consisting of a single-pixel of value 1 on a background of 0's, under the maps generated by the exponentiated matrices $\exp(\mathrm{i}\alpha\mathbf{P})$ (*top*), and $\exp(\mathrm{i}\alpha\mathbf{K})$ (*bottom*). The real, imaginary and absolute values of the transformed signal are shown for parameters $\alpha = 0, 0.1, 0.3$ and 0.5. The gray scale adjusts to the minimal (black) and maximal (white) values of the pixels with gray remaining zero.

3.2 The 'translation' map generated by $\mathbf{P}$

Next, consider the *momentum*, $\mathbf{M}^0_{\frac12,\frac12} = \mathbf{P}$ in (2), whose exponentiated action classically translates positions as $e^{\beta\partial_x} f(x) = f(x+\beta)$ in (10). In the finite model, the matrix $\exp(\mathrm{i}\beta\mathbf{P})$ acts on signals $\mathbf{f}$ through the well-known Wigner $\mathsf{SU}(2)$ *little-d* functions $d^j_{m,m'}(\beta)$ [5]. The exponential series in β acting on the pixel values is the map

$$
\begin{aligned}
f_m \mapsto f'_m(\beta) &= (e^{\mathrm{i}\beta\mathbf{P}}\mathbf{f})_m \\
&= f_m + \mathrm{i}\beta(\mathbf{P}\mathbf{f})_m - \tfrac{1}{2!}\beta^2(\mathbf{P}^2\mathbf{f})_m + \cdots \\
&= \sum_{m'=-j}^{j} (-1)^{m'-m} d^j_{m,m'}(\beta)\, f_{m'}.
\end{aligned} \tag{21}
$$

The action of $\exp(\mathrm{i}\beta\mathbf{P})$ is unitary and real, i.e., orthogonal. It is shown in Fig. 1 (top), where the object signal is the one-point unit signal $s_m^{(m^\circ)} := \delta_{m^\circ,m}$ at the central pixel $m^\circ = 0$; the result of the transformation is the *spot* $\mathbf{s}^{(m^\circ)\prime} = \{s_m^{(m^\circ)\prime}\}$ on the 1D screen. This is the analogue of the *point-transfer function* between m° and m in discrete mechanical systems. And again, if β in (21) is *small*, we only have to regard the first term in the series after the unity,

$$
\mathrm{i}(\mathbf{P}\mathbf{f})_m = \tfrac12 C^j_m\, f_{m+1} - \tfrac12 C^j_{-m}\, f_{m-1}, \tag{22}
$$

This is a centered, two-point discrete derivative (that limits to ∂_x as $j\to\infty$); the coefficients C^j_m in (4) are minimal at the extremes and maximal at the center of the m-position range.

For $m<0$ the first coefficient in (22) is larger than the second, so the series of $\exp(\mathrm{i}\alpha\mathbf{P})$ yields same-sign decreasing values that multiply the original spot signal; to its right $m>0$ the second coefficient is larger than the first, so the exponential series multiplies the spot signal by alternating signs that form an oscillatory wake. This is the finite analogue of the *translation* of pixel positions (10). The 'center of mass' of the signal moves to the left and leaves a real oscillating tail of alternating signs to its right, which in the $j\to\infty$ limit is zero. The absolute values are symmetric in m though.

3.3 The Fourier-Kravchuk map generated by $\mathbf{K}$

Also shown in Fig. 1 (bottom), is the exponentiated action of the mode generator matrix $\mathbf{K}$ in (3), which is symmetric and real. Its exponential $\exp(\mathrm{i}\alpha\mathbf{K})$ rotates between the plane of $\mathbf{Q}$ and $\mathbf{P}$, the position and momentum matrices. Thus, $\mathbf{K}$ is the generator of the fractional *Fourier-Kravchuk* transforms [3] that which act on the 1D signals $\mathbf{f}$ as,

$$
\begin{aligned}
f_m \mapsto f^{(\gamma)}_m &= (e^{\mathrm{i}\gamma\mathbf{K}}\mathbf{f})_m \\
&= f_m + \mathrm{i}\gamma(\mathbf{K}\mathbf{f})_m - \tfrac{1}{2!}\gamma^2(\mathbf{K}^2\mathbf{f})_m + \cdots \\
&= \sum_{m'=-j}^{j} d^j_{m,m'}(\gamma)\, f_{m'}.
\end{aligned} \tag{23}
$$

For small γ, the term after the unity shows that $\mathbf{K}$ acts as a position-dependent averager,

$$
(\mathbf{K}\mathbf{f})_m = \tfrac12 C^j_m\, f_{m+1} + \tfrac12 C^j_{-m}\, f_{m-1}. \tag{24}
$$

Had we realized the $\mathsf{su}(2)$ algebra (1)–(3) as generators of rotations on a 2-sphere in an ambient 3D (q,p,κ)-space, their exponentiated action would be that of *rigid* rotations of the sphere —the counterpart of translations and rotations of classical phase space.

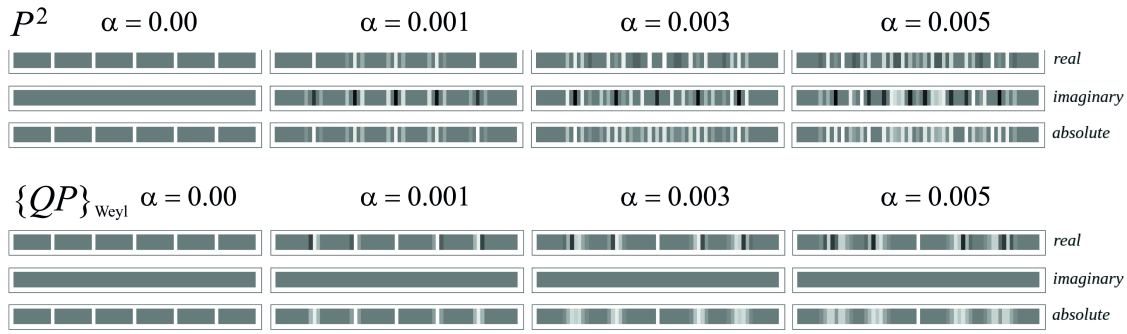

Fig. 2: Transformation of a 65-point signal consisting of five spot pixels of value 1 on a background of 0's, under the maps generated by the exponentiated matrices $\exp(\mathrm{i}\alpha\mathbf{P}^2)$ (*top*) and $\exp(\mathrm{i}\alpha\{\mathbf{QP}\}_{\text{Weyl}})$ (*bottom*). As in the previous figure, the real, imaginary and absolute values of the transformed signal are shown in the same gray scale; here the parameters are $\alpha = 0,\ 0.001,\ 0.003$ and 0.005.

3.4 On 'correction' of translations in $\mathsf{U}(N)$ maps

We can probe a finite optical system $\mathbf{A} \in \mathsf{U}(N)$ represented by an $N \times N$ unitary matrix, by recording its action on N one-point signals, $\mathbf{s}^{(m^\circ)} := \|\delta_{m,m^\circ}\|$, for all $m^\circ|_{-j}^{j}$. Each one will transform to a *spot* $\mathbf{A}\mathbf{s}^{(m^\circ)} = \mathbf{s}^{(A;m^\circ)} = \|s_m^{(A;m^\circ)}\|$, encompassing neighbouring pixels $m|_{-j}^{j}$. Since the set of all N one-point signals $\{\mathbf{s}^{(m^\circ)}\}_{m^\circ=-j}^{j}$ form an orthonormal basis under the inner product (19), so do their unitarily transformed spots $\mathbf{s}^{(A;m^\circ)}$. In principle, if we know all of the latter, we can reconstruct the $N \times N$ transformation matrix $\mathbf{A} = \|A_{m,m'}\| = \|(\mathbf{s}^{(m)}, \mathbf{A}\mathbf{s}^{(m')})\|$ which is a unitary matrix (18) with the set of N^2 cyclic parameters $\alpha = \{\alpha_{k,m}^{\sigma}\}$.

Consider first transformations generated by a linear combination of the three $\mathsf{SU}(2) \subset \mathsf{U}(N)$ generator matrices $\mathbf{M}^0_{\frac{1}{2},\pm\frac{1}{2}}$ and $\mathbf{M}^1_{0,0}$ at the top of the pyramids (14)–(15),

$$\mathbf{M}^{(1)}(a,b,c) = a\mathbf{Q} + b\mathbf{P} + c\mathbf{K} \tag{25}$$

acting on $\mathbf{s}^{(m)}$. Since this matrix is tri-diagonal, one-pixel object signals at m will diffuse to three pixels, s'_{m-1}, s'_m and s'_{m+1}. Using (20), (22) and (24), we can recover the parameters in (25) of the first-order approximation of $\exp(\mathrm{i}\alpha\mathbf{M}^{(1)})\,\mathbf{s} \approx \mathbf{s} + \mathrm{i}\alpha\mathbf{M}^{(1)}\,\mathbf{s} =: \mathbf{s}'$. Introducing the rank and weight indices, they can be expressed as

$$a(m) = \alpha^0_{\frac{1}{2},-\frac{1}{2}}(m) = \frac{\mathrm{i}}{m}\left(1 - \frac{s'_m}{s_m}\right), \quad (m \neq 0) \tag{26}$$

$$b(m) = \alpha^0_{\frac{1}{2},\frac{1}{2}}(m) = \frac{1}{s_m}\left(\frac{s'_{m-1}}{C^j_{m-1}} - \frac{s'_{m+1}}{C^j_{-m-1}}\right), \tag{27}$$

$$c(m) = \alpha^1_{0,0}(m) = -\frac{\mathrm{i}}{s_m}\left(\frac{s'_{m-1}}{C^j_{m-1}} + \frac{s'_{m+1}}{C^j_{-m-1}}\right), \tag{28}$$

where $s_m = 1$ for unit test signals and, when $m = \pm j$, the corresponding $s'_{\pm(j+1)}$ are obviated because $C^j_{\pm j} = 0$. (The fact that (26) excludes the $m = 0$ value can be supplemented by $\mathbf{M}^0_{0,0} = \mathbf{1}$ that generates an overall phase $e^{\mathrm{i}\alpha}\mathbf{s}^{(0)}$, so $a(0) = s'_0/s_0$.) Thus, when we test the transformation system with unit signals $(\mathbf{s}^{(m^\circ)})_m = \delta_{m^\circ,m}$ at all pixels $m^\circ|_{-j}^{j}$, we obtain the four $\mathsf{U}(2)$ lowest, 'linear' parameters of the system.

Only when the system matrix $\mathbf{A}(\alpha)$ in (18) is an exponential $\exp(\mathrm{i}\mathbf{M}^{(1)}(a,b,c))$ of the sum (25), will the parameters a, b, c have values independent of m. In this case, the $\mathsf{U}(2)$ phase and linear transformations due to translation can be *reversed* for any and all pixellated images $\mathbf{f}'$ on the finite array to recuperate the original $\mathbf{f} = \exp(-\mathrm{i}\mathbf{M}^{(1)})\mathbf{f}'$. Yet, if the parameters are *not* m-independent, we may choose their *average* values $\overline{a}, \overline{b}, \overline{c}$ over $m|_{-j}^{j}$ to build an $\mathbf{M}^{(1)}(\overline{a},\overline{b},\overline{c})$ to *eliminate* the two rightmost factors in the factored product (18) (i.e., linear transformation and overall phase). The remaining $2j - 1$ factors to the left in (18) will contain the departures of (26)–(28) from constancy as aberrations of the corrected first-order system. Thus, we continue with the action of higher-rank transformations.

4 Correction of quadratic and higher transformations

The finite analogue of the classical linear transformations (11) is generated by the five $N \times N$ 'quadratic' matrices $\mathbf{Q}^2$, $\mathbf{P}^2$, $\{\mathbf{QP}\}_{\text{Weyl}}$, $\{\mathbf{QK}\}_{\text{Weyl}}$ and $\{\mathbf{KP}\}_{\text{Weyl}}$ of rank $k = 1$, which can be analysed in terms similar to the linear ones above.

Again, $\exp(\mathrm{i}\alpha\mathbf{Q}^2)$ impresses a phase on the components of a signal $\mathbf{f}$; this is the quadratic approximation to a multiplication by a circular wavefront train. Regarding squared momentum,

$$(\mathbf{P}^2\mathbf{f})_m = -\tfrac{1}{4}C^j_m C^j_{m+1}\, f_{m+2} + \tfrac{1}{4}(C^j_m C^j_{-m-1} + C^j_{-m}C^j_{-m+1})\, f_m \\ -\tfrac{1}{4}C^j_{-m}C^j_{-m-1}\, f_{m-2}, \qquad (29)$$

it is represented by a real symmetric matrix with positive diagonal terms, zeroes on the second diagonals $m = m' \pm 1$, and smaller negative terms on the third diagonals $m = m' \pm 2$. The action of the i-exponential $\exp(\mathrm{i}\alpha\mathbf{P}^2)$ on a signal $\mathbf{f}$ will produce a symmetric diffusive oscillatory pattern, as a quantum mechanical free potential on peaked wavefunctions. In Fig. 2 we show the action of $\mathbf{P}^2$ on five test one-point signals distributed on the pixel array, and their corresponding spots for various values of the parameter α.

As to the matrix $\mathbf{K}^2 = -\mathbf{Q}^2 - \mathbf{P}^2 - j(j+1)\mathbf{1}$, comparing (24) with (22), its action will be that of (29) with all plus signs: symmetric and diffusive but not oscillatory. The matrix $\{\mathbf{QP}\}_{\rm Weyl} = \frac{1}{2}(\mathbf{QP}+\mathbf{PQ})$ will generate translations modulated by position, so that according to the parameter sign it compresses or expands the pixellated 1D image, as can be seen in Fig. 2, corresponding in the continuous classical limit with squeezing or expanding the positions of the spots on the pixel array.

For the quadratic elements generating transformations of rank $k = 1$ and aberrations order 2, the general form of their generator is

$$\mathbf{M}^{(2)}(\alpha) = \alpha^0_{1,1}\mathbf{P}^2 + \alpha^0_{1,0}\{\mathbf{QP}\}_{\rm Weyl} + \alpha^0_{1,-1}\mathbf{Q}^2 \\ +\alpha^1_{\frac12,\frac12}\{\mathbf{KP}\}_{\rm Weyl} + \alpha^1_{\frac12,-\frac12}\{\mathbf{KQ}\}_{\rm Weyl}. \qquad (30)$$

This matrix has five nonzero diagonals, $m' \in \{m, m\pm1, \pm2\}$, and its elements are

$$M^{(2)}_{m,m'} = -(\alpha^0_{1,1} + \mathrm{i}\alpha^1_{\frac12,\frac12})\,\tfrac14 C^j_m C^j_{m+1}\,\delta_{m+2,m'} \\ -(\mathrm{i}\alpha^0_{1,0} - \alpha^1_{\frac12,-\frac12})\,\tfrac14(2m+1)C^j_m\,\delta_{m+1,m'} \\ +\Big(\alpha^0_{1,1}\,\tfrac14(C^j_m C^j_{-m-1} + C^j_{-m}C^j_{m-1}) + \alpha^0_{1,-1}m^2\Big)\delta_{m,m'} \\ +(\mathrm{i}\alpha^0_{1,0} + \alpha^1_{\frac12,-\frac12})\,\tfrac14(2m-1)C^j_{-m}\,\delta_{m-1,m'} \\ -(\alpha^0_{1,1} - \mathrm{i}\alpha^1_{\frac12,\frac12}+)\,\tfrac14 C^j_{-m}C^j_{-m+1}\,\delta_{m-2,m'}. \qquad (31)$$

As previously, the action on the one-point test functions $(\mathbf{s}^{(m')})_m = \delta_{m,m'}$ that yield the aberrated data $\mathbf{s}'_m$, allows the inversion for the coefficients $\alpha^\sigma_{k,m}$ —all depending on the position m of the object unit test function,

$$\alpha^0_{1,1} = 2\mathrm{i}\Big(\frac{s_{m-2}}{C^j_{m-2}C^j_{m-1}} + \frac{s_{m+2}}{C^j_{-m-2}C^j_{-m-1}}\Big), \qquad (32)$$

$$\alpha^0_{1,0} = 2\Big(\frac{s_{m-1}}{(2m-1)C^j_{m-1}} - \frac{s_{m+1}}{(2m+1)C^j_{-m-1}}\Big), \qquad (33)$$

$$\alpha^0_{1,-1} = -\mathrm{i}\frac{C^j_{m-1}C^j_{-m} + C^j_{-m-1}C^j_m}{2m^2}\Big(\frac{s_{m-2}}{C^j_{m-2}C^j_{m-1}} + \frac{s_{m+2}}{C^j_{-m-2}C^j_{-m-1}}\Big) \\ +\mathrm{i}\frac{1-s_m}{m^2}, \qquad (34)$$

$$\alpha^1_{\frac12,\frac12} = -2\Big(\frac{s_{m-2}}{C^j_{m-2}C^j_{m-1}} - \frac{s_{m+2}}{C^j_{-m-2}C^j_{-m-1}}\Big), \qquad (35)$$

$$\alpha^1_{\frac12,-\frac12} = 2\mathrm{i}\Big(\frac{s_{m-1}}{(2m-1)C^j_{m-1}} + \frac{s_{m+1}}{(2m+1)C^j_{-m-1}}\Big), \qquad (36)$$

where $m \neq 0$ in (34). When we determine these coefficients, or their average values if they depend on m, we can implement the quadratic correction by applying $\exp(-\mathrm{i}\alpha\mathbf{M}^{(2)})$ after the linear correction has been made.

Next we can address transformations of rank $k = \frac32$ and aberration order 3, generated by linear combinations of $\mathbf{M}_{\frac32,m}(\alpha_{\frac32,m})$ in (17). The matrix is now 7-diagonal with elements

$$M^{(3)}_{m,m'} = (2\mathrm{i}\alpha^0_{\frac32,\frac32} - \alpha^1_{1,1})\,\tfrac{1}{16}C^j_m C^j_{m+1}C^j_{m+2}\,\delta_{m+3,m'} \\ -(\alpha^0_{\frac32,\frac12} + \mathrm{i}\alpha^1_{1,0})(m+1)\,\tfrac14 C^j_m C^j_{m+1}\,\delta_{m+2,m'} \\ -\Big((\mathrm{i}\alpha^0_{\frac32,\frac32} - \tfrac16\mathrm{i}\alpha^1_{1,1})\,\tfrac18 C^j_m(C^j_{m+1}C^j_{-m-2} \\ +C^j_m C^j_{-m-1} + C^j_{m-1}C^j_{-m}) \\ +(\mathrm{i}\alpha^0_{\frac32,-\frac12} - \alpha^1_{1,-1})(m^2+m+\tfrac13)\,\tfrac12 C^j_m\Big)\delta_{m+1,m'} \\ +\Big(\tfrac14\alpha^0_{\frac32,\frac12}\Big((m+\tfrac13)C^j_m C^j_{-m-1} + (m-\tfrac13)C^j_{-m}C^j_{m-1}\Big) \\ +\alpha^0_{\frac32,-\frac32}m^3\Big)\delta_{m,m'} \qquad (37) \\ +\Big((\mathrm{i}\alpha^0_{\frac32,\frac32} + \tfrac16\alpha^1_{1,1})\,\tfrac18 C^j_{-m}(C^j_m C^j_{-m-1} \\ +C^j_{m-1}C^j_{-m} + C^j_{m-2}C^j_{-m+1}) \\ +(\mathrm{i}\alpha^0_{\frac32,\frac12} + \alpha^1_{1,-1})(m^2-m+\tfrac13)\,\tfrac12 C^j_{-m}\Big)\delta_{m-1,m'} \\ -(\alpha^0_{\frac32,\frac12} - \mathrm{i}\alpha^1_{1,0})(m-1)\,\tfrac14 C^j_{-m}C^j_{-m+1}\,\delta_{m-2,m'} \\ -(2\mathrm{i}\alpha^0_{\frac32,\frac32} + \alpha^1_{1,1})\,\tfrac{1}{16}C^j_{-m}C^j_{-m+1}C^j_{-m+2}\,\delta_{m-3,m'}$$

By applying this transformation on one-point signals $\mathbf{s}^{(m)}$, we can obtain an approximation of the values of the aberration coefficients present in the signal. This is

$$\alpha^0_{\frac{3}{2},\frac{3}{2}} = 4\Big(\frac{s_{m-3}}{C^j_{m-3}C^j_{m-2}C^j_{m-1}} - \frac{s_{m+3}}{C^j_{-m-3}C^j_{-m-2}C^j_{-m-1}}\Big), \qquad (38)$$

$$\alpha^0_{\frac{3}{2},\frac{1}{2}} = 2\mathrm{i}\Big(\frac{s_{m-2}}{(m-1)C^j_{m-2}C^j_{m-1}} + \frac{s_{m+2}}{(m+1)C^j_{-m-2}C^j_{-m-1}}\Big), \qquad (39)$$

$$\begin{aligned}\alpha^0_{\frac{3}{2},-\frac{1}{2}} = {} & \frac{1}{(3m^4-m^2+\frac{1}{3})}\Bigg(-\frac{s_{m-3}}{C^j_{m-1}C^j_{m-2}C^j_{m-3}} \\ & \times\Big(2(m^2+m+\tfrac{1}{3})C^j_{-m+1}C^j_{m-2} \\ & \quad +(3m^2+m+1)(C^j_{m-1}C^j_{-m}+C^j_{-m-1}C^j_{m}) \\ & \quad +(m^2-m+\tfrac{1}{3})C^j_{-m-2}C^j_{m+1}\Big) \\ & -(3m^2+3m+1)\frac{s_{m-1}}{C^j_{m-1}}+(3m^2-3m+1)\frac{s_{m+1}}{C^j_{-m-1}} \\ & +\frac{s_{m+3}}{C^j_{-m-1}C^j_{-m-2}C^j_{-m-3}}\Big((m^2+m+\tfrac{1}{3})C^j_{-m+1}C^j_{m-2} \\ & \quad +(3m^2-m+1)(C^j_{m-1}C^j_{-m}+C^j_{-m-1}C^j_{m}) \\ & \quad +2(m^2-m+\tfrac{1}{3})C^j_{-m-2}C^j_{m+1}\Big)\Bigg), \qquad (40)\end{aligned}$$

$$\begin{aligned}\alpha^0_{\frac{3}{2},-\frac{3}{2}} = {} & \mathrm{i}\frac{1}{6m^3}\Big(\frac{s_{m-2}}{(m-1)C^j_{m-2}C^j_{m-1}} + \frac{s_{m+2}}{(m+1)C^j_{-m-2}C^j_{-m-1}}\Big) \\ & \times\Big((3m-1)C^j_{m-1}C^j_{-m}+(3m+1)C^j_{-m-1}C^j_{m}\Big) \\ & -\mathrm{i}\frac{s_{m-1}}{m^3} \qquad (41)\end{aligned}$$

$$\alpha^1_{1,1} = 4\mathrm{i}\Big(\frac{s_{m-3}}{C^j_{m-3}C^j_{m-2}C^j_{m-1}} + \frac{s_{m+3}}{C^j_{-m-3}C^j_{-m-2}C^j_{-m-1}}\Big), \qquad (42)$$

$$\alpha^1_{1,0} = 2\Big(\frac{s_{m-2}}{(m-1)C^j_{m-2}C^j_{m-1}} - \frac{s_{m+2}}{(m+1)C^j_{-m-2}C^j_{-m-1}}\Big),$$

$$\begin{aligned}\alpha^1_{1,-1} = {} & -\frac{\mathrm{i}}{(3m^4-m^2+\frac{1}{3})}\Bigg(\frac{s_{m-3}}{C^j_{m-1}C^j_{m-2}C^j_{m-3}} \\ & \times\Big(-(m^2-m+\tfrac{1}{3})C^j_{-m-2}C^j_{m+1} \\ & \quad +(m^2+3m+\tfrac{1}{3})(C^j_{m-1}C^j_{-m}+C^j_{-m-1}C^j_{m}) \\ & \quad +2(m^2+m+\tfrac{1}{3})C^j_{-m+1}C^j_{m-2}\Big) \\ & +(3m^2+3m+1)\frac{s_{m-1}}{C^j_{m-1}}+(3m^2-3m+1)\frac{s_{m+1}}{C^j_{-m-1}} \\ & +\frac{s_{m+3}}{C^j_{-m-1}C^j_{-m-2}C^j_{-m-3}}\Big(2(m^2-m+\tfrac{1}{3})C^j_{-m-2}C^j_{m+1} \\ & \quad +(m^2-3m+\tfrac{1}{3})(C^j_{m-1}C^j_{-m}+C^j_{-m-1}C^j_{m}) \\ & \quad -(m^2+m+\tfrac{1}{3})C^j_{-m+1}C^j_{m-2}\Big)\Bigg). \qquad (43)\end{aligned}$$

Except for the diagonal $\exp(\mathrm{i}\alpha\mathbf{Q}^u)$, the exponential of the matrices (1)–(3) will generally be full $N\times N$ matrices. Since the central elements of the matrices $\mathbf{P}^v$ and $\mathbf{K}^w$ grow as $\sim(\frac{1}{2}j)^{v+w}$, the values of the parameter α must be reduced by the inverse factor to keep the aberrations within a visually comparable scale, as done in Figs. 1 *vs.* 2. This motivates us to look at the first term of the exponential series, after the unity, to characterize the face of the aberrations assuming that they are indeed small.

5 Concluding remarks

The finite quantization scheme applied to images on 1D pixel arrays allows the classification of linear, quadratic and higher-order aberrations in correspondence with their continuous geometric optical counterparts. It also allows an iterative method to determine the coefficients that generate these transformations, rank by rank. Whether one decides to remove the lower ranks in the factored-product decomposition (18) and inquire on the aberrations in the remainder of that product, or use the extraction algorithm in (26)–(28) and (32)–(36), to determine a 'corrected' linear transformation, the factored-product expansion provides a well-defined parametrization of the unitary group $\mathsf{U}(N)$ based on small departures from the 'linear' $\mathsf{U}(2)$ subgroup.

This should be considered as an as-yet unexplored parametrization of the N^2-dimensional manifold of the unitary group $\mathsf{U}(N)$, whose best-known parametrization follows that of Euler angles in the N-dimensional orthogonal subgroup of matrices of unit determinant, $\mathsf{SO}(N)\supset\mathsf{SO}(N-1)\supset\cdots\supset\mathsf{SO}(3)\supset\mathsf{SO}(2)$, inserting phase transformations after each factor in that group chain [12]. The factored-product expansion coordinates $\{\alpha^\sigma_{k,m}\}$ however, do not compose simply under the concatenation of group elements, nor has the Haar measure over the group been written in terms of these coordinates. Further afield, since $\mathsf{U}(N)$ properly contains the discrete permutation group $\Pi(N)$ of $N!$ elements; a parametrization based on the permutation of pixel values, with small departures due to 'aberrations' seems possible but, to our knowledge, has not yet been attempted.

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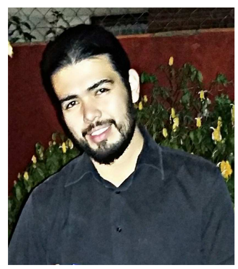
Kurt B. Wolf is presently Investigador Titular (equivalent to full professor) at the Universidad Nacional Autónoma de México, which he entered in 1971. He was the first director of Centro Internacional de Ciencias A.C. (1986-1994), organizing 14 national and international schools, workshops and conferences. He is author of: Integral Transforms in Science and Engineering (Plenum, New York, 1979) and Geometric Optics on Phase Space (Springer-Verlag, Heidelberg, 2004), and has been editor of 12 proceedings volumes. His work includes some 170 research articles in refereed journals, 58 chapters and in extenso contributions to proceedings, two books on scientific typography in Spanish and essays for the general public.



Kenan Uriostegui received the MSc degree from the Universidad Nacional Autónoma de México. His main interests include unitary transformations on discrete systems and advances in quantum optics. He is currently a PhD student at the Posgrado en Ciencias Física, UNAM, working under the supervision of Dr. K.B. Wolf.