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Study on a New (3+1)-Dimensional Extensions of the Konopelchenko-Dubrovsky Equation

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Abstract: In this work we introduce new (3+1)-dimensional extensions of the Konopelchenko-Dubrovsky (KD) equation. We use the simplified Hirota's method to study these new extensions. We derive the dispersion relation and the multiple soliton solutions for each developed model.

Keywords: Konopelchenko-Dubrovsky equation; dispersion relation; multiple soliton solutions.

1 Introduction

In [1], the Konopelchenko-Dubrovsky (KD) equation was presented in the form

$$u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y + 3au_x\partial_x^{-1}u_y = 0, \quad (1)$$

where a and b are real parameters, and ∂_x^{-1} is the inverse of ∂_x, with $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$, and

$$(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t)\,dt, \quad (2)$$

under the decaying condition at infinity.

Eq. (1) is a new nonlinear integrable evolution equation on two spatial and one temporal dimensions [2–4] The following equations

$$\psi_y + \psi_{xx} + \left(au + \frac{2b}{a}\right)\psi_x = 0, \quad (3)$$

and

$$\psi_t - 4\psi_{xxx} - 6\left(au - \frac{2b}{a}\right)\psi_{xx} + \left(3av - \frac{3}{2}a^2u^2 + 6bu - 12\frac{b^2}{a^2} - 3au_x\right)\psi_x = 0, \quad (4)$$

where $u_y = v_x$, are a Lax pair [3] for the KD equation (1). Konopelchenko and Dubrovsky [1] introduced (1) in connection with the nonlinear waves with a weak dispersion. For $u_y = 0$, equation (1) becomes the Gardner equation. However, For $a = 0$, equation (1) is the well-known Kadomtsev-Petviashvili (KP) equation. Moreover, for $b = 0$, equation (1) is the modified KP equation.

In [1], this equation was investigated by the inverse scattering transform method. The F-expansion method is used in [2] to investigate the KD equation. The main focus of these works in [1] and [2] was the solitary wave solutions.

To solve the KD equation (1), various methods [5]- [17] have been invested, such as the stan dard truncated Painlevé analysis, the tanh method and its generalizations, the the extended Riccati equation rational expansion method, the homotopy perturbation method, and many others.

Using the sense of the KD equation, we introduce the following (3+1)-dimensional extensions of the KD equation, given as

$$\begin{array}{l} u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y \\ +3au_x\partial_x^{-1}u_y - 3(\partial_x^{-1}u_z)_z + 3au_x\partial_x^{-1}u_z = 0, \end{array} \quad (5)$$

$$\begin{array}{l} u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y + 3au_x\partial_x^{-1}u_y \\ -3(\partial_x^{-1}u_z)_z + 3a\partial_x^{-1}u_z\partial_x^{-1}u_{yy} = 0, \end{array} \quad (6)$$

and

$$\begin{array}{l} u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y + 3au_x\partial_x^{-1}u_y \\ -3(\partial_x^{-1}u_z)_z + 3a\partial_x^{-1}u_z\partial_x^{-1}u_{zz} = 0, \end{array} \quad (7)$$

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will be referred to as the first, second, and third (3+1)-dimensional extended KD equations respectively, where a and b are real constants.

The objectives of this work are twofold. First, we seek to study the three new (3+1)-dimensions extensions by investing the simplified Hirota's method. Second, we aim to show that these equations give multiple soliton solutions for specific values of coefficients of the spatial variables x, y, and z. The three new models give distinct single- soliton solutions.

2 The first extended KD equation

In this section we study the first extended KD equation

$$u_t - u_{xxx} - 6buu_x + \tfrac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y + 3au_x\partial_x^{-1}u_y - 3(\partial_x^{-1}u_z)_z + 3au_x\partial_x^{-1}u_z = 0. \tag{8}$$

We first remove the integral term in (8) by introducing the potential

$$w(x,y,z,t) = u_x(x,y,z,t), \tag{9}$$

to carry (8) to the equation

$$w_{xt} - w_{xxxx} - 6bw_xw_{xx} + \tfrac{3}{2}a^2w_x^2w_{xx} - 3w_{yy} + 3aw_{xx}w_y - 3w_{zz} + 3aw_{xx}w_z = 0. \tag{10}$$

Substituting

$$w(x,y,z,t) = e^{\theta_i}, \theta_i = k_ix + r_iy + s_iz - \omega_it, \tag{11}$$

into the linear terms of (10), and solving the resulting equation for ω_i we obtain the dispersion relation as

$$\omega_i = -\frac{k_i^4 + 3r_i^2 + 3s_i^2}{k_i}, i = 1,2,3, \tag{12}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_ix + r_iy + s_iz + \left(\frac{k_i^4 + 3r_i^2 + 3s_i^2}{k_i}\right)t. \tag{13}$$

We then use the transformation

$$w(x,y,z,t) = R\left(\ln f(x,y,z,t)\right), \tag{14}$$

where the auxiliary function $f(x,y,z,t)$ reads

$$f(x,y,z,t) = 1 + e^{k_1x + r_1y + s_1z + \left(\frac{k_1^4 + 3r_1^2 + 3s_1^2}{k_1}\right)t}, \tag{15}$$

into Eq. (10) and solve to find that

$$R = \frac{2}{a}, \tag{16}$$

and for the single-soliton solution to exist, the coefficient s_1 must take the form

$$s_1 = -\frac{(ar_1 - 2bk_1 + ak_1^2)}{a}, \tag{17}$$

where k_1 and r_1 are left as free parameters. The last results give the dispersion relation (12) as

$$\omega_1 = -\left(\frac{k_1^4 + 3r_1^2}{k_1} + \frac{3(ar_1 - 2bk_1 + ak_1^2)^2}{a^2k_1}\right), i = 1,2,3. \tag{18}$$

In view of these last results, the single soliton solution for the (3+1)-dimensional extended KD equation (8) is given by

$$u(x,y,z,t) = \frac{2k_1e^{k_1x + r_1y - \frac{(ar_1 - 2bk_1 + ak_1^2)}{a}z + \left(\frac{k_1^4 + 3r_1^2}{k_1} + \frac{3(ar_1 - 2bk_1 + ak_1^2)^2}{a^2k_1}\right)t}}{a(1 + e^{k_1x + k_1y - \frac{(ar_1 - 2bk_1 + ak_1^2)}{a}z + \left(\frac{k_1^4 + 3r_1^2}{k_1} + \frac{3(ar_1 - 2bk_1 + ak_1^2)^2}{a^2k_1}\right)t})}, \tag{19}$$

obtained upon using the potential defined in (9).

For the two-soliton solutions, we use the auxiliary function $f(x,y,z,t)$ as

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2}, \tag{20}$$

where θ_1 and θ_2 are given in (13). By using this auxiliary function in (10), we find that the two soliton solutions exist only if

$$\begin{aligned} r_i &= k_i, \\ s_i &= -\frac{k_i(a - 2b + ak_i)}{a}, i = 1,2,3, \end{aligned} \tag{21}$$

where k_i is left as a free parameter.

To determine the two-soliton solutions explicitly, we substitute the obtained results into the formula $w(x,y,z,t) = \frac{2}{a}(\ln f(x,y,z,t))$, and then we use the potential $w(x,y,z,t) = u_x(x,y,z,t)$ as defined in (9).

To determine the three soliton solutions, we substitute the auxiliary function

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{22}$$

and proceed as before, we obtain the three-soliton solutions under the conditions (21).

3 The second extended KD equation

In this section we investigate the second extended KD equation

$$u_t - u_{xxx} - 6buu_x + \tfrac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y + 3au_x\partial_x^{-1}u_y - 3(\partial_x^{-1}u_z)_z + 3a\partial_x^{-1}u_z\partial_x^{-1}u_{yy} = 0. \tag{23}$$

To remove the integral term in (23) we use the potential

$$w(x,y,z,t) = u_x(x,y,z,t), \tag{24}$$

which gives the equation

$$w_{xt} - w_{xxxx} - 6bw_xw_{xx} + \tfrac{3}{2}a^2w_x^2w_{xx} - 3w_{yy} + 3aw_{xx}w_y - 3w_{zz} + 3aw_zw_{yy} = 0. \tag{25}$$

Substituting

$$w(x,y,z,t) = e^{\theta_i}, \theta_i = k_ix + r_iy + s_iz - \omega_it, \tag{26}$$

into the linear terms of (25), and solving the resulting equation for ω_i we obtain the dispersion relation as

$$\omega_i = -\frac{k_i^4 + 3r_i^2 + 3s_i^2}{k_i}, i = 1,2,3, \tag{27}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x + r_i y + s_i z + \left(\frac{k_i^4 + 3r_i^2 + 3s_i^2}{k_i}\right) t. \tag{28}$$

We next use the transformation

$$w(x,y,z,t) = R\left(\ln f(x,y,z,t)\right), \tag{29}$$

where the auxiliary function $f(x,y,z,t)$ reads

$$f(x,y,z,t) = 1 + e^{k_1 x + r_1 y + s_1 z + \left(\frac{k_1^4 + 3r_1^2 + 3s_1^2}{k_1}\right) t}, \tag{30}$$

into Eq. (25) and solve to find that

$$R = \frac{2}{a}, \tag{31}$$

and for the single-soliton solution to exist, the coefficient s_1 must take the form

$$s_1 = -\frac{k_1^2(ar_1 - 2bk_1 + ak_1^2)}{ar_1^2}, \tag{32}$$

where k_1 and r_1 are left as free parameters. The last results give the dispersion relation as

$$\omega_1 = -\left(\frac{k_1^4 + 3r_1^2}{k_1} + \frac{3k_1^4(ar_1 - 2bk_1 + ak_1^2)^2}{a^2 r_1^4}\right), i = 1,2,3. \tag{33}$$

In view of these last results, the single soliton solution for the (3+1)-dimensional extended KD equation (23) is given by

$$u(x,y,z,t) = \frac{2k_1 e^{k_1 x + r_1 y - \frac{k_1^2(ar_1 - 2bk_1 + ak_1^2)}{ar_1^2} z + \left(\frac{k_1^4 + 3r_1^2}{k_1} + \frac{3k_1^4(ar_1 - 2bk_1 + ak_1^2)^2}{a^2 r_1^4}\right) t}}{a\left(1 + e^{k_1 x + k_1 y - \frac{k_1^2(ar_1 - 2bk_1 + ak_1^2)}{ar_1^2} z + \left(\frac{k_1^4 + 3r_1^2}{k_1} + \frac{3k_1^4(ar_1 - 2bk_1 + ak_1^2)^2}{a^2 r_1^4}\right) t}\right)}, \tag{34}$$

obtained upon using the potential defined in (24).

For the two-soliton solutions, we the auxiliary function $f(x,y,z,t)$ as

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2}, \tag{35}$$

where θ_1 and θ_2 are given in (13). By using this auxiliary function in (25), we find that the two soliton solutions exist only if

$$\begin{aligned} r_i &= k_i, \\ s_i &= -\tfrac{k_i(a - 2b + ak_i)}{a}, i = 1,2,3, \end{aligned} \tag{36}$$

where k_i is left as a free parameter.

To determine the two-soliton solutions explicitly, we substitute the obtained results into the formula $w(x,y,z,t) = \frac{2}{a}(\ln f(x,y,z,t))$, and then we use the potential $w(x,y,z,t) = u_x(x,y,z,t)$ as defined in (24).

To determine the three-soliton solutions, we substitute the auxiliary function

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{37}$$

and proceed as before, we obtain the three-soliton solutions under the conditions (21). It is interesting to note that although the first and the second extended KD equations are different in the last term and the dispersion relation, but the second- and the third-soliton solutions are the same.

4 The third extended KD equation

In this section we study the third extended KD equation

$$u_t - u_{xxx} - 6buu_x + \tfrac{3}{2}a^2u^2u_x - 3(\partial_x^{-1}u_y)_y + 3au_x\partial_x^{-1}u_y - 3(\partial_x^{-1}u_z)_z + 3a\partial_x^{-1}u_z\partial_x^{-1}u_{zz} = 0, \tag{38}$$

Using the potential

$$w(x,y,z,t) = u_x(x,y,z,t), \tag{39}$$

carries (38) to the equation

$$w_{xt} - w_{xxxx} - 6bw_xw_{xx} + \tfrac{3}{2}a^2w_x^2w_{xx} - 3w_{yy} + 3aw_{xx}w_y - 3w_{zz} + 3aw_{zz}w_z = 0. \tag{40}$$

Proceeding as before, the dispersion relation as

$$\omega_i = -\frac{k_i^4 + 3r_i^2 + 3s_i^2}{k_i}, i = 1,2,3, \tag{41}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x + r_i y + s_i z + \left(\frac{k_i^4 + 3r_i^2 + 3s_i^2}{k_i}\right) t. \tag{42}$$

Using the transformation

$$w(x,y,z,t) = R\left(\ln f(x,y,z,t)\right), \tag{43}$$

where the auxiliary function $f(x,y,z,t)$ reads

$$f(x,y,z,t) = 1 + e^{k_1 x + r_1 y + s_1 z + \left(\frac{k_1^4 + 3r_1^2 + 3s_1^2}{k_1}\right) t}, \tag{44}$$

into Eq. (40) and solve to find that

$$R = \frac{2}{a}, \tag{45}$$

and for the single soliton solution to exist, the coefficient r_1 must take the form

$$r_1 = -\frac{(ak_1^4 + as_1^3 - 2bk_1^3)}{ak_1^2}, \tag{46}$$

where k_1 and s_1 are left as free parameters. The last results give the dispersion relation (41) to

$$\omega_1 = -\left(\frac{k_1^4+3s_1^2}{k_1} + \frac{3(ak_1^4+as_1^3-2bk_1^3)^2}{a^2k_1^5}\right), i = 1,2,3. \tag{47}$$

In view of these last results, the single-soliton solution for the (3+1)-dimensional extended KD equation (38) is given by

$$u(x,y,z,t) = \frac{2k_1 e^{k_1 x - \frac{(ak_1^4+as_1^3-2bk_1^3)}{ak_1^2} y + s_1 z + \left(\frac{k_1^4+3s_1^2}{k_1} + \frac{3(ak_1^4+as_1^3-2bk_1^3)^2}{a^2k_1^5}\right)t}}{a(1+e^{k_1 x - \frac{(ak_1^4+as_1^3-2bk_1^3)}{ak_1^2} y + s_1 z + \left(\frac{k_1^4+3s_1^2}{k_1} + \frac{3(ak_1^4+as_1^3-2bk_1^3)^2}{a^2k_1^5}\right)t}}, \tag{48}$$

obtained upon using the potential defined in (39).

For the two-soliton solutions, we the auxiliary function $f(x,y,z,t)$ as

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2}, \tag{49}$$

where θ_1 and θ_2 are given in (42). By using this auxiliary function in (40), we find that the two soliton solutions exist only if

$$\begin{aligned} s_i &= k_i, \\ r_i &= -\frac{k_i(a-2b+ak_i)}{a}, i = 1,2,3, \end{aligned} \tag{50}$$

where k_i is left as a free parameter.

To determine the two-soliton solutions explicitly, we substitute the obtained results into the formula $w(x,y,z,t) = \frac{2}{a}(\ln f(x,y,z,t))$, and then we use the potential $w(x,y,z,t) = u_x(x,y,z,t)$ as defined in (39).

To determine the three-soliton solutions, we substitute the auxiliary function

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{51}$$

and proceed as before, we obtain the three soliton solutions under the conditions (50).

5 Conclusion

In this work we introduce three (3+1)-dimensional extended KD equations. We show that the three extended forms give the same dispersion relation. The single-soliton solution for each extended form is distinct and appear for specific conditions of the coefficients of the spatial variables. However, the two- and three-soliton solutions are identical for the three-extended equations, and these solutions exist for specific values of the spatial variables coefficients.

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