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Unsteady Flow of a Microstretch Fluid through State Space Approach with Slip Conditions

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Abstract: This paper deals with the study of unsteady microstretch Couette and Poiseuille fluid flows between two infinite parallel plates. Analytical expressions for the velocity, microrotation and microstretch of the fluid flows are obtained using Laplace transform together with state space approach. The slip boundary conditions for both velocity and microrotation are applied at the boundaries. The inversion of Laplace transform of the flow field is obtained numerically using a standard numerical technique developed by Honig and Hirdes. The effects of the physical parameters on the velocity, microrotation and microstretch are discussed through graphs.

Keywords: Microstretch fluid; Slip condition; State space approach

1 Introduction

Microcontinuum field theory was introduced by Eringen [1] in 1964 to explain the concept of the motion of micro-elements taking into consideration the internal characteristics of the substructure particles. Microstretch fluids are a subclass of simple microfluids. These fluids have seven degrees of freedom, three for translation, three for rotation and one for stretch. Microelements of these fluids can stretch or contract in addition to being micropolar. Physically, the theory of microstretch fluid could have rich applications in many fields such as slurries, paper pulps, insect colonies, blood and other biological fluids [2].

The classical no-slip boundary conditions are applied to the Navier-Stokes equations assuming that the liquid molecules adjacent to the solid are stationary relative to the solid. This condition is not appropriate for the fluids with microstructure such as micropolar, microstretch, etc. A new boundary condition, which is named slip condition has been proposed by Navier [3]. This condition depends on the shear stress and permits the fluids to slip at a solid boundary. In other words, the tangential velocity of fluid relative to the solid at a point on its surface is proportional to the tangential stress at the point. The constant of proportionality between these two quantities may be termed as a coefficient of sliding friction, it is assumed to depend only on the nature of the fluid and solid surface. The slip boundary condition was used by several researchers in the classical Newtonian fluids [4,5,6,7], in the micropolar and microstretch fluids [8,9,10,11].

In recent years, many authors have discussed the micropolar fluid flow problems between two parallel plates. Faltas et al [13] discussed the problem of unsteady unidirectional Poiseuille flow with no-slip and no-spin boundary conditions. Solutions of steady micropolar Couette and Poiseuille flow can be found in 1968 [12]. The problem of Couette flow bounded by two infinite parallel plates was considered by Ashmawy [14], assuming a linear slip boundary condition on the upper and lower plates. However, the microstretch fluids flows have received a little attention from researchers. Ariman [15] discussed the poiseuille flow between parallel plates in microstretch fluid. Iesan derived a uniqueness theorem for an incompressible microstretch fluid [16]. Eringen studied the steady flow of an incompressible microstretch fluid in circular arteries [17]. The state space approach is a mathematical model of a physical system and it is applicable to solve some problems in fluid dynamics. The state space approach was employed by Devakar and Iyengar [18] to discuss the Stokes first problem of a micropolar fluid with no-slip and no-spin conditions. They also used the same technique in [19,20] to investigate the Couette and Poiseuille motion of micropolar fluid assuming that one of the plate moves suddenly while the other is at rest. Slayi and Ashmawy

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applied this technique to obtain the solution of velocity and microrotation in Laplace domain to the unsteady slip flow of a micropolar fluid bounded by two parallel plates [21]. In 2016, time-dependent slip flow of a micropolar fluid was studied in [22].

In this paper, we consider the problem of unsteady motion of microstretch fluid flow through two infinite parallel plates. Two different cases are discussed. First, we suppose that the lower plate moves with some velocity. In the second part, the motion of two parallel plates is induced by the pressure gradient. The slip condition for both velocity and microrotation is applied at the boundaries. The problem has been solved in the Laplace domain using state space approach method. The inverse transforms are obtained using a numerical technique to get the velocity, micro-rotation and microstretch in space-time domain. The results are presented and discussed graphically.

2 Basic and constitutive equations of microstretch fluid

The equations concerning the flow of an incompressible microstretch fluid with no body loads are given by

$$\nabla\cdot\mathbf{q}=0, \tag{1}$$

$$\rho\frac{d\mathbf{q}}{dt}=\nabla(\lambda_0\varphi-p)+\kappa\nabla\times\nu-(\mu+\kappa)\nabla\times\nabla\times\mathbf{q}, \tag{2}$$

$$\rho j\frac{d\nu}{dt}=-2\kappa\nu+\kappa\nabla\times\mathbf{q}-\gamma_0\nabla\times\nabla\times\nu+(\alpha_0+\beta_0+\gamma_0)\nabla(\nabla\cdot\nu), \tag{3}$$

$$\frac{1}{2}\rho j\frac{d\varphi}{dt}=a_0\nabla\cdot\nabla\varphi+\pi_0-\lambda_0(\nabla\cdot\mathbf{q})-\lambda_1\varphi \tag{4}$$

The constitutive equations for the stresses, couple stresses and internal microstretch force density are given as

$$t_{ij}=(\lambda q_{r,r}+\lambda_0\varphi-p)\delta_{ij}+\mu q_{i,j}+(\mu+\kappa)q_{j,i}-\kappa\varepsilon_{ijk}\nu_k \tag{5}$$

$$m_{ij}=\alpha_0\nu_{r,r}\delta_{ij}+\beta_0\nu_{i,j}+\gamma_0\nu_{j,i}-b_0\varepsilon_{ijk}\varphi_{,k} \tag{6}$$

$$m_k=a_0\varphi_{,k}+b_0\varepsilon_{ijk}\nu_{i,j} \tag{7}$$

where ρ and j represent the fluid density and gyration parameters, respectively. They are assumed to be constants. Also, δ_{ij} and ε_{ijk} are denoting, respectively, Kronecker delta function and the alternating tensor. The vectors $\mathbf{q}$ and ν are respresenting, respectively, the velocity and microrotation of the fluid flow. φ denotes the microsretch scalar function. p denotes the pressure of fluid at any point. The material constants (μ, κ, λ_0, λ_1, a_0) represent the viscosity coefficients and (α_0 , β_0 , γ_0) represent the gyro-viscosity coefficients.

3 Unsteady Couette flow of microstretch fluid

Let us consider the incompressible microstretch fluid bounded by two horizontal parallel plates separated by a distance h. The motion is assumed to be unsteady. Initially, The two plates are fixed. When $t\succ 0$, the lower plate position changes and moves along the x-direction by a time-dependent velocity of magnitude $U f(t)$ while the upper plate is held fixed. Assuming that the pressure gradient is zero. Using the Cartesian coordinates (x,y,z), the components of velocity, microrotation and microstretch, respectively, have the following forms $\mathbf{q}=(u(y,t),0,0)$, $\nu=(0,0,\omega(y,t))$ and $\varphi=\varphi(y,t)$.

The proposed initial and boundary conditions are

$$u(y,0)=0 \quad \omega(y,0)=0 \quad \text{and} \quad \varphi(y,0)=0. \tag{8}$$

$$\beta_1(u(0,t)-U f(t))=\tau_{yx}(0,t), \quad \xi_1\omega(0,t)=m_{yz}(0,t) \tag{9}$$

$$\beta_2(u(h,t))=-\tau_{yx}(h,t), \quad \xi_2\omega(h,t)=-m_{yz}(h,t) \tag{10}$$

$$\varphi(h,t)=\varphi(0,t)=0 \tag{11}$$

where β_1 and β_2 are the velocity slip parameters of the lower and upper plates. Also, ξ_1 and ξ_2 represent the microrotation parameters of the two plates. These parameters are varying from zero to infinity and are assumed to depend only on the nature of the fluid and the material plates.

Let us introduce the following non-dimensional variables

$$\hat{y}=\frac{y}{h},\quad \hat{u}=\frac{u}{U},\quad \hat{t}=\frac{U}{h}t,\quad \hat{\omega}=\frac{h}{U}\omega,\quad \hat{\tau}_{yx}=\frac{h}{U\mu}\tau_{yx},$$

$$\hat{m}_{yz}=\frac{h^2}{\beta_0 U}m_{yz},\quad \hat{m}_k=\frac{m_k}{\pi_0 h},\quad \hat{\varphi}=\frac{a_0}{\pi_0 h^2}\varphi$$

Using the above variables and dropping hats for convenience, the differential equations (1-4) become

$$R\frac{\partial u}{\partial t}=m\frac{\partial\omega}{\partial y}+\frac{\partial^2 u}{\partial y^2} \tag{12}$$

$$\frac{R}{n_2}\frac{\partial\omega}{\partial t}=-2n\omega-n\frac{\partial u}{\partial y}+\frac{\partial^2\omega}{\partial y^2} \tag{13}$$

$$\frac{Rm_1}{n_2}\frac{\partial\varphi}{\partial t} = -m_2\varphi + 1 + \frac{\partial^2\varphi}{\partial y^2} \tag{14}$$

where

$$R = \frac{\rho uh}{(\mu+\kappa)}, \quad n_2 = \frac{2+K}{2(1+K)}, \quad m = \frac{K}{1+K}, \quad n = K\frac{h^2}{\gamma_0}$$

$$K = \frac{\kappa}{\mu}, \quad m_1 = \frac{\gamma}{2a_0}, \quad m_2 = \frac{\lambda_1 h^2}{a_0}.$$

The initial and boundary conditions ($8-11$) in terms of dimensionless quantities can be written as

$$u(y,0) = 0 \quad \text{and} \quad \omega(y,0) = 0 \quad \varphi(y,0) = 0. \tag{15}$$

$$\alpha_1(u(0,t) - f(t)) = \tau_{yx}(0,t), \quad \eta_1\omega(0,t) = m_{yz}(0,t) \tag{16}$$

$$-\alpha_2(u(1,t)) = \tau_{yx}(1,t), \quad -\eta_2\omega(1,t) = m_{yz}(1,t) \tag{17}$$

$$\varphi(0,t) = \varphi(1,t) = 0 \tag{18}$$

where

$$\alpha_1 = \frac{h\beta_1}{\mu}, \quad \alpha_2 = \frac{h\beta_2}{\mu}, \quad \eta_1 = \frac{h\xi_1}{\gamma_0}, \quad \eta_2 = \frac{h\xi_2}{\gamma_0}$$

And the expressions of the non-dimensional stress, couple stress and microstretch inertia components are

$$\tau_{yx}(y,t) = (1+K)\frac{\partial u(y,t)}{\partial y} + K\omega(y,t) \tag{19}$$

$$m_{yz} = \frac{\gamma_0}{\beta_0}\frac{\partial\omega(y,t)}{\partial y} \tag{20}$$

$$m_y = \frac{\partial\varphi}{\partial y}, \quad m_x = \frac{-b_0 U}{\pi_0 h^3}\frac{\partial\omega}{\partial y} \tag{21}$$

We now introduce the Laplace transform, defined by the relation

$$\bar{F}(y,s) = \int_0^\infty e^{-st}F(y,t)dt \tag{22}$$

The differential equations (12-14) reduce to,

$$\frac{\partial^2\bar{u}}{\partial y^2} + m\frac{\partial\bar{\omega}}{\partial y} - Rs\bar{u} = 0 \tag{23}$$

$$\frac{\partial^2\bar{\omega}}{\partial y^2} - n\frac{\partial\bar{u}}{\partial y} - a\bar{\omega} = 0 \tag{24}$$

$$\frac{\partial^2\bar{\varphi}}{\partial y^2} - b\bar{\varphi} + \frac{1}{s} = 0 \tag{25}$$

where

$$a = (2n + \frac{Rs}{n_2}) \qquad b = (m_2 + \frac{Rm_1 s}{n_2})$$

The boundary conditions are taking the forms

$$\alpha_1(\bar{u}(0,s) - \bar{f}(s)) = (1+K)\bar{u}'(0,s) + K\bar{\omega}(0,s), \tag{26}$$

$$\eta_1\bar{\omega}(0,s) = \bar{\omega}'(0,s), \quad \bar{\varphi}(0,s) = 0. \tag{27}$$

$$-\alpha_2(\bar{u}(1,s)) = (1+K)\bar{u}'(0,s) + K\bar{\omega}(1,s), \tag{28}$$

$$-\eta_2\bar{\omega}(1,s) = \bar{\omega}'(1,s), \quad \bar{\varphi}(1,s) = 0. \tag{29}$$

We now apply state space approach to the problem, and then the differential equations (23-25) can be expressed as matrix form as

$$\frac{d}{dy}\bar{V}(y,s) = A(s)\bar{V}(y,s) + \bar{B}(s); \tag{30}$$

where

$$A(s) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ Rs & 0 & 0 & 0 & -m & 0 \\ 0 & a & 0 & n & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \end{pmatrix}, \bar{V}(y,s) = \begin{pmatrix} \bar{u}(y,s) \\ \bar{\omega}(y,s) \\ \bar{\varphi}(y,s) \\ \bar{u}'(y,s) \\ \bar{\omega}'(y,s) \\ \bar{\varphi}'(y,s) \end{pmatrix}, \bar{B}(s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{-1}{s} \end{pmatrix}$$

where $\bar{V}(y,s)$ denotes the vector that contains the components of velocity, microrotation and microstretch and their derivatives. The formal solution of the system (30) takes the following form

$$\bar{V}(y,s) = \exp[A(s)y]\bar{V}(0,s) + \mathbf{H}(\mathbf{y},\mathbf{s}) \tag{31}$$

where

$$\mathbf{H}(\mathbf{y},\mathbf{s}) = \exp[A(s)y]\int_{z=0}^{y}\exp[-A(s)z]\bar{B}(s)\,dz \tag{32}$$

To compute the matrix $\exp[A(s)y]$, we compute first the characteristic equation of the matrix $A(s)$, then we get

$$k^6 - (Rs + a - mn + b)k^4 + (Rs(b+a) + b(a-mn)k^2 - Rsab = 0$$

Solving the above-mentioned characteristic equation, we obtain the roots $\pm k_1$, $\pm k_2$ and $\pm k_3$. Then, we use the Maclaurin series expansion of $\exp[A(s)y]$ given by

$$\exp[A(s)y] = \sum_{r=0}^{\infty}\frac{[A(s)y]^r}{r!} \tag{33}$$

We apply the Cayley-Hamilton theorem to write the infinite series (33) in the following form

$$\exp[A(s)y] = L(y,s) = c_0 I + c_1 A + c_2 A^2 + c_3 A^3 + c_4 A^4 + c_5 A^5 \quad (34)$$

where I is the unit matrix of order 6. $(c_0 - c_5)$ are parameters obtained in terms of the variables y and s. Then, by replacing the matrix A with the roots $\pm k_1, \pm k_2$ and $\pm k_3$ which are obtained before in equation (34), hence we get the following system of linear equations

$$\exp[\pm k_i y] = c_0 \pm c_1 k_i \pm c_2 k_i \pm c_3 k_i \pm c_4 k_i \pm c_5 k_i \quad i = 1,2,3.$$

After solving this system, we can evaluate the scalar coefficient $(c_0 - c_5)$, then we can obtain the entries $(L_{ij}; i, j = 1-6)$ of the matrix $L(y,s)$ after substituting A, A^2, A^3, A^4 and A^5 in equation (34). To obtain the explicit solution of $\mathbf{H}(\mathbf{y},\mathbf{s})$, we compute the $\exp[-A(s)y]$, and then evaluate the integral in equation (32).
The Maclaurin series expansion of $\exp[-A(s)y]$ is given by

$$\exp[-A(s)y] = \sum_{r=0}^{\infty} \frac{(-1)^r [A(s)y]^r}{r!} \quad (35)$$

Applying also here Cayley-Hamilton theorem to write the higher powers of A in terms of I, A, A^2, A^3, A^4 and A^5, then infinite series can be written in the following form

$$\exp[-A(s)y] = c_0 I - c_1 A + c_2 A^2 - c_3 A^3 + c_4 A^4 - c_5 A^5 \quad (36)$$

Then we can write $exp[-A(s)y]$ in terms of $L_{ij}, i, j = 1-6$.
Hence, equation (31) can be written in the form

$$\bar{u}(y,s) = L_{11}\bar{u}(0,s) + L_{12}\bar{\omega}(0,s) + L_{13}\bar{\varphi}(0,s) + L_{14}\bar{u}'(0,s) + L_{15}\bar{\omega}'(0,s) + L_{16}\bar{\varphi}'(0,s) + H_1(y,s) \quad (37)$$

$$\bar{\omega}(y,s) = L_{21}\bar{u}(0,s) + L_{22}\bar{\omega}(0,s) + L_{23}\bar{\varphi}(0,s) + L_{24}\bar{u}'(0,s) + L_{25}\bar{\omega}'(0,s) + L_{26}\bar{\varphi}'(0,s) + H_2(y,s) \quad (38)$$

$$\bar{\varphi}(y,s) = L_{31}\bar{u}(0,s) + L_{32}\bar{\omega}(0,s) + L_{33}\bar{\varphi}(0,s) + L_{34}\bar{u}'(0,s) + L_{35}\bar{\omega}'(0,s) + L_{36}\bar{\varphi}'(0,s) + H_3(y,s) \quad (39)$$

$$\bar{u}'(y,s) = L_{41}\bar{u}(0,s) + L_{42}\bar{\omega}(0,s) + L_{43}\bar{\varphi}(0,s) + L_{44}\bar{u}'(0,s) + L_{45}\bar{\omega}'(0,s) + L_{46}\bar{\varphi}'(0,s) + H_4(y,s) \quad (40)$$

$$\bar{\omega}'(y,s) = L_{51}\bar{u}(0,s) + L_{52}\bar{\omega}(0,s) + L_{53}\bar{\varphi}(0,s) + L_{54}\bar{u}'(0,s) + L_{55}\bar{\omega}'(0,s) + L_{56}\bar{\varphi}'(0,s) + H_5(y,s) \quad (41)$$

$$\bar{\varphi}'(y,s) = L_{61}\bar{u}(0,s) + L_{62}\bar{\omega}(0,s) + L_{63}\bar{\varphi}(0,s) + L_{64}\bar{u}'(0,s) + L_{65}\bar{\omega}'(0,s) + L_{66}\bar{\varphi}'(0,s) + H_6(y,s) \quad (42)$$

We now apply the boundary conditions on the above equations satisfied at $y = 0$, we get

$$\bar{\varphi}(0,s) = 0 \quad (43)$$

$$\bar{u}(0,s) = \frac{(1+K)\bar{u}'(0,s) + K\bar{\omega}(0,s)}{\alpha_1} + \bar{f}(s) \quad (44)$$

$$\bar{\omega}(0,s) = \frac{\bar{\omega}'(0,s)}{\eta_1} \quad (45)$$

After substituting the equations (43-45) in the equations (37-42), then we obtain the solution in terms of $\bar{u}'(0,s), \bar{\omega}'(0,s)$ and $\bar{\varphi}'(0,s)$. To obtain these three components, we apply the remaining boundary conditions given in equations (28-29) at $y = 1$ on the above equations. Therefore, after some calculations and rearrangement we get the derivatives in terms of $\bar{f}(s)$ and L^1_{ij}'s, the values of $L(y,s)$ at $y = 1$. Finally, after substituting the values of $\bar{u}'(0,s)$, $\bar{\omega}'(0,s)$ and $\bar{\varphi}'(0,s)$ in equations (37-42), we can obtain easily the expressions of velocity, microrotation and microstretch and their derivatives in Laplace domain.

4 The numerical inversion of Laplace transform

The numerical inversion technique was developed by Honig and Hirdes [11] to invert Laplace transfotm. The components of the velocity, microrotation and microstretch are obtained in the physical domain. Utilizing this numerical technique, the inverse Laplace transform of the function $\bar{g}(s)$ is approximated by the formula

$$g(t) = \frac{\exp(ct)}{T}\left[\frac{1}{2}\bar{g}(c) + Re\left(\sum_{k=1}^{N} \bar{g}\left(c + \frac{ik\pi}{T}\right)\exp\left(\frac{ik\pi t}{T}\right)\right)\right]$$

where $0 < t < 2T$, $i = \sqrt{-1}$, ε is a small positive number that corresponds to the degree of accuracy required and N is sufficiently large integer chosen such that,

$$\exp(ct) Re\left[\bar{g}\left(c + \frac{iN\pi}{T}\right)\exp\left(\frac{iN\pi t}{T}\right)\right] < \varepsilon,$$

The parameter c is a positive free parameter that must be greater than real parts of all singularities of $\bar{g}(s)$.

5 Numerical results and discussions

We apply the above mentioned technique to the obtained results by assuming that the moving plate is suddenly moved with constant velocity; $f(t) = H(t)$, where $H(t)$ is the Heaviside step function. From Fig 1, 2 and 3, we observe that the velocity, microrotation and microstretch increase in time t and the steady state is obtained at large value of time. In fig.4, we study the effect of microstretch parameter, we conclude that this parameter has no significant influence on the velocity and microrotation while as we increase the microstretch parameter, we find that there is an increasing effect on microstretch function.

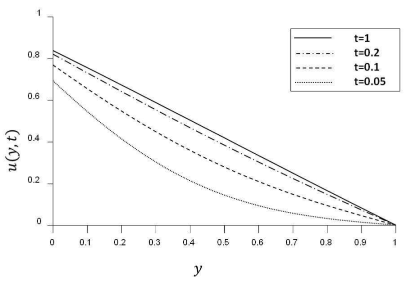

Fig. 1: variation of velocity versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

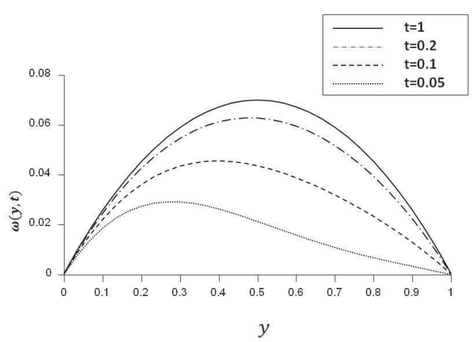

Fig. 2: variation of microrotation versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

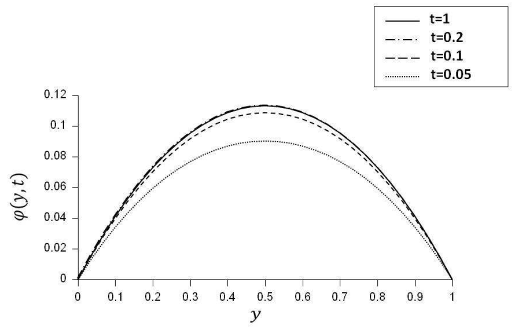

Fig. 3: variation of microstretch versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

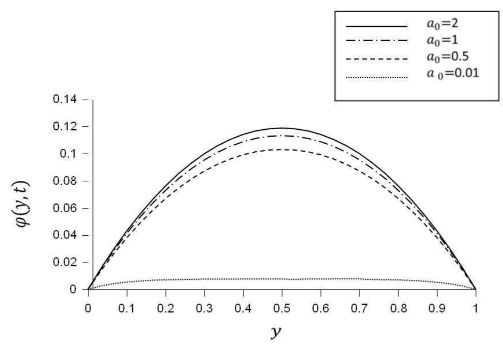

Fig. 4: variation of microstretch versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

6 Flow due to the induced pressure gradient

In this part, we assume that the incompressible microstretch flow starts due to a sudden pressure gradient. Using the Cartesian coordinates (x,y,z), the components of velocity, microrotation and microstretch, respectively, have the following forms, respectively, $\mathbf{q} = (u(y,t), 0, 0)$, $v = (0, 0, \omega(y,t))$ and $\varphi = \varphi(y,t)$.

The initial and slip boundary conditions applied to the problem are assumed to be

$$u(y,0) = 0 \quad \omega(y,0) = 0 \quad \text{and} \quad \varphi(y,0) = 0. \tag{46}$$

$$\beta_1 u(0,t) = \tau_{yx}(0,t), \quad \xi_1 \omega(0,t) = m_{yz}(0,t) \tag{47}$$

$$\beta_2(u(h,t)) = -\tau_{yx}(h,t), \quad \xi_2 \omega(h,t) = -m_{yz}(h,t) \tag{48}$$

$$\varphi(h,t) = \varphi(0,t) = 0 \tag{49}$$

where β_1 and β_2 are the velocity slip parameters of the lower and upper plates. Also, ξ_1 and ξ_2 represent the microrotation parameters of the two plates. These parameters are varying from zero to infinity and are assumed to depend only on the nature of the fluid and the material plates.
We now define the dimensionless quantities as

$$\hat{y} = \frac{y}{h}, \quad \hat{x} = \frac{x}{h}, \quad \hat{u} = \frac{\rho h}{(\mu+\kappa)} u, \quad \hat{t} = \frac{(\mu+\kappa)}{\rho h^2} t,$$

$$\hat{\omega} = \frac{\rho \kappa h^2}{(\mu+\kappa)^2}\omega, \quad \hat{\tau}_{yx} = \frac{\rho h^2}{(\mu+\kappa)^2}\tau_{yx}, \quad \hat{m}_{yz} = \frac{\rho \kappa h^3}{\gamma_0(\mu+\kappa)^2} m_{yz}$$

$$\hat{p} = \frac{\rho h^2}{(\mu+\kappa)^2} p, \quad \hat{m}_k = \frac{m_k}{\pi_0 h}, \quad \hat{\varphi} = \frac{a_0}{\pi_0 h^2}\varphi$$

Using the non-dimensional variables, dropping hats for convenience and after introducing the Laplace transform, the differential equations (1-4) become

$$\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial \bar{\omega}}{\partial y} - s\bar{u} - \frac{\partial \bar{p}}{\partial x} = 0 \tag{50}$$

$$\frac{\partial^2 \bar{\omega}}{\partial y^2} - f\frac{\partial \bar{u}}{\partial y} - d\bar{\omega} = 0 \tag{51}$$

$$\frac{\partial^2 \bar{\varphi}}{\partial y^2} - e\bar{\varphi} + \frac{1}{s} = 0 \tag{52}$$

where

$$d = (g + ws), \quad e = (m_2 + m_1 ws),$$

$$f = \frac{K^2 h^2}{\gamma_0(\mu+\kappa)}, \quad g = \frac{2\kappa h^2}{\gamma_0}, \quad w = \frac{(\mu+\kappa)}{\gamma_0} j,$$

$$K = \frac{\kappa}{\mu}, \quad m_1 = \frac{\gamma_0}{2a_0}, \quad m_2 = \frac{\lambda_1 h^2}{a_0},$$

The non-dimensional boundary conditions (47-49) in Laplace domain are

$$\alpha_1 \bar{u}(0,s) = (1+K)[\bar{u}'(0,s) + \bar{\omega}(0,s)], \tag{53}$$

$$\eta_1 \bar{\omega}(0,s) = \bar{\omega}'(0,s), \quad \bar{\varphi}(0,s) = 0. \tag{54}$$

$$-\alpha_2(\bar{u}(1,s)) = (1+K)[\bar{u}'(1,s) + \bar{\omega}(1,s)], \tag{55}$$

$$-\eta_2 \bar{\omega}(1,s) = \bar{\omega}'(0,s), \bar{\varphi}(1,s) = 0 \tag{56}$$

where

$$\alpha_1 = \frac{h\beta_1}{\mu}, \quad \alpha_2 = \frac{h\beta_2}{\mu}, \quad \eta_1 = \frac{h\xi_1}{\gamma_0}, \quad \eta_2 = \frac{h\xi_2}{\gamma_0}$$

Following the state space technique as is used before, and then the equations (50-52) can be written in matrix form as

$$\frac{d}{dy}\bar{V}(y,s) = D(s)\bar{V}(y,s) + \bar{E}(s); \tag{57}$$

where

$$D(s) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ s & 0 & 0 & 0 & -1 & 0 \\ 0 & d & 0 & f & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 \end{pmatrix}$$

$$\bar{V}(y,s) = \begin{pmatrix} \bar{u}(y,s) \\ \bar{\omega}(y,s) \\ \bar{\varphi}(y,s) \\ \bar{u}'(y,s) \\ \bar{\omega}'(y,s) \\ \bar{\varphi}'(y,s) \end{pmatrix}, \bar{E}(s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\bar{\varphi}(s) \\ 0 \\ \frac{-1}{s} \end{pmatrix}$$

where $\bar{V}(y,s)$denotes the vector that contains the components of velocity, microrotation and microstretch and their derivatives. The formal solution of the matrix differential equation (50-52) can be written as

$$\bar{V}(y,s) = \exp[D(s)y]\bar{V}(0,s) + \mathbf{G}(\mathbf{y},\mathbf{s}) \tag{58}$$

where

$$\mathbf{G}(\mathbf{y},\mathbf{s}) = \exp[D(s)y]\int_{z=0}^{y} \exp[-D(s)z]E(\bar{s}), dz \tag{59}$$

Here, we use the same strategies as before. First, we find the characteristic equation of the matrix $D(s)$ given by

$$\vartheta^6 - (s+d-f+e)\vartheta^4 + (s(e+d)+e(d-f))\vartheta^2 - sde = 0$$

After solving the above-mentioned characteristic equation, we can obtain the roots $\pm\vartheta_1$, $\pm\vartheta_2$ and $\pm\vartheta_3$. The Maclaurin series expansion of $\exp[D(s)y]$ is given by

$$\exp[D(s)y] = \sum_{r=0}^{\infty} \frac{[D(s)y]^r}{r!} \tag{60}$$

Then, afterwards, we apply the Cayley-Hamilton theorem to write the infinite series (60) in the following form

$$\begin{aligned} \exp[D(s)y] &= \ell(y,s) \\ &= d_0 I + d_1 D + d_2 D^2 + d_3 D^3 + d_4 D^4 + d_5 D^5 \\ &. \end{aligned} \tag{61}$$

where I is the sixth order unit matrix and $(d_0 - d_5)$ are parameters depending on y and s. The characteristic roots $\pm\vartheta_1, \pm\vartheta_2$ and $\pm\vartheta_3$ satsify the equation (61), hence, by replacing the matrix D with its characteristic roots, we obtain the following system of linear equations

$$\exp[\pm\vartheta_i y] = d_0 \pm d_1\vartheta_i \pm d_2\vartheta_i \pm d_3\vartheta_i \pm d_4\vartheta_i \pm d_5\vartheta_i \quad i = 1,2,3.$$

After solving this system, we can evaluate the scalar coefficient $d_0 - d_5$, then we can obtain the entries $(\ell_{ij}; i, j = 1-6)$ of the matrix $\ell(y,s)$ after substituting D, D^2, D^3, D^4, D^5 in equation (61). The explicit solution of $\mathbf{G}(\mathbf{y},\mathbf{s})$ can be determined by using the Maclaurin series expansion of $\exp[D(s)y]$ and then by computing the integral in equation (67).

The Maclaurin series expansion of $\exp[-D(s)y]$ is given by

$$\exp[-D(s)y] = \sum_{r=0}^{\infty} \frac{(-1)^r [D(s)y]^r}{r!}$$

Applying Cayley-Hamilton theorem, the infinite series can be written in the form

$$\exp[-D(s)y] = d_0 I - d_1 D + d_2 D^2 - d_3 D^3 + d_4 D^4 - d_5 D^5 \tag{62}$$

Then we can write $exp[-D(s)y]$ in terms of $\ell_{ij}, i, j = 1-6$.

Hence, equation (58) can be written in the form

$$\begin{aligned}\bar{u}(y,s) &= \ell_{11}\bar{u}(0,s) + \ell_{12}\bar{\omega}(0,s) + \ell_{13}\bar{\varphi}(0,s) + \ell_{14}\bar{u}'(0,s)\\ &+ \ell_{15}\bar{\omega}'(0,s) + \ell_{16}\bar{\varphi}'(0,s) + G_1(y,s)\end{aligned} \tag{63}$$

$$\begin{aligned}\bar{\omega}(y,s) &= \ell_{21}\bar{u}(0,s) + \ell_{22}\bar{\omega}(0,s) + \ell_{23}\bar{\varphi}(0,s) + \ell_{24}\bar{u}'(0,s)\\ &+ \ell_{25}\bar{\omega}'(0,s) + \ell_{26}\bar{\varphi}'(0,s) + G_2(y,s)\end{aligned} \tag{64}$$

$$\begin{aligned}\bar{\varphi}(y,s) &= \ell_{31}\bar{u}(0,s) + \ell_{32}\bar{\omega}(0,s) + \ell_{33}\bar{\varphi}(0,s) + \ell_{34}\bar{u}'(0,s)\\ &= \ell_{35}\bar{\omega}'(0,s) + \ell_{36}\bar{\varphi}'(0,s) + G_3(y,s)\end{aligned} \tag{65}$$

$$\begin{aligned}\bar{u}'(y,s) &= \ell_{41}\bar{u}(0,s) + \ell_{42}\bar{\omega}(0,s) + \ell_{43}\bar{\varphi}(0,s) + \ell_{44}\bar{u}'(0,s)\\ &= \ell_{45}\bar{\omega}'(0,s) + \ell_{46}\bar{\varphi}'(0,s) + G_4(y,s)\end{aligned} \tag{66}$$

$$\begin{aligned}\bar{\omega}'(y,s) &= \ell_{51}\bar{u}(0,s) + \ell_{52}\bar{\omega}(0,s) + \ell_{53}\bar{\varphi}(0,s) + \ell_{54}\bar{u}'(0,s)\\ &= \ell_{55}\bar{\omega}'(0,s) + \ell_{56}\bar{\varphi}'(0,s) + G_5(y,s)\end{aligned} \tag{67}$$

$$\begin{aligned}\bar{\varphi}'(y,s) &= \ell_{61}\bar{u}(0,s) + \ell_{62}\bar{\omega}(0,s) + \ell_{63}\bar{\varphi}(0,s) + \ell_{64}\bar{u}'(0,s)\\ &= \ell_{65}\bar{\omega}'(0,s) + \ell_{66}\bar{\varphi}'(0,s) + G_6(y,s)\end{aligned} \tag{68}$$

We now apply the boundary conditions (53-54) satisfied at $y = 0$,

$$\bar{\varphi}(0,s) = 0 \tag{69}$$

$$\bar{u}(0,s) = \frac{(1+K)[\bar{u}'(0,s) + \bar{\omega}(0,s)]}{\alpha_1} \tag{70}$$

$$\bar{\omega}(0,s) = \frac{\bar{\omega}'(0,s)}{\eta_1} \tag{71}$$

After substituting the equations above in the equations (63-68), then we obtain the solution in terms of $\bar{u}'(0,s)$,$\bar{\omega}'(0,s)$ and $\bar{\varphi}'(0,s)$. To obtain these three components, we apply the remaining boundary conditions given in equations (55-56) at $y = 1$ on the above equations. Therefore after some calculations and rearrangement we get the derivatives in terms of ℓ_{ij}^1's, the values of $\ell(y,s)$ at $y = 1$. Finally, after substituting the values of $\bar{u}'(0,s)$, $\bar{\omega}'(0,s)$ and $\bar{\varphi}'(0,s)$ in equations (63-68), we can obtain easily the expressions of velocity, micro-rotation and microstretch and their derivatives in Laplace domain.

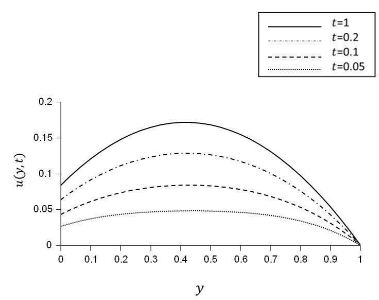

Fig. 5: variation of velocity versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

7 Numerical results and discussions

We have computed u(y,t), $\omega(y,t)$ and $\varphi(y,t)$ for different values of parameters and we represent the results graphically. We consider that the dimensionless pressure gradient is given by

$$\frac{-\partial p}{\partial x} = H(t)$$

Where H(t) is the Heaviside unit step function.

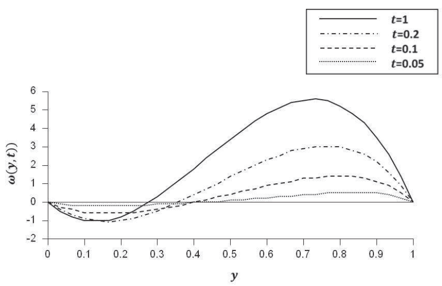

Fig. 6: variation of microrotation versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

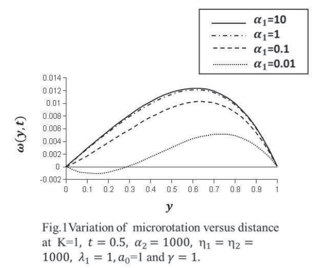

Fig. 9: variation of microrotation versus distance for $t = 0.5$, $\alpha_2 = \infty$, $\eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

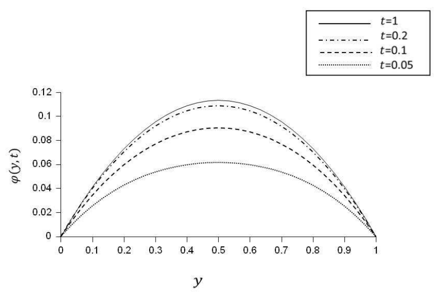

Fig. 7: variation of microstretch versus distance for $\alpha_1 = 10$, $\alpha_2 = \eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

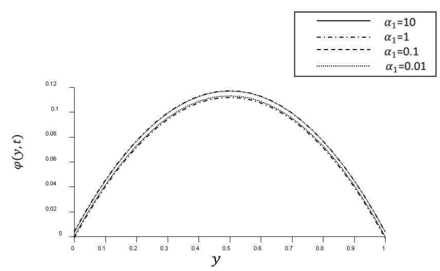

Fig. 10: variation of microstetch versus distance for $t = 0.5$, $\alpha_2 = \infty$, $\eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

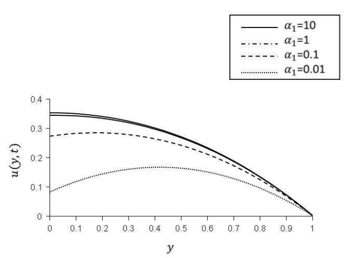

Fig. 8: variation of velocity versus distance for $t = 0.5$, $\alpha_2 = \infty$, $\eta_1 = \eta_2 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $K = 1$.

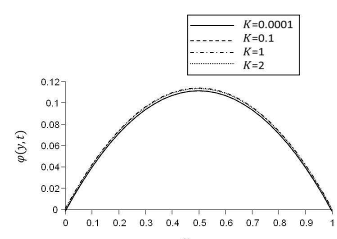

Fig. 11: variation of microstretch versus distance for $\alpha_1 = \eta 2 = 1$, $\alpha_2 = \eta_1 \to \infty$, $\lambda_1 = 1$, $a_0 = 1$, $\gamma_0 = 1$ and $t = 0.5$

In Figure 5, 6 and 7, we study the variation of velocity, microrotation and microstretch with distance for different times, we can conclude that their values increase monotonically with time. The velocity, microrotation and microstretch distributions for different values of the velocity slip parameter are discussed. We conclude that this parameter does not affect microstretch in Fig. 10, while it has a considerable effect on the velocity and microrotation in Figs.8 and 9. Finally we discuss the effect of micropolarity parameter. In addition, we also observe that this parameter has no effect on microstretch as in Fig.11, but it has a considerable effect on velocity and microrotation.

8 Conclusion

The unsteady motion of an incompressible microstretch fluid flow between two infinite parallel plates is investigated through the state space approach. Two cases are studied; in the first case, the fluid motion is caused by the time-dependent motion of the lower plate with no pressure gradient while the upper plate is kept fixed. In the second case, the fluid motion is induced by a constant pressure gradient where the two plates are set stationary. The slip boundary condition is applied for both velocity and microrotation on the two plane boundaries. The Laplace transform technique is employed to obtain the flow field functions analytically in the Laplace domain using state space method. The Laplace transform is inverted numerically and the flow field functions are represented graphically. The effect of the physical parameters such as the slip parameters, micropolarity constant and microstretch coefficients are discussed numerically through graphs. The classical case of no slip can be recovered as a special case of the present work when the slip parameters approach infinity. In addition, it is concluded that micropolarity parameter has a small influence on the microstretch function while its effect on the velocity and microrotation functions is remarkable.

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