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Caputo and Canavati Fractional Approximation by Choquet Integrals

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Abstract: Here we consider the quantitative Caputo and Canavati fractional approximation of positive sublinear operators to the unit operator. At the beginning we perform the investigation of the fractional rate of the convergence of the Bernstein-Kantorovich-Choquet and Bernstein-Durrweyer-Choquet polynomial Choquet-integral operators. After that we discuss the very general comonotonic positive sublinear operators based on the representation theorem of Schmeidler (1986) [1]. We end with the approximation by the very general direct Choquet-integral form positive sublinear operators. All fractional approximations are presented via inequalities implying the modulus of continuity of the approximated function fractional order derivative.

Keywords: Jackson type inequality, Choquet integral, modulus of continuity, comonotonicity of functions and operators, Bernstein-Kantorovich-Choquet and Bernstein-Durrmeyer-Choquet operators, fractional derivative.

1 Introduction

In 1953 G. Choquet [2] defined the capacities as well as his integral. At the beginning these were utilized to statistical mechanics and potential theory, and they gave rise to the investigation of non-additive measure theory. The ideas proposed by Choquet attracted economists especially after the seminal manuscript of Shapley (1953) [3] regarding the investigation of cooperative games. Schmeidler utilized them [4] in a model of choice with non-additive beliefs and since then the capacities and Choquet integrals became main stream to decision theorists.

This integral has huge applications to decision making under risk and uncertainty, in finance, in economics and in portofolio problems.

Our motivation is inspired by foundations of Bayesian decision theory and subjective probability.

Due to the paramount importance of Choquet integral, we investigate the related positive sublinear operators approximation, part of it is exhibited in this paper in the sense of Caputo and Canavati fractional derivatives.

The manuscript contains five chapters. During the first four chapters named Background - I, Background - II, Background - III and Background - IV, respectively, we introduce the requested basic theoretical results. Finally, the chapter five presents the original results of the manuscript.

2 Background - I

Some details about the Choquet integral are given below, see also [5].

Definition 1. *Consider $\Omega \neq \varnothing$ and let $\mathscr{C}$ be a σ-algebra of subsets in Ω.*

(i) (see, e.g., [6], p. 63) The set function $\mu : \mathscr{C} \to [0, +\infty]$ is called a monotone set function (or capacity) if $\mu(\varnothing) = 0$ and $\mu(A_1) \leq \mu(B_1)$ for all $A_1, B_1 \in \mathscr{C}$, with $A_1 \subset B_1$. μ is called submodular if

$$\mu(A_1 \cup B_1) + \mu(A_1 \cap B_1) \leq \mu(A_1) + \mu(B_1), \text{ for all } A_1, B_1 \in \mathscr{C}.$$

μ is called bounded if $\mu(\Omega) < +\infty$ and normalized if $\mu(\Omega) = 1$.

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(ii) (see, e.g., [6], p. 233, or [2]) If μ is a monotone set function on $\mathscr{C}$ and if $f_1 : \Omega \to \mathbb{R}$ is $\mathscr{C}$-measurable (for any Borel subset $B_1 \subset \mathbb{R}$ it follows $f_1^{-1}(B_1) \in \mathscr{C}$), then for any $A_1 \in \mathscr{C}$, the Choquet integral becomes

$$(C)\int_{A_1} f_1 d\mu = \int_0^{+\infty} \mu\left(F_\beta\left(f_1\right) \cap A_1\right) d\beta + \int_{-\infty}^{0} \left[\mu\left(F_\beta\left(f_1\right) \cap A_1\right) - \mu\left(A_1\right)\right] d\beta,$$

where we utilise the notation $F_\beta(f_1) = \{\omega \in \Omega : f_1(\omega) \geq \beta\}$. We recall that if $f_1 \geq 0$ on A_1, then in the above formula we get $\int_{-\infty}^{0} = 0$.

The classical Riemann integral are the integrals on the right-hand side.

f_1 is named Choquet integrable on A_1 if $(C)\int_{A_1} f_1 d\mu \in \mathbb{R}$.

Several properties of this integral are given below.

*Remark.*If $\mu : \mathscr{C} \to [0,+\infty]$ represents a monotone set function, then we have:

(i) For all $a \geq 0$ we have $(C)\int_A a f_1 d\mu = a \cdot (C)\int_{A_1} f_1 d\mu$ (if $f_1 \geq 0$ then see, e.g., [6], Theorem 11.2, (5), p. 228 and if f_1 is arbitrary sign, then see, e.g., [14], p. 64, Proposition 5.1, (ii)).

(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [6], pp. 232-233, or [14], p. 65) $(C)\int_{A_1}(f_1 + c)\, d\mu = (C)\int_{A_1} f_1 d\mu + c \cdot \mu(A_1)$.

If μ is submodular too, then for all f_1, g_1 of arbitrary sign and lower bounded, we get ([14], p. 75, Theorem 6.3)

$$(C)\int_{A_1}(f_1 + g_1)\, d\mu \leq (C)\int_{A_1} f_1 d\mu + (C)\int_{A_1} g_1 d\mu.$$

(iii) If $f_1 \leq g$ on A_1 then $(C)\int_{A_1} f_1 d\mu \leq (C)\int_{A_1} g_1 d\mu$ (see, e.g., [6], p. 228, Theorem 11.2, (3) if $_1, g_1 \geq 0$ and p. 232 if f_1, g_1 are of arbitrary sign).

(iv) Let $f_1 \geq 0$. If $A_1 \subset B$ then $(C)\int_{A_1} f_1 d\mu \leq (C)\int_{B_1} f d\mu$. In addition, if μ is finitely subadditive, then

$$(C)\int_{A_1 \cup B_1} f d\mu \leq (C)\int_{A_1} f_1 d\mu + (C)\int_{B_1} f_1 d\mu.$$

(v) We conclude that $(C)\int_{A_1} 1 \cdot d\mu(t) = \mu(A_1)$.

(vi) $\mu(A_1) = \gamma(M(A_1))$, $\gamma : [0,1] \to [0,1]$ denotes an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and M is a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\varnothing) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of normalized, monotone and submodular set functions ([14], pp. 16-17, Example 2.1). Such set functions μ are also called distorsions of countably normalized, additive measures (or distorted measures). For a simple example, we can take $\gamma(t) = \frac{2t}{1+t}$, $\gamma(t) = \sqrt{t}$.

If γ function increases, concave and fulfils only $\gamma(0) = 0$, then for any bounded Borel measure m, $\mu(A_1) = \gamma(m(A_1))$ gives a simple example of bounded, monotone and submodular set function.

(vii) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_{A_1} f_1 d\mu$ reduces to the usual Lebesgue type integral (see, e.g., [14], p. 62, or [6], p. 226).

(viii) If $f_1 \geq 0$, then $(C)\int_{A_1} f_1 d\mu \geq 0$.

(ix) Let $\mu = \sqrt{M}$, where M represents the Lebesgue measure on $[0,+\infty)$, then μ denotes a monotone and submodular set function, furthermore μ is strictly positive, see [19].

(x) If $\Omega = \mathbb{R}^N$, $N \in \mathbb{N}$, we call μ strictly positive if $\mu(A_1) > 0$, for any open subset $A_1 \subseteq \mathbb{R}^N$.

Definition 2.*([16]) For the $\Omega \neq \varnothing$, the power set $\mathscr{P}(\Omega)$ denotes the family of all subsets of Ω.*

(i) A function $\lambda : \Omega \to [0,1]$ fulfilling $\sup\{\lambda(s) : s \in \Omega\} = 1$, is called possibility distribution on Ω.

(ii) $P : \mathscr{P}(\Omega) \to [0,1]$ is called possibility measure, if it fulfills $P(\varnothing) = 0$, $P(\Omega) = 1$, and $P(\cup_{i\in I} A_i) = \sup\{P(A_i) : i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. We recall that if $A_1, B_1 \subset \Omega$, $A_1 \subset B_1$, then the last property implies $P(A_1) \leq P(B_1)$ and that $P(A_1 \cup B_1) \leq P(A_1) + P(B_1)$.

Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda : \mathscr{P}(\Omega) \to [0,1]$, $P_\lambda(A_1) = \sup\{\lambda(s) : s \in A_1\}$, $A_1 \subset \Omega$. In addition if $f_1 : \Omega \to \mathbb{R}_+$, then the possibilistic integral of f_1 on $A_1 \subset \Omega$ with respect to P_λ is defined by $(Pos)\int_{A_1} f_1 dP_\lambda = \sup\{f_1(t)\lambda(t) : t \in A_1\}$ (see [16], chapter 1).

We recall that any possiblity measure μ is normalized, monotone and submodular. By using $\mu(A_1 \cup B_1) = \max\{\mu(A_1), \mu(B_1)\}$ we obtain monotonicity, and from $\mu(A_1 \cap B_1) \leq \min\{\mu(A_1), \mu(B_1)\}$ we get the submodularity.

3 Background - II

From Caputo fractional calculus:

We need

Definition 3. *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $g \in AC^n([a,b])$ (space of functions g with $g^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We denote left Caputo fractional derivative (see [15], p. 40, [18], [20]) the function*

$$D_{*a}^{\nu} g(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} g^{(n)}(t)\, dt, \quad \forall\, x \in [a,b], \tag{1}$$

where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.

We set $D_{*a}^{0} g(x) = g(x)$, $\forall\, x \in [a,b]$.

Lemma 1. *([9]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $g \in C^{n-1}([a,b])$ and $g^{(n)} \in L_{\infty}([a,b])$. Then $D_{*a}^{\nu} g(a) = 0$.*

We need

Definition 4. *(see also [8], [17], [18]) Let $g \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} g(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} g^{(m)}(\zeta)\, d\zeta, \quad \forall\, x \in [a,b]. \tag{2}$$

We set $D_{b-}^{0} g(x) = g(x)$.

Lemma 2. *([9]) Let $g \in C^{m-1}([a,b])$, $g^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} g(b) = 0$.*

Convention 1 *We suppose that*

$$D_{*x_0}^{a} g(x) = 0, \text{ for } x < x_0, \tag{3}$$

and

$$D_{x_0-}^{\alpha} g(x) = 0, \text{ for } x > x_0, \tag{4}$$

for all $x, x_0 \in [a,b]$.

We have

Proposition 2. *[9] Let $g \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} g(x)$ is continuous in $x \in [a,b]$.*

Proposition 3. *[9] Let $g \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} g(x)$ is continuous in $x \in [a,b]$.*

For $g \in C([a,b])$ we define the (first) modulus of continuity:

$$\omega_1(g,\delta) := \sup_{\substack{x,y \in [a,b]: \\ |x-y| \leq \delta}} |g(x) - g(y)|, \delta > 0. \tag{5}$$

The modulus of continuity $\omega_1(g,\delta)$ is defined the same way for bounded functions, see (5), and it is finite.

We have

Remark. [9] Let $g \in C^{n-1}([a,b])$, $g^{(n)} \in L_{\infty}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. We get

$$\omega_1\left(D_{*a}^{\nu} g, \delta\right) \leq \frac{2\left\|g^{(n)}\right\|_{\infty}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{6}$$

Let $g \in C^{m-1}([a,b])$, $g^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha} g, \delta\right) \leq \frac{2\left\|g^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{7}$$

It means $\omega_1\left(D^{\nu}_{*a}g,\delta\right)$, $\omega_1\left(D^{\alpha}_{b-}g,\delta\right)$ are finite.

Thus, $D^{\nu}_{*a}g$ and $D^{\alpha}_{b-}g$ are bounded, from

$$\left|D^{\nu}_{*a}g\left(x\right)\right| \leq \frac{\left\|g^{(n)}\right\|_{\infty}}{\Gamma\left(n-\nu+1\right)}\left(b-a\right)^{n-\nu}, \quad \forall\, x\in\left[a,b\right], \tag{8}$$

see [9].

We need

Definition 5. *Let $D^{\alpha}_{x_0}g$ denote any of $D^{\alpha}_{x_0-}g$, $D^{\alpha}_{*x_0}g$, and $\delta>0$. We set*

$$\omega_1\left(D^{\alpha}_{x_0}g,\delta\right) := \max\left\{\omega_1\left(D^{\alpha}_{x_0-}g,\delta\right)_{[a,x_0]}, \omega_1\left(D^{\alpha}_{*x_0}g,\delta\right)_{[x_0,b]}\right\}, \tag{9}$$

where $x_0\in\left[a,b\right]$. We discuss the moduli of continuity over $\left[a,x_0\right]$ and $\left[x_0,b\right]$, respectively.

We have

Definition 6. *We consider $C_+\left(\left[a,b\right]\right) := \left\{g:\left[a,b\right]\to\mathbb{R}_+,\text{ continuous functions}\right\}$. Let $L_N : C_+\left(\left[a,b\right]\right)\to C_+\left(\left[a,b\right]\right)$, operators, $\forall\, N\in\mathbb{N}$, obeying*

(i)

$$L_N\left(\alpha g\right)=\alpha L_N\left(g\right), \ \forall\alpha\geq 0, \forall g\in C_+\left(\left[a,b\right]\right), \tag{10}$$

(ii) if $g,h\in C_+\left(\left[a,b\right]\right): g\leq h$, then

$$L_N\left(g\right)\leq L_N\left(h\right), \ \forall N\in\mathbb{N}, \tag{11}$$

(iii)

$$L_N\left(g+h\right)\leq L_N\left(g\right)+L_N\left(h\right), \ \forall\, g,h\in C_+\left(\left[a,b\right]\right). \tag{12}$$

We denote $\left\{L_N\right\}_{N\in\mathbb{N}}$ positive sublinear operators.

We have

Theorem 1. *([12] Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $x_0\in\left[a,b\right]\subset\mathbb{R}$, $f\in AC^m\left(\left[a,b\right],\mathbb{R}_+\right)$, and $g^{(m)}\in L_{\infty}\left(\left[a,b\right]\right)$. We suggest that $g^{(k)}\left(x_0\right)=0$, $k=1,\ldots,m-1$. Let $L_N: C_+\left(\left[a,b\right]\right)\to C_+\left(\left[a,b\right]\right)$, $\forall\, N\in\mathbb{N}$, be positive sublinear operators, obeying $L_N\left(1\right)=1$, $\forall\, N\in\mathbb{N}$. Thus, we get*

$$\left|L_N\left(g\right)\left(x_0\right)-g\left(x_0\right)\right| \leq \frac{\omega_1\left(D^{\alpha}_{x_0}g,\delta\right)}{\Gamma\left(\alpha+1\right)}\cdot$$

$$\left[L_N\left(\left|\cdot-x_0\right|^{\alpha}\right)\left(x_0\right)+\frac{L_N\left(\left|\cdot-x_0\right|^{\alpha+1}\right)\left(x_0\right)}{\left(\alpha+1\right)\delta}\right], \tag{13}$$

$\delta>0$, $\forall\, N\in\mathbb{N}$.

We recall that (13) is true for $\alpha>1$, $\alpha\notin\mathbb{N}$.

Corollary 1. *([12]) Let $0<\alpha<1$, $x_0\in\left[a,b\right]\subset\mathbb{R}$, $g\in AC\left(\left[a,b\right],\mathbb{R}_+\right)$, and $g'\in L_{\infty}\left(\left[a,b\right]\right)$. Let $L_N: C_+\left(\left[a,b\right]\right)\to C_+\left(\left[a,b\right]\right)$, $\forall\, N\in\mathbb{N}$, be positive sublinear operators, obeying $L_N\left(1\right)=1$, $\forall\, N\in\mathbb{N}$. Thus (13) is true.*

We also need:

Theorem 2. *([12]) Let $0<\alpha<1$, $x_0\in\left[a,b\right]\subset\mathbb{R}$, $g\in AC\left(\left[a,b\right],\mathbb{R}_+\right)$, and $g'\in L_{\infty}\left(\left[a,b\right]\right)$. Let L_N from $C_+\left(\left[a,b\right]\right)$ into itself be positive sublinear operators, such that $L_N\left(1\right)=1$, $\forall\, N\in\mathbb{N}$. Suppose that $L_N\left(\left|\cdot-x_0\right|^{\alpha+1}\right)\left(x_0\right)>0$, $\forall\, N\in\mathbb{N}$. Then*

$$\left|L_N\left(f\right)\left(x_0\right)-g\left(x_0\right)\right| \leq$$

$$\frac{\left(\alpha+2\right)\omega_1\left(D^{\alpha}_{x_0}g,\left(L_N\left(\left|\cdot-x_0\right|^{\alpha+1}\right)\left(x_0\right)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma\left(\alpha+2\right)}\left(L_N\left(\left|\cdot-x_0\right|^{\alpha+1}\right)\left(x_0\right)\right)^{\frac{\alpha}{\alpha+1}}. \tag{14}$$

We mention some notions from Canavati fractional calculus.
Then we need

*Remark.*I) Here see [7], pp. 7-10.
Let $x, x_0 \in [a,b]$ obey $x \geq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, such that $p = [\nu]$, $[\cdot]$ the integral part, $\alpha = \nu - p$ $(0 < \alpha < 1)$.
Let $g \in C^p([a,b])$ and define

$$\left(J_\nu^{x_0} g\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} g(t)\,dt, \ \ x_0 \leq x \leq b. \tag{15}$$

as the left generalized fractional Riemann-Liouville integral.
Γ represents Gamma function.
We have $\left(J_\nu^{x_0} g\right)(x_0) = 0$. We define $\left(J_\nu^{x_0} g\right)(x) = 0$ for $x < x_0$. By [7], p. 388, $\left(J_\nu^{x_0} g\right)(x)$ is a continuous function in x, for a fixed x_0.
We define the subspace $C_{x_0+}^\nu([a,b])$ of $C^p([a,b])$:

$$C_{x_0+}^\nu([a,b]) := \left\{ g \in C^p([a,b]) : J_{1-\alpha}^{x_0} g^{(p)} \in C^1([x_0,b]) \right\}. \tag{16}$$

So let $g \in C_{x_0+}^\nu([a,b])$, we define the left generalized ν-fractional derivative of g over $[x_0,b]$ as

$$D_{x_0+}^\nu f = \left(J_{1-\alpha}^{x_0} g^{(p)} \right)', \tag{17}$$

that is

$$\left(D_{x_0+}^\nu g\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x_0}^x (x-t)^{-\alpha} g^{(p)}(t)\,dt, \tag{18}$$

which exists for $f \in C_{x_0+}^\nu([a,b])$, for $a \leq x_0 \leq x \leq b$ [13].
II) Here see [8], [4].
Let $x, x_0 \in [a,b]$ obey $x \leq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, fulfilling $p = [\nu]$, $\alpha = \nu - p$ $(0 < \alpha < 1)$.
Let $g \in C^p([a,b])$ and define

$$\left(J_{x_0-}^\nu g\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^{x_0} (z-x)^{\nu-1} g(z)\,dz, \ \ a \leq x \leq x_0. \tag{19}$$

the right generalized Riemann-Liouville fractional integral.
Define the subspace of functions

$$C_{x_0-}^\nu([a,b]) := \left\{ g \in C^p([a,b]) : J_{x_0-}^{1-\alpha} g^{(p)} \in C^1([a,x_0]) \right\}. \tag{20}$$

Define the right generalized ν-fractional derivative of g over $[a,x_0]$ as

$$D_{x_0-}^\nu g = (-1)^{p-1} \left(J_{x_0-}^{1-\alpha} g^{(p)} \right)'. \tag{21}$$

Notice that

$$J_{x_0-}^{1-\alpha} g^{(p)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^{x_0} (z-x)^{-\alpha} g^{(p)}(z)\,dz, \tag{22}$$

exists for $g \in C_{x_0-}^\nu([a,b])$, and

$$\left(D_{x_0-}^\nu g\right)(x) = \frac{(-1)^{p-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^{x_0} (z-x)^{-\alpha} g^{(p)}(z)\,dz. \tag{23}$$

I.e.

$$\left(D_{x_0-}^\nu g\right)(x) = \frac{(-1)^{p-1}}{\Gamma(p-\nu+1)} \frac{d}{dx} \int_x^{x_0} (z-x)^{p-\nu} g^{(p)}(z)\,dz, \tag{24}$$

which exists for $g \in C_{x_0-}^\nu([a,b])$, for $a \leq x \leq x_0 \leq b$.

We make

Definition 7.*Let $\overline{D}_{x_0}^{\nu} g$ denote any of $D_{x_0-}^{\nu} f$, $D_{x_0+}^{\nu} g$, and $\delta > 0$. We denote*

$$\omega_1\left(\overline{D}_{x_0}^{\nu} g, \delta\right) := \max\left\{\omega_1\left(D_{x_0-}^{\nu} g, \delta\right)_{[a,x_0]}, \omega_1\left(D_{x_0+}^{\nu} g, \delta\right)_{[x_0,b]}\right\}, \tag{25}$$

where $x_0 \in [a,b]$. The moduli of continuity are discussed over $[a,x_0]$ and $[x_0,b]$, respectively.

We will use

Theorem 3.*[11] Let $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x_0 \in [a,b]$ and $g : [a,b] \to \mathbb{R}_+$, $g \in C_{x_0+}^{\nu}([a,b]) \cap C_{x_0-}^{\nu}([a,b])$. Suppose that $g^{(k)}(x_0) = 0$, $k = 1,...,p-1$, and $\left(D_{x_0+}^{\nu} g\right)(x_0) = \left(D_{x_0-}^{\nu} g\right)(x_0) = 0$. Let $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, be positive sublinear operators, obeying $L_N(1) = 1$, $\forall N \in \mathbb{N}$. As a result we have*

$$|L_N(g)(x_0) - g(x_0)| \leq \frac{\omega_1\left(\overline{D}_{x_0}^{\nu} g, \delta\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[L_N\left(|\cdot - x_0|^{\nu}\right)(x_0) + \frac{L_N\left(|\cdot - x_0|^{\nu+1}\right)(x_0)}{(\nu+1)\delta}\right], \tag{26}$$

$\delta > 0$, $\forall N \in \mathbb{N}$.

4 Background - III

We mention

Definition 8.*[19] Let $I = [0,1]$, $\mathscr{B}_I$ the σ-algebra of all Borel measurable subsets of I, $(\Gamma_{N,x})_{N\in\mathbb{N},\, x\in I}$ will be the collection of the family $\Gamma_{N,x} = \{\mu_{N,k,x}\}_{k=0}^{N}$, of monotone, submodular and strictly positive set functions $\mu_{N,k,x}$ on $\mathscr{B}_I$.*

Let $f_1 : [0,1] \to \mathbb{R}_+$ be a B_I-measurable function which is bounded, and call $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, for any $x \in [0,1]$.

The expression of Bernstein-Kantorovich-Choquet operators is

$$K_{N,\Gamma_{N,x}}(f_1)(x) = \sum_{k=0}^{N} p_{N,k}(x) \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f_1(t)\, d\mu_{N,k,x}(t)}{\mu_{N,k,x}\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \quad \forall x \in [0,1]. \tag{27}$$

If $\mu_{N,k,x} = \mu$, for all N,x,k, we denote $K_{N,\Gamma_{N,x}}(f_1) := K_{N,\mu}(f_1)$.

Theorem 4.*[19] Assume that $\mu_{N,k,x} = \mu := \sqrt{M}$, for all N,k and x, where M represents the Lebesgue measure on $[0,1]$. Then*

$$\left|K_{N,\mu}(f_1)(x) - f_1(x)\right| \leq 2\omega_1\left(f_1, \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}\right), \tag{28}$$

$\forall N \in \mathbb{N}$, $x \in [0,1]$, $f_1 \in C_+([0,1])$, *above ω_1 is over $[0,1]$.*

*Remark.*With the help of [19] we conclude

$$K_{N,\mu}(|\cdot - x|)(x) \leq \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall N \in \mathbb{N}. \tag{29}$$

For $m > 1$, we conclude that $|\cdot - x|^{m-1} \leq 1$, it implies

$$|\cdot - x|^m = |\cdot - x|\,|\cdot - x|^{m-1} \leq |\cdot - x|,$$

thus

$$K_{N,\mu}\left(|\cdot-x|^{m}\right)(x)\leq K_{N,\mu}\left(|\cdot-x|\right)(x),$$

that is

$$K_{N,\mu}\left(|\cdot-x|^{m}\right)(x)\leq\frac{\sqrt{x(1-x)}}{\sqrt{N}}+\frac{1}{N},\ \ \forall\, x\in[0,1],\ N\in\mathbb{N},m\geq1. \tag{30}$$

We conclude that $K_{N,\mu}(1)=1$, $\forall\, N\in\mathbb{N}$.

$K_{N,\mu}$ operators are positive sublinear operators from $C_{+}([0,1])$ into itself.

Definition 9.*[5] We discuss measures of possibility. Denoting $p_{N,k}(x)=\binom{N}{k}x^{k}(1-x)^{N-k}$, let us define*

$$\lambda_{N,k}(t):=\frac{p_{N,k}(t)}{k^{k}N^{-N}(N-k)^{N-k}\binom{N}{k}}=\frac{t^{k}(1-t)^{N-k}}{k^{k}N^{-N}(N-k)^{N-k}},\ \ k=0,...,N. \tag{31}$$

We suppose that $0^{0}=1$. As a result the cases $k=0$, and $k=N$ are interesting. By using the root $\frac{k}{N}$ of $p_{N,k}'(x)$, we get

$$\max\{p_{N,k}(t):t\in[0,1]\}=k^{k}N^{-N}(N-k)^{N-k}\binom{N}{k},$$

which says that each $\lambda_{N,k}$ is a possibility distribution on $[0,1]$.

Denoting by $P_{\lambda_{N,k}}$ the possibility measure induced by $\lambda_{N,k}$ and $\Gamma_{n,x}:=\Gamma_{N}:=\{P_{\lambda_{N,k}}\}_{k=0}^{N}$ (that is Γ_{N} is independent of x), we define the nonlinear Bernstein-Durrmeyer-Choquet polynomial operators with respect to the set functions in Γ_{N} as

$$D_{N,\Gamma_{N}}(f_{1})(x):=\sum_{k=0}^{N}p_{N,k}(x)\frac{(C)\int_{0}^{1}f_{1}(t)t^{k}(1-t)^{N-k}dP_{\lambda_{N,k}}(t)}{(C)\int_{0}^{1}t^{k}(1-t)^{N-k}dP_{\lambda_{N,k}}(t)}, \tag{32}$$

$\forall\, x\in[0,1]$, $N\in\mathbb{N}$, $f_{1}\in C_{+}([0,1])$.

*Remark.*Above $P_{\lambda_{N,k}}$ is bounded, monotone, submodular and strictly positive, $N\in\mathbb{N}$, $k=0,1,...,N$. Notice that $D_{N,\Gamma_{N}}(1)=1$, $\forall\, N\in\mathbb{N}$.

$D_{N,\Gamma_{N}}$ operators are positive sublinear operators mapping $C_{+}([0,1])$ into itself.

We mention

Theorem 5.*[5] For every $f\in C_{+}([0,1])$, $x\in[0,1]$ and $N\in\mathbb{N}-\{1\}$, we have*

$$\left|D_{N,\Gamma_{N}}(f_{1})(x)-f_{1}(x)\right|\leq2\omega_{1}\left(f,\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2}\sqrt{x}}{\sqrt{N}}+\frac{1}{N}\right), \tag{33}$$

where ω_{1} is on $[0,1]$.

*Remark.*From [5] we conclude

$$D_{N,\Gamma_{N}}\left(|\cdot-x|\right)(x)\leq\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2}\sqrt{x}}{\sqrt{N}}+\frac{1}{N},\ \ \forall\, N\in\mathbb{N}-\{1\}. \tag{34}$$

Let $m>1$, notice that $|\cdot-x|^{m-1}\leq1$, we get

$$|\cdot-x|^{m}=|\cdot-x|\,|\cdot-x|^{m-1}\leq|\cdot-x|,$$

hence

$$D_{N,\Gamma_{N}}\left(|\cdot-x|^{m}\right)(x)\leq D_{N,\Gamma_{N}}\left(|\cdot-x|\right)(x),$$

that is

$$D_{N,\Gamma_{N}}\left(|\cdot-x|^{m}\right)(x)\leq\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2}\sqrt{x}}{\sqrt{N}}+\frac{1}{N}, \tag{35}$$

$\forall\, N\in\mathbb{N}-\{1\},\forall\, x\in[0,1],m\geq1$.

We have

*Remark.*For $x \in [0,1]$, we have $\max(x(1-x)) = \frac{1}{4}$, at $x = \frac{1}{2}$. Thus,

$$\frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N} \le \frac{1}{2\sqrt{N}} + \frac{1}{N}, \tag{36}$$

$\forall\, x \in [0,1],\ \forall\, N \in \mathbb{N}$.

We also have

$$\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N} \le \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}, \tag{37}$$

$\forall\, x \in [0,1],\ \forall\, N \in \mathbb{N} - \{1\}$.

Corollary 2.*(to Theorem 4) We have*

$$\left\|K_{N,\mu}(f_1) - f_1\right\|_\infty \le 2\omega_1\left(f_1, \frac{1}{2\sqrt{N}} + \frac{1}{N}\right), \tag{38}$$

$\forall\, N \in \mathbb{N},\ f \in C_+([0,1])$.

Corollary 3.*(to Theorem 5) We have*

$$\|D_{N,\Gamma_N}(f_1) - f_1\|_\infty \le 2\omega_1\left(f_1, \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right), \tag{39}$$

$\forall\, N \in \mathbb{N} - \{1\},\ f_1 \in C_+([0,1])$.

The Bernstein-Kantorovich-Choquet operators $K_{N,\mu}$, where $\mu := \sqrt{M}$, with M the Lebesgue measure on $[0,1]$ can be written as:

$$K_{N,\mu}(f_1)(x) = \sum_{k=0}^{N} p_{N,k}(x)\, \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f_1(t)\, d\mu(t)}{\mu\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \tag{40}$$

$\forall\, x \in [0,1],\ \forall\, N \in \mathbb{N},\ f_1 \in C_+([0,1])$.

5 Background - IV

The following definitions are given.

Definition 10.*Let Ω be a set, and let $f_1, g_1 : \Omega \to \mathbb{R}$ be bounded functions. f_1 and g_1 are comonotonic, if for every $\omega, \omega' \in \Omega$,*

$$\left(f_1(\omega) - f_1\left(\omega'\right)\right)\left(g_1(\omega) - g_1\left(\omega'\right)\right) \ge 0. \tag{41}$$

Theorem 6.*[1] Let $\mathscr{L}_\infty(\mathscr{A})$ be the vector space of $\mathscr{A}$-measurable bounded real valued functions on Ω, where $\mathscr{A} \subset 2^\Omega$ is a σ-algebra. For $\Gamma : \mathscr{L}_\infty(\mathscr{A}) \to \mathbb{R}$, suppose that for $f_1, g_1 \in \mathscr{L}_\infty(\mathscr{A})$:*

(i) $\Gamma(cf) = c\Gamma(f_1)$, $\forall\, c > 0$,

(ii) $f_1 \le g$, implies $\Gamma(f_1) \le \Gamma(g_1)$,

and

(iii) $\Gamma(f_1 + g_1) = \Gamma(f_1) + \Gamma(g_1)$, for any comonotonic f_1, g_1.

As a result $\gamma(A) := \Gamma(1_A)$, $\forall\, A \in \mathscr{A}$, defines a finite monotone set function on $\mathscr{A}$, and Γ is the Choquet integral with respect to γ, i.e.

$$\Gamma(f_1) = (C)\int_\Omega f_1(t)\, d\gamma(t),\ \ \forall\, f_1 \in \mathscr{L}_\infty(\mathscr{A}). \tag{42}$$

Here 1_A means the characteristic function on A.

We give

*Remark.*We assume here that $[a,b]\subset\mathbb{R}$, $\mathscr{B}=\mathscr{B}([a,b])$ is the Borel σ-algebra on $[a,b]$, and $\mathscr{L}_\infty(\mathscr{B})$ is the vector space of $\mathscr{B}$-measurable bounded real valued functions on $[a,b]$. Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $\mathscr{L}_\infty(\mathscr{B})$ into $C_+([a,b])$, and $x\in[a,b]$. That is here L_N fulfills the positive homogenuity, monotonicity and subadditivity properties, see (10)-(12).

Assume $L_N(1)=1$, $\forall\, N\in\mathbb{N}$. Clearly here $\mathscr{L}_\infty(\mathscr{B})\supset C_+([a,b])$. We treat $L_N|_{C_+([a,b])}$, just denoted as L_N, $\forall\, N\in\mathbb{N}$.

$L_N(\cdot)(x):\mathscr{L}_\infty(\mathscr{B})\to\mathbb{R}$ is a functional, $\forall\, N\in\mathbb{N}$.The properties are given below:

(i)

$$L_N(cf)(x)=cL_N(f_1)(x),\ \forall\, c>0,\ \forall\ f_1\in\mathscr{L}_\infty(\mathscr{B}),\tag{43}$$

(ii)

$$f_1\le g,\text{ implies } L_N(f_1)(x)\le L_N(g_1)(x),\ \text{ where } f_1,g_1\in\mathscr{L}_\infty(\mathscr{B}),\tag{44}$$

and

(iii)

$$L_N(f+g)(x)\le L_N(f)(x)+L_N(g)(x),\ \ \forall\, f,g\in\mathscr{L}_\infty(\mathscr{B}).\tag{45}$$

For comonotonic $f,g\in\mathscr{L}_\infty(\mathscr{B})$, we suppose that

$$L_N(f+g)(x)=L_N(f)(x)+L_N(g)(x).\tag{46}$$

Thus L_N is called comonotonic.

With the help of Theorem 6 we conclude:

$$\gamma_{N,x}(A):=L_N(1_A)(x),\ \forall\, A\in\mathscr{B},\forall\, N\in\mathbb{N},\tag{47}$$

defines a finite monotone set function on $\mathscr{B}$, and

$$L_N(f)(x)=(C)\int_a^b f_1(t)\,d\gamma_{N,x}(t),\tag{48}$$

$\forall\ f_1\in\mathscr{L}_\infty(\mathscr{B})$, $\forall\, N\in\mathbb{N}$.

We conclude that (47) is true for any $f_1\in C_+([a,b])$. Furthermore $\gamma_{N,x}$ is normalized, namely $\gamma_{N,x}([a,b])=1$, $\forall\, N\in\mathbb{N}$.

Thus, we have:

*Remark.*Assume that $[a,b]\subset\mathbb{R}$, $\mathscr{B}=\mathscr{B}([a,b])$ is the Borel σ-algebra on $[a,b]$. For each $N\in\mathbb{N}$ and each $x\in[a,b]$ we have the monotone set functions $\mu_{N,x};\mathscr{B}\to\mathbb{R}_+$. We suppose that all $\mu_{N,x}$ are normalized, namely $\mu_{N,x}([a,b])=1$, and submodular. Below we discuss the operators $T_N:C_+([a,b])\to C_+([a,b])$ written as

$$T_N(f_1)(x)=(C)\int_{[a,b]} f_1(t)\,d\mu_{N,x}(t),\tag{49}$$

$\forall\, N\in\mathbb{N}$, $\forall\, x\in[a,b]$.

I $\mu_{N,x}$ are chosen in such a way that $T_N(C_+([a,b]))\subseteq C_+([a,b])$.

We conclude that

(i)

$$T_N(\alpha f)(x)=\alpha T_N(f_1)(x),\ \forall\ \alpha\ge 0,\tag{50}$$

(ii)

$$f_1\le g,\text{ implies } T_N(f_1)(x)\le T_N(g_1)(x),\tag{51}$$

and

(iii)

$$T_N(f_1+g_1)(x)\le T_N(f_1)(x)+T_N(g_1)(x),\tag{52}$$

$\forall\, N\in\mathbb{N}$, $\forall\, x\in[a,b]$, $\forall\ f_1,g_1\in C_+([a,b])$.

T_N are positive sublinear operators (10)-(12). Besides $T_N(1)=1$, $\forall\, N\in\mathbb{N}$.

6 Main Results

We give

Theorem 7. *Let $\alpha > 1$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $x_0 \in [0,1]$, $f \in AC^m([0,1], \mathbb{R}_+)$, and $f^{(m)} \in L_\infty([0,1])$. Furthermore we suppose that $f^{(k)}(x_0) = 0$, $k = 1, \ldots, m-1$. Then*

$$\left|K_{N,\mu}(f)(x_0) - f(x_0)\right| \leq \frac{\omega_1\left(D_{x_0}^{\alpha} f, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \quad \forall\, N \in \mathbb{N}. \tag{53}$$

If $N \to \infty$, we obtain that $K_{N,\mu}(f)(x_0) \to f(x_0)$.

Proof. From (13) we conclude ($\delta > 0$)

$$\left|K_{N,\mu}(f)(x_0) - f(x_0)\right| \leq \frac{\omega_1\left(D_{x_0}^{\alpha} f, \delta\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[K_{N,\mu}\left(|\cdot - x_0|^{\alpha}\right)(x_0) + \frac{K_{N,\mu}\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)}{(\alpha+1)\delta}\right] \overset{(30)}{\leq}$$

$$\frac{\omega_1\left(D_{x_0}^{\alpha} f, \delta\right)}{\Gamma(\alpha+1)}\left[\left(\frac{\sqrt{x_0(1-x_0)}}{\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)\delta}\left(\frac{\sqrt{x_0(1-x_0)}}{\sqrt{N}} + \frac{1}{N}\right)\right] \overset{(36)}{\leq} \tag{54}$$

$$\frac{\omega_1\left(D_{x_0}^{\alpha} f, \delta\right)}{\Gamma(\alpha+1)}\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)\delta}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)\right]$$

(setting $\delta := \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}$, then $\delta^{\alpha+1} = \frac{1}{2\sqrt{N}} + \frac{1}{N}$, and $\delta^{\alpha} = \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}$)

$$= \frac{\omega_1\left(D_{x_0}^{\alpha} f, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)}\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \tag{55}$$

proving the claim.

We continue with

Theorem 8. *Let $\nu > 1$, $\longleftarrow \nu \notin \mathbb{N}$, $p = [\nu]$, $x_0 \in [0,1]$ and $f : [0,1] \to \mathbb{R}_+$, $f \in C_{x_0+}^{\nu}([0,1]) \cap C_{x_0-}^{\nu}([0,1])$. Assume that $f^{(k)}(x_0) = 0$, $k = 1, \ldots, p-1$, and $\left(D_{x_0+}^{\nu} f\right)(x_0) = \left(D_{x_0-}^{\nu} f\right)(x_0) = 0$. Then*

$$\left|K_{N,\mu}(f)(x_0) - f(x_0)\right| \leq \frac{\omega_1\left(\overline{D}_{x_0}^{\nu} f, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\nu+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\nu}{\nu+1}}\right], \quad \forall\, N \in \mathbb{N}. \tag{56}$$

If $N \to \infty$, we get that $K_{N,\mu}(f)(x_0) \to f(x_0)$.

Proof. By (26) we get ($\delta > 0$)

$$\left|K_{N,\mu}(f)(x_0) - f(x_0)\right| \leq \frac{\omega_1\left(\overline{D}_{x_0}^{\nu} f, \delta\right)}{\Gamma(\nu+1)} \cdot \tag{57}$$

$$\left[K_{N,\mu}\left(|\cdot - x_0|^{\nu}\right)(x_0) + \frac{K_{N,\mu}\left(|\cdot - x_0|^{\nu+1}\right)(x_0)}{(\nu+1)\delta}\right] \overset{\text{(by (30), (36))}}{\leq}$$

$$\frac{\omega_1\left(\overline{D}_{x_0}^{\nu} f, \delta\right)}{\Gamma(\nu+1)}\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\nu+1)\delta}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)\right]$$

(setting $\delta := \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\nu+1}}$, then $\delta^{\nu+1} = \frac{1}{2\sqrt{N}} + \frac{1}{N}$, and $\delta^{\nu} = \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\nu}{\nu+1}}$)

$$= \frac{\omega_1\left(\overline{D}_{x_0}^{\nu} f, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)}\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\nu+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\nu}{\nu+1}}\right], \tag{58}$$

proving the claim.

We present

Theorem 9. *Let $\alpha > 1$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $x_0 \in [0,1]$, $f \in AC^m([0,1], \mathbb{R}_+)$, and $f^{(m)} \in L_\infty([0,1]) : f^{(k)}(x_0) = 0$, $k = 1, \ldots, m-1$. Then*

$$|D_{N,\Gamma_N}(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{\alpha} f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \quad \forall\, N \in \mathbb{N} - \{1\}. \tag{59}$$

If $N \to \infty$, we get that $D_{N,\Gamma_N}(f)(x_0) \to f(x_0)$.

Proof. By (13) we obtain ($\delta > 0$)

$$|D_{N,\Gamma_N}(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{\alpha} f, \delta\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[D_{N,\Gamma_N}\left(|\cdot - x_0|^{\alpha}\right)(x_0) + \frac{D_{N,\Gamma_N}\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)}{(\alpha+1)\delta}\right] \overset{\text{(by (35), (37))}}{\leq}$$

$$\frac{\omega_1\left(D_{x_0}^{\alpha} f, \delta\right)}{\Gamma(\alpha+1)}\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)\delta}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)\right]$$

(setting $\delta := \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}$)

$$= \frac{\omega_1\left(D_{x_0}^{\alpha} f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \tag{60}$$

proving the claim.

We also present

Theorem 10. *Let $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x_0 \in [0,1]$ and $f : [0,1] \to \mathbb{R}_+$, $f \in C^{\nu}_{x_0+}([0,1]) \cap C^{\nu}_{x_0-}([0,1])$. Suppose that $f^{(k)}(x_0) = 0$, $k = 1,...,p-1$, and $\left(D^{\nu}_{x_0+} f\right)(x_0) = \left(D^{\nu}_{x_0-} f\right)(x_0) = 0$. Then*

$$|D_{N,\Gamma_N}(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(\overline{D}^{\nu}_{x_0} f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\nu+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\nu}{\nu+1}}\right], \ \forall N \in \mathbb{N} - \{1\}. \tag{61}$$

If $N \to \infty$, we get that $D_{N,\Gamma_N}(f)(x_0) \to f(x_0)$.

Proof. By (26) we get ($\delta > 0$)

$$|D_{N,\Gamma_N}(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(\overline{D}^{\nu}_{x_0} f, \delta\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[D_{N,\Gamma_N}\left(|\cdot - x_0|^{\nu}\right)(x_0) + \frac{1}{(\nu+1)\delta} D_{N,\Gamma_N}\left(|\cdot - x_0|^{\nu+1}\right)(x_0)\right] \overset{\text{(by (35), (37))}}{\leq}$$

$$\frac{\omega_1\left(\overline{D}^{\nu}_{x_0} f, \delta\right)}{\Gamma(\nu+1)}\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\nu+1)\delta}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)\right]$$

(setting $\delta := \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\nu+1}}$)

$$= \frac{\omega_1\left(\overline{D}^{\nu}_{x_0} f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\nu+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\nu}{\nu+1}}\right], \tag{62}$$

proving the claim.

Based on Theorem 6 and Remark 5 we present

Theorem 11. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC^m([a,b], \mathbb{R}_+)$, and $f^{(m)} \in L_\infty([a,b])$. Furthermore we suppose that $f^{(k)}(x_0) = 0$, $k = 1,...,m-1$. Let $L_N : \mathscr{L}_\infty(\mathscr{B}([a,b])) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, be positive sublinear comonotonic operators, obeying $L_N(1) = 1$, $\forall N \in \mathbb{N}$. Then*

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D^{\alpha}_{x_0} f, \delta\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[\left((C)\int_a^b |t - x_0|^{\alpha} d\gamma_{N,x_0}(t)\right) + \frac{1}{(\alpha+1)\delta}\left((C)\int_a^b |t - x_0|^{\alpha+1} d\gamma_{N,x_0}(t)\right)\right], \tag{63}$$

$\delta > 0$, $\forall N \in \mathbb{N}$.

Proof. By Theorem 1.

Theorem 12. *Let $0 < \alpha < 1$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC([a,b], \mathbb{R}_+)$, and $f' \in L_\infty([a,b])$. Let $L_N : \mathscr{L}_\infty(\mathscr{B}([a,b])) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, be positive sublinear comonotonic operators, fulfilling $L_N(1) = 1$, $\forall N \in \mathbb{N}$. Suppose that $(C)\int_a^b |t - x_0|^{\alpha+1} d\gamma_{N,x_0}(t) > 0$, $\forall N \in \mathbb{N}$. Then*

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{(\alpha+2)\,\omega_1\left(D^{\alpha}_{x_0} f, \left((C)\int_a^b |t - x_0|^{\alpha+1} d\gamma_{N,x_0}(t)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)}$$

$$\left((C)\int_a^b |t-x_0|^{\alpha+1} d\gamma_{N,x_0}(t)\right)^{\frac{\alpha}{\alpha+1}}, \tag{64}$$

$\forall\, N \in \mathbb{N}$.

If $(C)\int_a^b |t-x_0|^{\alpha+1} d\gamma_{N,x_0}(t) \to 0$, *then* $L_N(f)(x_0) \to f(x_0)$, *as* $N \to \infty$.

*Proof.*By Theorem 2.

Theorem 13.*Let* $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x_0 \in [a,b]$ *and* $f : [a,b] \to \mathbb{R}_+$, $f \in C^{\nu}_{x_0+}([a,b]) \cap C^{\nu}_{x_0-}([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1,\ldots,p-1$, *and* $\left(D^{\nu}_{x_0+} f\right)(x_0) = \left(D^{\nu}_{x_0-} f\right)(x_0) = 0$. *Let* $L_N : \mathscr{L}_\infty(\mathscr{B}([a,b])) \to C_+([a,b])$, $\forall\, N \in \mathbb{N}$, *be positive sublinear comonotonic operators, fulfilling* $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. *Then*

$$|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(\overline{D}^{\nu}_{x_0} f, \delta\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[(C)\int_a^b |t-x_0|^{\nu} d\gamma_{N,x_0}(t) + \frac{1}{(\nu+1)\delta}\left((C)\int_a^b |t-x_0|^{\nu+1} d\gamma_{N,x_0}(t)\right)\right], \tag{65}$$

$\forall\, N \in \mathbb{N}$.

*Proof.*By Theorem 3.

We finish by giving related results based on Remark 5.

We give

Theorem 14.*Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC^m([a,b],\mathbb{R}_+)$, *and* $f^{(m)} \in L_\infty([a,b])$. *Furthermore we suppose that* $f^{(k)}(x_0) = 0$, $k = 1,\ldots,m-1$. *Then*

$$|T_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D^{\alpha}_{x_0} f, \delta\right)}{\Gamma(\alpha+1)} \cdot$$

$$\left[\left((C)\int_a^b |t-x_0|^{\alpha} d\mu_{N,x_0}(t)\right) + \frac{1}{(\alpha+1)\delta}\left((C)\int_a^b |t-x_0|^{\alpha+1} d\mu_{N,x_0}(t)\right)\right], \tag{66}$$

$\delta > 0$, $\forall\, N \in \mathbb{N}$.

*Proof.*By Theorem 1.

Theorem 15.*Let* $0 < \alpha < 1$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC([a,b],\mathbb{R}_+)$, *and* $f' \in L_\infty([a,b])$. *Assume that* $(C)\int_a^b |t-x_0|^{\alpha+1} d\mu_{N,x_0}(t) > 0$, $\forall\, N \in \mathbb{N}$. *Then*

$$|T_N(f)(x_0) - f(x_0)| \leq \frac{(\alpha+2)\,\omega_1\left(D^{\alpha}_{x_0} f, \left((C)\int_a^b |t-x_0|^{\alpha+1} d\mu_{N,x_0}(t)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)}$$

$$\left((C)\int_a^b |t-x_0|^{\alpha+1} d\mu_{N,x_0}(t)\right)^{\frac{\alpha}{\alpha+1}}, \tag{67}$$

$\forall\, N \in \mathbb{N}$.

If $(C)\int_a^b |t-x_0|^{\alpha+1} d\mu_{N,x_0}(t) \to 0$, *then* $T_N(f)(x_0) \to f(x_0)$, *as* $N \to \infty$.

*Proof.*By Theorem 2.

Theorem 16.*Let* $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x_0 \in [a,b]$ *and* $f : [a,b] \to \mathbb{R}_+$, $f \in C^{\nu}_{x_0+}([a,b]) \cap C^{\nu}_{x_0-}([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1,\ldots,p-1$, *and* $\left(D^{\nu}_{x_0+} f\right)(x_0) = \left(D^{\nu}_{x_0-} f\right)(x_0) = 0$. *Then*

$$|T_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(\overline{D}^{\nu}_{x_0} f, \delta\right)}{\Gamma(\nu+1)} \cdot$$

$$\left[(C)\int_a^b |t-x_0|^{\nu} d\mu_{N,x_0}(t) + \frac{1}{(\nu+1)\delta}\left((C)\int_a^b |t-x_0|^{\nu+1} d\mu_{N,x_0}(t)\right)\right], \tag{68}$$

$\delta > 0$, $\forall\, N \in \mathbb{N}$.

*Proof.*By Theorem 3.

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