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Transient Analysis of a Non-Preemptive Priority Queueing System

V. S. S. Yadavalli[1,*], *Diatha Krishna Sundar*[2], *Swaminathan Udayabaskaran*[3] *and C. T. Dora Pravina*[3]

[1] Department of Industrial & Systems Engineering, University of Pretoria, South Africa
[2] Indian Institute of Management, Bangalore 560076, India
[3] Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, Tamilnadu 600062, India

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Abstract: A single server queueing system is considered in which two types of customers arrive according to independent Poisson processes. Customers of type 1 are of priority nature and the other customers of type 2 are of non-priority. Type 1 customers have non-preemptive priority over type 2 customers. Assuming that service times for both types of customers have exponential distribution with mean $1/\mu$, we obtain explicit expressions for the transient solution for the state probability distribution. We deduce the steady-state joint distribution of the number of customers of type 1 and customers of type 2 and also obtain performance measures of the system.

Keywords: Single server queue, non-preemptive priority, transient solution, performance measure

1 Introduction

Queueing systems with priority customers have been studied very extensively in the past. Priority queueing systems (queueing systems dealing with priority customers) arise in variety of phenomena such as telephone switching systems, computer systems, communication systems and health-care systems. McMillan [1] mentions the use of priority queueing systems in cellular mobile networks. Choi and Chang [2] provide several examples such as telephone in the restaurant, subscriber line modules of telephone exchanges, communication protocols and channel allocation scheme in wireless networks. The monograph of Ng and Soong [3] brings out various applications of priority queues in communication networks. The monograph of Jaiswal [4] provides an excellent unified account of queueing systems under priority disciplines. Cobham [5] introduced $M/G/1$ queueing system with priority assignments in waiting line problems and obtained expression for the average elapsed time between the arrival in the line of a unit of a given priority and its admission to the facility for servicing. White and Christie [6] defined preemptive priority and studied queueing systems with preemptive priorities or with breakdown. Miller [7] analysed the stationary behaviour of a priority queue in which different type of customers arrive at a service facility and each customer has a relative priority for order of service. Jaiswal [8] obtained the queue length probability generating function for a preemptive resume priority queue with Poisson arrivals and general service time distributions by using supplementary variable method. Jaiswal [9] obtained time-dependent solution of the head-of-the-line priority queue characterized by Poisson arrivals and general service-time distributions. Yeo [10] studied preemptive priority queues with K classes of customers with a preemptive repeat and a preemptive resume policy. Takacs [11] studied priority queues in which service is rendered with privileged interruptions or without interruption. Chang [12] studied a single server queueing system with non-preemptive and preemptive-resume priorities. Vasicek [13] studied preemptive priority queues with single server in which the preempted items do not return to service and are lost. Madan [14] studied a queueing system in which a mechanical service channel serves the units one by one on a priority basis and is subject to occasional breakdowns. Miller [15] obtained steady-state distributions of exponential single server (preemptive and nonpreemptive) priority queues with two classes of customers by using

* Corresponding author e-mail: sarma.yadavalli@up.ac.za

Neut's theory of matrix-geometric invariant probability vectors. Choi et al. [16] studied a $M/M/1$ queue with impatient customers of higher priority. Brandt and Brandt [17] studied a two-class $M/M/1$ queueing system under preemptive resume and impatience of the prioritized customers. Drekic and Woolford [18] studied a single-server preemptive priority queue with balking. Iravani and Balcioglu [19] studied three different priority queues with impatient customers. Using matrix-geometric method, Krishnamoorthy and Manjunath [20] studied priority queues generated through customer induced service interruption. Aibatov [21] considered a single server queueing system with preemptive resume service discipline and unreliable server and obtained the limit distribution of the number of customers in the system. The above survey indicates that transient analysis of priority queueing systems has not been studied in many cases. In this paper, we provide a new approach for obtaining the transient analysis of a non-preemptive priority queueing system and also deduce the steady-state solution for the queueing system.

The paper is organised as follows: In Section 2, we describe the model. Section 3 provides the governing equations of the model. The transient solution is obtained in Section 4. Section 5 contains the steady-state distribution. Performance measures are obtained in Section 6. A numerical illustration is provided in Section 7. Section 8 provides a conclusion.

2 2. Model description

Customers are of two types, priority and non-priority customers. These customers arrive independently to a service station, priority customers arrive according to a Poisson process with rate λ_1 and non-priority customers arrive according to a Poisson process with rate λ_2. There is a single server and the service time of each customer is exponentially distributed with mean $\frac{1}{\mu}$. All arriving customers queue-up when the server is busy. We assume that the buffer capacity is infinite. Priority customers have non-preemptive priority over non-priority customers in service. Let $S(t)$ be the state of the server at time t. We define

$$S(t) = \begin{cases} 0 \text{ if the sever is idle;} \\ 1 \text{ if the server is busy with a priority customer;} \\ 2 \text{ if the server is busy with a non-priority customer.} \end{cases}$$

Let $X_1(t)$ denote the number of priority customers in the system at time t and $X_2(t)$ denote the number of non-priority customers in the system at time t. Then the three dimensional stochastic process $\{(X_1(t), X_2(t), S(t)), t \geq 0\}$ is a Markov process. The state space of the process is

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5,$$

where

$$\Omega_1 = \{(0,0,0)\}, \Omega_2 = \{(i,0,1)|i = 1,2,\cdots\},$$

$$\Omega_3 = \{(0,j,2)|j = 1,2,\cdots\},$$

$$\Omega_4 = \{(i,j,1)|i,j = 1,2,\cdots\}, \Omega_5 = \{(i,j,2)|i,j = 1,2,\cdots\}.$$

3 Governing Equations

We assume that $X_1(0) = 0, X_2(0) = 0, S(0) = 0$. We define the joint probability distribution of the queueing system by

$$P(i,j,k,t) = Pr\{(X_1(t), X_2(t), S(t)) = (i,j,k)\},$$

where $(i,j,k) \in \Omega$. Using probability laws, we obtain the following cases:

Case 1: $X_1(t) = 0, X_2(t) = 0, S(t) = 0$

Since the system started at the state $(0,0,0)$ at time $t = 0$, we have the following three mutually exclusive and exhaustive events:

(i) No customer arrived up to time t;

(ii) The system occupied the state $(1,0,1)$ at some time u before t, the server completed the service in $(u, u+du)$ and no customer arrived in (u,t);

(iii) The system occupied the state $(0,1,2)$ at some time u before t, the server completed the service in $(u, u+du)$ and no customer arrived in (u,t).

Consequently, we obtain the renewal type integral equation

$$P(0,0,0,t) = e^{-(\lambda_1+\lambda_2)t}$$

$$+\mu \int_0^t [P(1,0,1,u) + P(0,1,2,u)] e^{-(\lambda_1+\lambda_2)(t-u)} du.$$

Using the notation $f(t)\copyright g(t) = \int_0^t f(u)g(t-u)du$ for the convolution of two functions $f(t)$ and $g(t)$, the above equation (1) can be written as

$$P(0,0,0,t) = e^{-(\lambda_1+\lambda_2)t} + \mu[P(1,0,1,t) + P(0,1,2,t)] \copyright e^{-(\lambda_1+\lambda_2)t}. \tag{3.1}$$

Case 2: $X_1(t) = 1, X_2(t) = 0, S(t) = 1$

To be in the state $(1,0,1)$ at time t, there are three mutually exclusive and exhaustive events:

(i) $(0,0,0)$ at some time u before t, a priority customer arrived in $(u, u+du)$ and no arrival and no service took place in (u,t);

(ii) $(2,0,1)$ at some time u before t, the service of a priority customer was completed in $(u, u+du)$ and no arrival and no service took place in (u,t);

(iii) $(1,1,2)$ at some time u before t, the service of the non-priority customer was completed in $(u, u+du)$ and no arrival and no service took place in (u,t).

Consequently, we have

$$P(1,0,1,t) = [P(0,0,0,t)\lambda_1 + P(2,0,1,t)\mu + P(1,1,2,t)\mu] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \tag{3.2}$$

In the same way, we obtain the other cases.

Case 3: $X_1(t) = i, X_2(t) = 0, S(t) = 1, i \geq 2$

$$P(i,0,1,t) = [P(i-1,0,1,t)\lambda_1 + P(i+1,0,1,t)\mu$$

$$+P(i,1,2,t)\mu] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \tag{3.3}$$

Case 4: $X_1(t) = 1, X_2(t) = j, S(t) = 1, j \geq 1$

$$P(1,j,1,t) = [P(1,j-1,1,t)\lambda_2 + P(2,j,1,t)\mu + P(1,j+1,2,t)\mu] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \quad (3.4)$$

Case 5: $X_1(t) = i, X_2(t) = j, S(t) = 1, i \geq 2, j \geq 1$

$$P(i,j,1,t) = [P(i,j-1,1,t)\lambda_2 + P(i-1,j,1,t)\lambda_1 + P(i+1,j,1,t)\mu + P(i,j+1,2,t)\mu] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \quad (3.5)$$

Case 6: $X_1(t) = 0, X_2(t) = 1, S(t) = 2$

$$P(0,1,2,t) = [P(0,2,2,t)\mu + P(1,1,1,t)\mu + P(0,0,0,t)\lambda_2] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \quad (3.6)$$

Case 7: $X_1(t) = 0, X_2(t) = j, S(t) = 2, j \geq 2$

$$P(0,j,2,t) = [P(0,j+1,2,t)\mu + P(0,j-1,2,t)\lambda_2 + P(1,j,1,t)\mu] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \quad (3.7)$$

Case 8: $X_1(t) = i, X_2(t) = 1, S(t) = 2, i \geq 1$

$$P(i,1,2,t) = P(i-1,1,2,t)\lambda_1 \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \quad (3.8)$$

Case 9: $X_1(t) = i, X_2(t) = j, S(t) = 2, i \geq 1, j \geq 2$

$$P(i,j,2,t) = [P(i,j-1,2,t)\lambda_2 + P(i-1,j,2,t)\lambda_1] \copyright e^{-(\lambda_1+\lambda_2+\mu)t}. \quad (3.9)$$

4 Transient solution

We denote the Laplace transform of $P(i,j,k,t)$ by

$$P^*(i,j,k,\theta) = L[P(i,j,k,t)] = \int_0^\infty e^{-\theta t} P(i,j,k,t)dt.$$

Taking Laplace transform, equations (3.1)-(3.9) yield

$$(\theta+\lambda_1+\lambda_2)P^*(0,0,0,\theta) = 1 + \mu[P^*(1,0,1,\theta) + P^*(0,1,2,\theta)], \quad (4.1)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(1,0,1,\theta) = P^*(0,0,0,\theta)\lambda_1 + P^*(2,0,1,\theta)\mu + P^*(1,1,2,\theta)\mu, \quad (4.2)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(i,0,1,\theta) = P^*(i-1,0,1,\theta)\lambda_1 + P^*(i+1,0,1,\theta)\mu + P^*(i,1,2,\theta)\mu, i \geq 2, \quad (4.3)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(1,j,1,\theta) = P^*(1,j-1,1,\theta)\lambda_2 + P^*(2,j,1,\theta)\mu + P^*(1,j+1,2,\theta)\mu, j \geq 1, \quad (4.4)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(i,j,1,\theta) = P^*(i,j-1,1,\theta)\lambda_2 + P^*(i-1,j,1,\theta)\lambda_1$$
$$+P^*(i+1,j,1,\theta)\mu + P^*(i,j+1,2,\theta)\mu, i \geq 2, j \geq 1, \quad (4.5)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(0,1,2,\theta) = P^*(0,2,2,\theta)\mu + P^*(1,1,1,\theta)\mu + P^*(0,0,0,\theta)\lambda_2, \quad (4.6)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(0,j,2,\theta) = P^*(0,j+1,2,\theta)\mu + P^*(0,j-1,2,\theta)\lambda_2 + P^*(1,j,1,\theta)\mu, j \geq 2, \quad (4.7)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(i,1,2,\theta) = P^*(i-1,1,2,\theta)\lambda_1, i \geq 1, \quad (4.8)$$

$$(\theta+\lambda_1+\lambda_2+\mu)P^*(i,j,2,\theta) = P^*(i,j-1,2,\theta)\lambda_2 + P^*(i-1,j,2,\theta)\lambda_1, i \geq 1, j \geq 2. \quad (4.9)$$

We define

$$G^{(1)*}(u,v,\theta) = \sum_{i=1}^\infty \sum_{j=0}^\infty P^*(i,j,1,\theta)u^i v^j,$$

$$G^{(2)*}(u,v,\theta) = \sum_{i=0}^\infty \sum_{j=1}^\infty P^*(i,j,2,\theta)u^i v^j.$$

$$G^*(u,v,\theta) = G^{(1)*}(u,v,\theta) + G^{(2)*}(u,v,\theta) + P^*(0,0,0,\theta).$$

Then, we have

$$G^{(1)*}(1,1,\theta) + G^{(2)*}(1,1,\theta) + P^*(0,0,0,\theta) = \frac{1}{\theta}. \quad (4.10)$$

We also define

$$G_i^{(1)*}(v,\theta) = \sum_{j=0}^\infty P^*(i,j,1,\theta)v^j, i = 1,2,\cdots; G_i^{(2)*}(v,\theta) = \sum_{j=1}^\infty P^*(i,j,2,\theta)v^j, i = 0,1,2,\cdots.$$

Then, we get

$$G^{(1)*}(u,v,\theta) = \sum_{i=1}^\infty G_i^{(1)*}(v,\theta)u^i, G^{(2)*}(u,v,\theta) = \sum_{i=0}^\infty G_i^{(2)*}(v,\theta)u^i.$$

By using (4.1), (4.6) and (4.7), we get

$$\left[\theta+\lambda_1+\lambda_2(1-v)+\mu\left(1-\frac{1}{v}\right)\right] G_0^{(2)*}(v,\theta) =$$

$$\mu G_1^{(1)*}(v,\theta) - [\theta+\lambda_1+\lambda_2(1-v)] P^*(0,0,0,\theta) + 1. \quad (4.11)$$

By using (4.8) and (4.9), we get

$$[\theta+\lambda_1+\lambda_2(1-v)+\mu]G_i^{(2)*}(v,\theta) = \lambda_1 G_{i-1}^{(2)*}(v,\theta), i \geq 1. \quad (4.12)$$

Setting $i = 1$ in (4.12), we obtain

$$[\theta+\lambda_1+\lambda_2(1-v)+\mu]G_1^{(2)*}(v,\theta) = \lambda_1 G_0^{(2)*}(v,\theta). \quad (4.13)$$

By using (4.2) and (4.4), we get

$$[\theta+\lambda_1+\lambda_2(1-v)+\mu]G_1^{(1)*}(v,\theta) = \lambda_1 P^*(0,0,0,\theta) + \mu G_2^{(1)*}(v,\theta) + \frac{\mu}{v} G_1^{(2)*}(v,\theta). \quad (4.14)$$

By using (4.3) and (4.5), we get

$$[\theta+\lambda_1+\lambda_2(1-v)+\mu]G_i^{(1)*}(v,\theta) = \lambda_1 G_{i-1}^{(1)*}(v,\theta) + \mu G_{i+1}^{(1)*}(v,\theta) + \frac{\mu}{v} G_i^{(2)*}(v,\theta), i \geq 2. \quad (4.15)$$

By using (4.12), we get

$$[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu]G^{(2)*}(u,v,\theta) = [\theta+\lambda_1+\lambda_2(1-v)+\mu]G_0^{(2)*}(v,\theta). \quad (4.16)$$

From (4.16), we obtain

$$G^{(2)*}(u,v,\theta) = \frac{[\theta+\lambda_1+\lambda_2(1-v)+\mu]G_0^{(2)*}(v,\theta)}{[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu]}. \quad (4.17)$$

By using (4.14) and (4.15), we get

$$\left[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu\left(1-\frac{1}{u}\right)\right] G^{(1)*}(u,v,\theta) = \lambda_1 u P^*(0,0,0,\theta)$$
$$+\frac{\mu}{v} G^{(2)*}(u,v,\theta) - \left[\mu G_1^{(1)*}(v,\theta) + \frac{\mu}{v} G_0^{(2)*}(v,\theta)\right]. \quad (4.18)$$

From (4.11), we get

$$\frac{\mu}{v} G_0^{(2)*}(v,\theta) + \mu G_1^{(1)*}(v,\theta) = [\theta+\lambda_1+\lambda_2(1-v)+\mu] G_0^{(2)*}(v,\theta)$$
$$+ [\theta+\lambda_1+\lambda_2(1-v)] P^*(0,0,0,\theta) - 1. \quad (4.19)$$

Substituting (4.19) into (4.18), we get

$$\left[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu\left(1-\frac{1}{u}\right)\right] G^{(1)*}(u,v,\theta) = \lambda_1 u P^*(0,0,0,\theta) + \frac{\mu}{v} G^{(2)*}(u,v,\theta)$$
$$- [\theta+\lambda_1+\lambda_2(1-v)+\mu] G_0^{(2)*}(v,\theta) - [\theta+\lambda_1+\lambda_2(1-v)] P^*(0,0,0,\theta) + 1. \quad (4.20)$$

Substituting (4.17) into (4.20), we get

$$\left[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu\left(1-\frac{1}{u}\right)\right]G^{(1)*}(u,v,\theta)=$$

$$\frac{\mu}{v}\,\frac{[\theta+\lambda_1+\lambda_2(1-v)+\mu]G_0^{(2)*}(v,\theta)}{[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu]}$$

$$-\left[\theta+\lambda_1+\lambda_2(1-v)+\mu\right]G_0^{(2)*}(v,\theta)-\left[\theta+\lambda_1(1-u)+\lambda_2(1-v)\right]P^*(0,0,0,\theta)+1. \quad (4.21)$$

From (4.21), we get

$$G^{(1)*}(u,v,\theta)=\frac{NG^{(1)*}(u,v,\theta)}{DG^{(1)*}(u,v,\theta)},$$

where

$$NG^{(1)*}(u,v,\theta)=1-\{\theta+\lambda_1(1-u)+\lambda_2(1-v)\}P^*(0,0,0,\theta)$$

$$+\{\theta+\lambda_1+\lambda_2(1-v)+\mu\}\left\{\frac{\mu}{v}\,\frac{1}{[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu]}-1\right\}G_0^{(2)*}(v,\theta), \quad (4.22)$$

$$DG^{(1)*}(u,v,\theta)=\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu\left(1-\frac{1}{u}\right). \quad (4.23)$$

The function $G^{(1)*}(u,v,\theta)$ converges when Re $\theta>0$, $|u|<1$ and $|v|<1$. So $NG^{(1)*}(u,v,\theta)$ must vanish wherever $DG^{(1)*}(u,v,\theta)$ vanishes in the region $\mathfrak{D}$ defined by Re $\theta>0$, $|u|<1$ and $|v|<1$. As a quadratic in u, the denominator $DG^{(1)*}(u,v,\theta)$ of (4.21) has two roots

$$u_1(v)=\frac{\theta+\lambda_1+\lambda_2(1-v)+\mu-\sqrt{[\theta+\lambda_1+\lambda_2(1-v)+\mu]^2-4\lambda_1\mu}}{2\lambda_1}, \quad (4.24)$$

$$u_2(v)=\frac{\theta+\lambda_1+\lambda_2(1-v)+\mu+\sqrt{[\theta+\lambda_1+\lambda_2(1-v)+\mu]^2-4\lambda_1\mu}}{2\lambda_1}. \quad (4.25)$$

These roots satisfy the following conditions

$$|u_1(v)|<|u_2(v)|, \quad (4.26)$$

$$u_1(v)+u_2(v)=\frac{\theta+\lambda_1+\lambda_2(1-v)+\mu}{2\lambda_1}, \quad (4.27)$$

$$u_1(v)u_2(v)=\frac{\mu}{\lambda_1}, \quad (4.28)$$

$$\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu\left(1-\frac{1}{u}\right)=\frac{\lambda_1}{u}(u-u_1(v))(u_2(v)-u), \quad (4.29)$$

$$\theta+\lambda_1(1-u_1(v))+\lambda_2(1-v)+\mu=\frac{\mu}{u_1(v)}. \quad (4.30)$$

When $|u|=1, |v|<1$ and Re $\theta>0$, we have

$$|\lambda_1u^2+\mu|\le\lambda_1+\mu<|\theta+\lambda_1+\lambda_2(1-v)+\mu|$$

$$=|[\theta+\lambda_1+\lambda_2(1-v)+\mu]u|. \quad (4.31)$$

Consequently, by Rouche's theorem, $DG^{(1)*}(u,v,\theta)$ has only one zero in the region $\mathfrak{D}$. Since $|u_1(v)|<|u_2(v)|$, that zero is $u_1(v)$. Hence $NG^{(1)*}(u_1(v),v,\theta)=0$. This leads to the equation

$$1-\{\theta+\lambda_1(1-u_1(v))+\lambda_2(1-v)\}P^*(0,0,0,\theta)$$

$$+\{\theta+\lambda_1+\lambda_2(1-v)+\mu\}\left\{\frac{\mu}{v}\,\frac{1}{[\theta+\lambda_1(1-u_1(v))+\lambda_2(1-v)+\mu]}-1\right\}G_0^{(2)*}(v,\theta)=0. \quad (4.32)$$

Using (4.30) in (4.32) and simplifying, we get

$$G_0^{(2)*}(v,\theta)=\frac{v\left[\mu\left\{1-u_1(v)\right\}P^*(0,0,0,\theta)-u_1(v)\right]}{u_1(v)\{\theta+\lambda_1+\lambda_2(1-v)+\mu\}\left(u_1(v)-v\right)}. \quad (4.33)$$

The function $G_0^{(2)*}(v,\theta)$ converges when Re $\theta>0$ and $|v|<1$. As before, by invoking Rouche's theorem, we get

$$\mu\left\{1-u_1(v_1)\right\}P^*(0,0,0,\theta)-u_1(v_1)=0, \quad (4.34)$$

where v_1 is a root of the equation

$$u_1(v)-v=0 \quad (4.35)$$

such that $|v_1|<1$. Simplifying (4.35), we get

$$(\lambda_1+\lambda_2)v^2-(\theta+\lambda_1+\lambda_2+\mu)v+\mu=0. \quad (4.36)$$

Solving (4.36), we get

$$v_1=\frac{\theta+\lambda_1+\lambda_2+\mu-\sqrt{(\theta+\lambda_1+\lambda_2+\mu)^2-4(\lambda_1+\lambda_2)\mu}}{2(\lambda_1+\lambda_2)}. \quad (4.37)$$

Substituting $v=v_1$ in (4.24), we get

$$u_1(v_1)=v_1. \quad (4.38)$$

Equation (4.38) agrees with the fact that v_1 is a root of $u(v)=v$. From (4.34), we get

$$P^*(0,0,0,\theta)=\frac{v_1}{\mu\left(1-v_1\right)}. \quad (4.39)$$

Consequently, (4.33) becomes

$$G_0^{(2)*}(v,\theta)=\frac{\mu v}{\{\theta+\lambda_1+\lambda_2(1-v)+\mu\}u_1(v)v_1}\left(\frac{v_1-u_1(v)}{u_1(v)-v}\right)P^*(0,0,0,\theta). \quad (4.40)$$

From (4.24), we get

$$u_1(v)-v_1=\frac{2\lambda_2v_1(v-v_1)}{\{\theta+\lambda_1+\lambda_2+\mu-(2\lambda_1v_1+\lambda_2v)+\sqrt{[\theta+\lambda_1+\lambda_2(1-v)+\mu]^2-4\lambda_1\mu}\}}. \quad (4.41)$$

$$u_1(v)-v=\frac{2(v-v_1)\{(\lambda_1+\lambda_2)vv_1-\mu\}}{v_1\left\{\theta+\lambda_1+\lambda_2+\mu-(2\lambda_1+\lambda_2)v+\sqrt{[\theta+\lambda_1+\lambda_2(1-v)+\mu]^2-4\lambda_1\mu}\right\}}, \quad (4.42)$$

Consequently, we obtain

$$\frac{v_1-u_1(v)}{u_1(v)-v}=-\frac{u_1(v)-v_1}{u_1(v)-v}$$

$$=\frac{\lambda_2v_1^2\left\{\theta+\lambda_1+\lambda_2+\mu-(2\lambda_1+\lambda_2)v+\sqrt{[\theta+\lambda_1+\lambda_2(1-v)+\mu]^2-4\lambda_1\mu}\right\}}{\{\mu-(\lambda_1+\lambda_2)vv_1\}\{\theta+\lambda_1+\lambda_2+\mu-(2\lambda_1v_1+\lambda_2v)+\sqrt{[\theta+\lambda_1+\lambda_2(1-v)+\mu]^2-4\lambda_1\mu}\}}. \quad (4.43)$$

Substituting (4.43) in (4.40), we obtain

$$G_0^{(2)*}(v,\theta)=\frac{\lambda_2vv_1y\left(-2\lambda_1v+y\right)}{2\{\mu-(\lambda_1+\lambda_2)vv_1\}x\left(-2\lambda_1v_1+y\right)}P^*(0,0,0,\theta), \quad (4.44)$$

where

$$x=\theta+\lambda_1+\lambda_2(1-v)+\mu, \quad (4.45)$$

$$y=x+\sqrt{x^2-4\lambda_1\mu}. \quad (4.46)$$

From (4.39), we get

$$P^*(0,0,0,\theta)=\frac{1}{\mu}\sum_{n=0}^{\infty}v_1^{n+1}. \quad (4.47)$$

Taking inverse Laplace transform of (4.47), we get

$$P(0,0,0,t)=\frac{1}{\mu}\sum_{n=0}^{\infty}L^{-1}\left[v_1^{n+1}\right]. \tag{4.48}$$

From Abramowitz and Stegun [22], we get

$$\phi_n(t)=L^{-1}\left[v_1^n\right]=e^{-(\lambda_1+\lambda_2+\mu)t}L^{-1}\left[\left(\frac{\theta-\sqrt{\theta^2-4(\lambda_1+\lambda_2)\mu}}{2(\lambda_1+\lambda_2)}\right)^n\right]$$

$$=e^{-(\lambda_1+\lambda_2+\mu)t}\left(\frac{\mu}{\lambda_1+\lambda_2}\right)^{n/2}\frac{I_{n+1}\left(2t\sqrt{(\lambda_1+\lambda_2)\mu}\right)}{t}. \tag{4.49}$$

Putting (4.47) into (4.46), we get

$$P(0,0,0,t)=\frac{1}{\mu}\sum_{n=0}^{\infty}\phi_{n+1}(t). \tag{4.50}$$

When $|v|<1$, we find $|(\lambda_1+\lambda_2)vv_1|<\mu, 0<|2\lambda_1 v|<y$. Proceeding to expand (4.44) in power series in v, we first obtain

$$G_0^{(2)*}(v,\theta)=\frac{\lambda_2 vv_1}{\{\mu-(\lambda_1+\lambda_2)vv_1\}}\left[1-\frac{\lambda_1 v}{x}+\frac{\lambda_1 v_1}{x}+\sum_{k=1}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}\frac{2^{2k-1}\lambda_1^k\mu^k}{x^{2k}}\right.$$

$$\left.-\frac{v}{x}\sum_{k=1}^{\infty}\frac{2^k\lambda_1^{k+1}v_1^k}{y^k}+\frac{1}{x}\sum_{k=1}^{\infty}\frac{2^k\lambda_1^{k+1}v_1^{k+1}}{y^k}\right]P^*(0,0,0,\theta). \tag{4.51}$$

Further, we obtain

$$\frac{1}{x}=\frac{1}{\theta+\lambda_1+\lambda_2(1-v)+\mu}=\frac{1}{\theta+\lambda_1+\lambda_2+\mu-\lambda_2 v}=\sum_{r=0}^{\infty}\frac{\lambda_2^r v^r}{(\theta+\lambda_1+\lambda_2+\mu)^{r+1}}, \tag{4.52}$$

$$\frac{1}{x^{2k}}=\frac{1}{\{\theta+\lambda_1+\lambda_2(1-v)+\mu\}^{2k}}=\sum_{r=0}^{\infty}(-1)^r\binom{-2k}{r}\frac{\lambda_2^r v^r}{(\theta+\lambda_1+\lambda_2+\mu)^{r+2k}}, \tag{4.53}$$

$$\frac{1}{y^k}=\sum_{r=0}^{\infty}\frac{u_r(\gamma;k)}{r!}(x-\gamma)^r=\sum_{r=0}^{\infty}(-1)^r\frac{u_r(\gamma;k)}{r!}\lambda_2^r v^r, \tag{4.54}$$

$$\frac{1}{xy^k}=\sum_{i=0}^{\infty}\sum_{r=0}^{i}\frac{(-1)^{i-r}\lambda_2^i u_{i-r}(\gamma;k)}{(i-r)!(\theta+\lambda_1+\lambda_2+\mu)^{r+1}}v^i. \tag{4.55}$$

where

$$u_0(x,k)=\frac{1}{y^k},\quad \gamma=\theta+\lambda_1+\lambda_2+\mu,$$

$$u_r(\gamma;k)=\left[\frac{d^r}{dx^r}\left(\frac{1}{y^k}\right)\right]_{x=\gamma}, r=0,1,2,\cdots.$$

Using the Leibnitz rule of successive differentiation (see Kaplan [23]), we obtain

$$u_{r+2}(\gamma;k)=\frac{(k^2-r^2)u_r(\gamma;k)-(2r+1)\gamma u_{r+1}(\gamma;k)}{(\gamma^2-4\lambda_1\mu)}, r=0,1,2,\cdots. \tag{4.56}$$

The recurrence relation (4.56) is easily solved by using the conditions

$$u_0(\gamma;k)=\frac{1}{(\gamma+\sqrt{\gamma^2-4\lambda_1\mu})^k}, \tag{4.57}$$

$$u_1(\gamma;k)=-\frac{ku_0(\gamma;k)}{\sqrt{\gamma^2-4\lambda_1\mu}}. \tag{4.58}$$

Substituting (4.52)-(4.55) in (4.51) and simplifying, we obtain

$$G_0^{(2)*}(v,\theta)=[T_1-T_2+T_3+T_4-T_5+T_6]P^*(0,0,0,\theta), \tag{4.59}$$

where

$$T_1=\frac{\lambda_2 v_1}{\mu}v+\sum_{j=2}^{\infty}\frac{\lambda_2(\lambda_1+\lambda_2)^{j-1}v_1^j}{\mu^j}v^j, \tag{4.60}$$

$$T_2=\sum_{j=2}^{\infty}\sum_{m=1}^{j-1}\frac{\lambda_1\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}v_1^m}{\mu^m(\theta+\lambda_1+\lambda_2+\mu)^{j-m}}v^j, \tag{4.61}$$

$$T_3=\frac{\lambda_1\lambda_2 v_1^2}{\mu(\theta+\lambda_1+\lambda_2+\mu)}v+\sum_{j=2}^{\infty}\sum_{m=1}^{j}\frac{\lambda_1\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}v_1^{m+1}}{\mu^m(\theta+\lambda_1+\lambda_2+\mu)^{j-m+1}}v^j, \tag{4.62}$$

$$T_4=\sum_{k=1}^{\infty}\frac{(-1)^k\lambda_1^k\lambda_2\binom{\frac{1}{2}}{k}2^{2k-1}\mu^{k-1}v_1}{(\theta+\lambda_1+\lambda_2+\mu)^{2k}}v$$

$$+\sum_{j=2}^{\infty}\sum_{m=1}^{j}\sum_{k=1}^{\infty}\frac{(-1)^{j+k-m}\lambda_1^k\lambda_2^{j-m+1}\binom{\frac{1}{2}}{k}\binom{-2k}{j-m}2^{2k-1}(\lambda_1+\lambda_2)^{m-1}\mu^{k-m}v_1^m}{(\theta+\lambda_1+\lambda_2+\mu)^{j+2k-m}}v^j, \tag{4.63}$$

$$T_5=\sum_{j=2}^{\infty}\sum_{n=1}^{j-1}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n-1}2^k\lambda_1^{k+1}\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}u_{j-n-1}(\gamma;k)v_1^{m+k}}{(j-n-1)!\mu^m(\theta+\lambda_1+\lambda_2+\mu)^{n-m+1}}v^j, \tag{4.64}$$

$$T_6=\sum_{k=1}^{\infty}\frac{2^k\lambda_1^{k+1}\lambda_2 u_0(\gamma;k)v_1^{k+2}}{\mu(\theta+\lambda_1+\lambda_2+\mu)}v$$

$$+\sum_{j=2}^{\infty}\sum_{n=1}^{j}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n}2^k\lambda_1^{k+1}\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}u_{j-n}(\gamma;k)v_1^{m+k+1}}{\mu^m(j-n)!(\theta+\lambda_1+\lambda_2+\mu)^{n-m+1}}v^j. \tag{4.65}$$

Equating the coefficient of v^j in (4.59),
(i) for $j=1$, we get

$$P^*(0,1,2,\theta)=P^*(0,0,0,\theta)\left[\frac{\lambda_2 v_1}{\mu}+\frac{\lambda_1\lambda_2 v_1^2}{\mu(\theta+\lambda_1+\lambda_2+\mu)}\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{(-1)^k\lambda_1^k\lambda_2\binom{\frac{1}{2}}{k}2^{2k-1}\mu^{k-1}v_1}{(\theta+\lambda_1+\lambda_2+\mu)^{2k}}+\sum_{k=1}^{\infty}\frac{2^k\lambda_1^{k+1}\lambda_2 u_0(\gamma;k)v_1^{k+2}}{\mu(\theta+\lambda_1+\lambda_2+\mu)}\right]. \tag{4.66}$$

(ii) for $j=2,3,\cdots$, we get

$$P^*(0,j,2,\theta)=P^*(0,0,0,\theta)\left[\frac{\lambda_2(\lambda_1+\lambda_2)^{j-1}v_1^j}{\mu^j}-\sum_{m=1}^{j-1}\frac{\lambda_1\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}v_1^m}{\mu^m(\theta+\lambda_1+\lambda_2+\mu)^{j-m}}\right.$$

$$+\sum_{m=1}^{j}\frac{\lambda_1\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}v_1^{m+1}}{\mu^m(\theta+\lambda_1+\lambda_2+\mu)^{j-m+1}}$$

$$+\sum_{m=1}^{j}\sum_{k=1}^{\infty}\frac{(-1)^{j+k-m}\lambda_1^k\lambda_2^{j-m+1}\binom{\frac{1}{2}}{k}\binom{-2k}{j-m}2^{2k-1}(\lambda_1+\lambda_2)^{m-1}\mu^{k-m}v_1^m}{(\theta+\lambda_1+\lambda_2+\mu)^{j+2k-m}}$$

$$-\sum_{n=1}^{j-1}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n-1}2^k\lambda_1^{k+1}\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}u_{j-n-1}(\gamma;k)v_1^{m+k}}{(j-n-1)!\mu^m(\theta+\lambda_1+\lambda_2+\mu)^{n-m+1}}$$

$$\left.+\sum_{n=1}^{j}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n}2^k\lambda_1^{k+1}\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}u_{j-n}(\gamma;k)v_1^{m+k+1}}{\mu^m(j-n)!(\theta+\lambda_1+\lambda_2+\mu)^{n-m+1}}\right]. \tag{4.67}$$

From Abramowitz and Stegun [22] and Schiff [24], we have the inverse Laplace transforms

$$L^{-1}\left[\left(\theta-\sqrt{\theta^2-a^2}\right)^{\nu}\right]=\frac{\nu a^{\nu}}{t}I_{\nu}(at),$$

$$L^{-1}\left[\frac{\left(\theta-\sqrt{\theta^2-a^2}\right)^{\nu}}{\sqrt{\theta^2-a^2}}\right]=a^{\nu}I_{\nu}(at),$$

$$L^{-1}\left[\frac{1}{(\theta^2-a^2)^k}\right]=\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}I_{k-\frac{1}{2}}(at).$$

Setting $\kappa_r(t;k)=L^{-1}[u_r(\gamma;k)]$, we get

$$\kappa_0(t;k)=L^{-1}[u_0(\gamma;k)]=e^{-\gamma t}\left(\frac{k}{a^k}\right)\left(\frac{I_k(at)}{t}\right),$$

$$\kappa_1(t;k)=L^{-1}\left[u_1(\gamma;k)\right]=-e^{-\gamma t}\left(\frac{k}{a^k}\right)I_k(at),$$

$$\kappa_{r+2}(t;k)=e^{-\gamma t}\left[(k^2-r^2)\frac{\sinh at}{a}©\kappa_r(t;k)-(2r+1)\cosh at©\kappa_{r+1}(t;k)\right], r=0,1,2,\cdots,$$

where $a^2 = 4\lambda_1\mu$. Taking inverse transform of (4.66), we get

$$P(0,1,2,t) = P(0,0,0,t) © \frac{1}{\mu}\Big[\lambda_2\phi_1(t) + \lambda_1\lambda_2\phi_2(t) © e^{-(\lambda_1+\lambda_2+\mu)t} + \sum_{k=1}^{\infty}\frac{(-1)^k\lambda_1^k\lambda_2\binom{\frac{1}{2}}{k}2^{2k-1}\mu^k\phi_1(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{2k-1}}{(2k-1)!} + \sum_{k=1}^{\infty}2^k\lambda_1^{k+1}\lambda_2\kappa_0(t;k) © \phi_{k+2}(t) © e^{-(\lambda_1+\lambda_2+\mu)t}\Big]. \tag{4.68}$$

Taking inverse transform of (4.67), we get

$$\begin{aligned}P(0,j,2,t) = P(0,0,0,t) © \Bigg[&\frac{\lambda_2(\lambda_1+\lambda_2)^{j-1}\phi_j(t)}{\mu^j} - \sum_{m=1}^{j-1}\frac{\lambda_1\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}\phi_m(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{j-m-1}}{\mu^m(j-m-1)!} \\ &+ \sum_{m=1}^{j}\frac{\lambda_1\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}\phi_{m+1}(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{j-m}}{\mu^m(j-m)!} \\ &+ \sum_{m=1}^{j}\sum_{k=1}^{\infty}\frac{(-1)^{j+k-m}\lambda_1^k\lambda_2^{j-m+1}\binom{\frac{1}{2}}{k}\binom{-2k}{j-m}2^{2k-1}(\lambda_1+\lambda_2)^{m-1}\mu^{k-m}\phi_m(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{j+2k-m-1}}{(j+2k-m-1)!} \\ &- \sum_{n=1}^{j-1}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n-1}2^k\lambda_1^{k+1}\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}\kappa_{j-n-1}(t,k) © \phi_{m+k}(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{n-m}}{(j-n-1)!(n-m)!\mu^m} \\ &+ \sum_{n=1}^{j}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n}2^k\lambda_1^{k+1}\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}\kappa_{j-n}(t,k) © \phi_{m+k+1}(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{n-m}}{\mu^m(j-n)!(n-m)!}\Bigg].\end{aligned} \tag{4.69}$$

Taking (4.17) and simplifying, we get

$$G^{(2)*}(u,v,\theta) = \sum_{j=1}^{\infty}P^*(0,j,2,\theta)v^j + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\frac{\binom{i+k-1}{k}\lambda_1^i\lambda_2^kP^*(0,j-k,2,\theta)}{(\theta+\lambda_1+\lambda_2+\mu)^{i+k}}u^iv^j. \tag{4.70}$$

Equating the coefficient of $u^iv^j, i = 1,2,\cdots; j = 1,2,\cdots$ in (4.70), we get

$$P^*(i,j,2,\theta) = \sum_{k=0}^{j-1}\frac{\binom{i+k-1}{k}\lambda_1^i\lambda_2^kP^*(0,j-k,2,\theta)}{(\theta+\lambda_1+\lambda_2+\mu)^{i+k}}. \tag{4.71}$$

Taking inverse Laplace transform of (4.71), we get

$$P(i,j,2,t) = \sum_{k=0}^{j-1}\frac{\binom{i+k-1}{k}\lambda_1^i\lambda_2^ke^{-(\lambda_1+\lambda_2+\mu)t}t^{i+k-1}}{(i+k-1)!} © P(0,j-k,2,t), i,j = 1,2,\cdots. \tag{4.72}$$

Taking (4.21) and simplifying, we get

$$G^{(1)*}(u,v,\theta) = \frac{1}{(\mu-\lambda_1uu_1)}\Big[\lambda_1uu_1P^*(0,0,0,\theta) + \frac{\lambda_1\mu uu_1}{v}\frac{\{\theta+\lambda_1+\lambda_2(1-v)+\mu\}}{[\theta+\lambda_1(1-u)+\lambda_2(1-v)+\mu][\theta+\lambda_1(1-u_1)+\lambda_2(1-v)+\mu]}G_0^{(2)*}(v,\theta)\Big]. \tag{4.73}$$

Expanding in power series, equation (4.73) gives

$$G^{(1)*}(u,v,\theta) = \sum_{i=1}^{\infty}\frac{\lambda_1^iu^iu_1^i}{\mu^i}P^*(0,0,0,\theta) + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{r=1}^{i}\sum_{k=1}^{j+1}\sum_{n=0}^{\infty}\frac{\binom{i+j+n-r-k+1}{j-k+1}\lambda_1^{i+n}\lambda_2^{j-k+1}u_1^{r+n}u^iv^j}{\mu^{r-1}(\theta+\lambda_1+\lambda_2+\mu)^{i+j+n-r-k+2}}P^*(0,k,2,\theta). \tag{4.74}$$

Equating $u^iv^0, i = 1,2,\cdots$, (4.74) yields

$$P^*(i,0,1,\theta) = \frac{\lambda_1^iu_1^i}{\mu^i}P^*(0,0,0,\theta) + \left(\sum_{r=1}^{i}\sum_{n=0}^{\infty}\frac{\lambda_1^{i+n}u_1^{r+n}}{\mu^{r-1}(\theta+\lambda_1+\lambda_2+\mu)^{i+n-r+1}}\right)P^*(0,1,2,\theta), i = 1,2,\cdots. \tag{4.75}$$

Equating $u^iv^j, i,j = 1,2,\cdots$, (4.74) yields

$$P^*(i,j,1,\theta) = \sum_{r=1}^{i}\sum_{k=1}^{j+1}\sum_{n=0}^{\infty}\frac{\binom{i+j+n-r-k+1}{j-k+1}\lambda_1^{i+n}\lambda_2^{j-k+1}u_1^{r+n}}{\mu^{r-1}(\theta+\lambda_1+\lambda_2+\mu)^{i+j+n-r-k+2}}P^*(0,k,2,\theta), i,j = 1,2,\cdots. \tag{4.76}$$

Taking inverse Laplace transform of (4.75), we get

$$P(i,0,1,t) = \frac{\lambda_1^i\phi_i(t)}{\mu^i} © P(0,0,0,t) + \sum_{r=1}^{i}\sum_{n=0}^{\infty}\frac{\lambda_1^{i+n}\phi_{r+n}(t) © e^{-(\lambda_1+\lambda_2+\mu)t}t^{i+n-r} © P(0,1,2,t)}{\mu^{r-1}(i+n-r)!}, i = 1,2,\cdots. \tag{4.77}$$

Taking inverse Laplace transform of (4.76), we get, for $i = 1,2,\cdots; j = 1,2,\cdots$,

$$P(i,j,1,t) = \sum_{r=1}^{i}\sum_{k=1}^{j+1}\sum_{n=0}^{\infty}\frac{\binom{i+j+n-r-k+1}{j-k+1}\lambda_1^{i+n}\lambda_2^{j-k+1}\phi_{r+n}(t)}{\mu^{r-1}(i+j+n-r-k+1)!} © e^{-(\lambda_1+\lambda_2+\mu)t}t^{i+j+n-r-k+1} © P(0,k,2,t). \tag{4.78}$$

Equations (4.50), (4.68), (4.69), (4.72), (4.77) and (4.78) provide explicit expressions for the transient solutions of the queueing system.

5 Steady-state distribution

We define the steady-state probabilities

$$\pi(0,0,0) = \lim_{t\to\infty}P(0,0,0,t),$$

$$\pi(i,j,1) = \lim_{t\to\infty}P(i,j,1,t), i = 1,2,\cdots, j = 0,1,2,\cdots;$$

$$\pi(i,j,2) = \lim_{t\to\infty}P(i,j,2,t), i = 0,1,2,\cdots, j = 1,2,\cdots.$$

By the final value theorem of Laplace transform, we have

$$\pi(0,0,0) = \lim_{\theta\to 0}\theta P(0,0,0,\theta),$$

$$\pi(i,j,1) = \lim_{\theta\to 0}\theta P(i,j,1,\theta), i = 1,2,\cdots; j = 0,1,2,\cdots;$$

$$\pi(i,j,2) = \lim_{\theta\to 0}\theta P(i,j,2,\theta), i = 0,1,2,\cdots; j = 1,2,\cdots.$$

Using the condition $\mu > \lambda_1 + \lambda_2$, (4.34) leads to

$$\{v_1\}_{\theta\to 0} = 1. \tag{5.1}$$

By using (5.1), (4.36) gives

$$\{u_1(v_1)\}_{\theta\to 0} = 1. \tag{5.2}$$

Put $\alpha = \left\{\frac{du_1(v_1)}{d\theta}\right\}_{\theta\to 0}$ and $\beta = \left\{\frac{dv_1}{d\theta}\right\}_{\theta\to 0}$. By using (4.37) and (4.38), we get

$$(\lambda_1 - \mu)\alpha + \lambda_2\beta = 1, \tag{5.3}$$

$$\lambda_1\alpha + (\lambda_2 - \mu)\beta = 1. \tag{5.4}$$

Solving (5.3) and (5.4), we get

$$\alpha = \beta = \frac{-1}{\mu - \lambda_1 - \lambda_2}. \tag{5.5}$$

Putting $\rho_1 = \frac{\lambda_1}{\mu}$ and $\rho_2 = \frac{\lambda_2}{\mu}$, we write (5.5) as

$$\alpha = \beta = \frac{1}{\mu(\rho_1+\rho_2-1)}. \tag{5.6}$$

By using (4.42) and L$'$Hospital rule, and (5.6), we get

$$\pi(0,0,0) = \lim_{\theta\to 0} \frac{\theta}{\theta+\lambda_1+\lambda_2-(\lambda_1+\lambda_2)v_1}$$
$$= \frac{1}{1-(\lambda_1+\lambda_2)\beta} = 1-\rho_1-\rho_2. \tag{5.7}$$

By using (4.66), we get

$$\pi(0,1,2) = \pi(0,0,0)\Bigg[\frac{\lambda_2}{\mu} + \frac{\lambda_1\lambda_2}{\mu(\lambda_1+\lambda_2+\mu)}$$
$$+\sum_{k=1}^{\infty} \frac{(-1)^k\lambda_1^k\lambda_2\binom{\frac{1}{2}}{k}2^{2k-1}\mu^{k-1}}{(\lambda_1+\lambda_2+\mu)^{2k}} + \sum_{k=1}^{\infty}\frac{2^k\lambda_1^{k+1}\lambda_2 z_0(v;k)}{\mu(\lambda_1+\lambda_2+\mu)}\Bigg], \tag{5.8}$$

where

$$v = \lambda_1+\lambda_2+\mu,$$
$$z_0(v,r) = \lim_{\theta\to 0} u_0(\gamma,r) = (v+\sqrt{v^2-4\lambda_1\mu})^{-r}.$$

By using (4.67), we get

$$\pi(0,j,2) = \pi(0,0,0)\Bigg[\frac{\lambda_2(\lambda_1+\lambda_2)^{j-1}}{\mu^j} - \sum_{m=1}^{j-1}\frac{\lambda_1\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}}{\mu^m(\lambda_1+\lambda_2+\mu)^{j-m}}$$
$$+\sum_{m=1}^{j}\frac{\lambda_1\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}}{\mu^m(\lambda_1+\lambda_2+\mu)^{j-m+1}}$$
$$+\sum_{m=1}^{j}\sum_{k=1}^{\infty}\frac{(-1)^{j+k-m}\lambda_1^k\lambda_2^{j-m+1}\binom{\frac{1}{2}}{k}\binom{-2k}{j-m}2^{2k-1}(\lambda_1+\lambda_2)^{m-1}\mu^{k-m}}{(\lambda_1+\lambda_2+\mu)^{j+2k-m}}$$
$$-\sum_{n=1}^{j-1}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n-1}2^k\lambda_1^{k+1}\lambda_2^{j-m}(\lambda_1+\lambda_2)^{m-1}z_{j-n-1}(v;k)}{(j-n-1)!\mu^m(\lambda_1+\lambda_2+\mu)^{n-m+1}}$$
$$+\sum_{n=1}^{j}\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{j-n}2^k\lambda_1^{k+1}\lambda_2^{j-m+1}(\lambda_1+\lambda_2)^{m-1}z_{j-n}(v;k)}{\mu^m(j-n)!(\lambda_1+\lambda_2+\mu)^{n-m+1}}\Bigg], j=2,3,\cdots, \tag{5.9}$$

where

$$z_1(v,r) = \lim_{\theta\to 0} u_1(\gamma,r) = -\frac{r}{\sqrt{v^2-4\lambda_1\mu}(v+\sqrt{v^2-4\lambda_1\mu})^r},$$

$$z_k(v,r) = \lim_{\theta\to 0} u_k(\gamma,r) = \frac{\{r^2-(k-2)^2\}z_{k-2}(v,r)-(2k-3)vz_{k-1}(v,r)}{v^2-4\lambda_1\mu}, k=2,3,\cdots.$$

By using (4.71), we get

$$\pi(i,j,2) = \sum_{k=0}^{j-1}\frac{\binom{i+k-1}{k}\lambda_1^i\lambda_2^k\pi(0,j-k,2)}{(\lambda_1+\lambda_2+\mu)^{i+k}}, i,j=1,2,\cdots. \tag{5.10}$$

By using (4.75), we get

$$\pi(i,0,1) = \frac{\lambda_1^i}{\mu^i}\pi(0,0,0) + \left(\sum_{r=1}^{i}\sum_{n=0}^{\infty}\frac{\lambda_1^{i+n}}{\mu^{r-1}(\lambda_1+\lambda_2+\mu)^{i+n-r+1}}\right)\pi(0,1,2), i=1,2,\cdots. \tag{5.11}$$

By using (4.76), we get

$$\pi(i,j,1) = \sum_{r=1}^{i}\sum_{k=1}^{j+1}\sum_{n=0}^{\infty}\frac{\binom{i+j+n-r-k+1}{j-k+1}\lambda_1^{i+n}\lambda_2^{j-k+1}}{\mu^{r-1}(\lambda_1+\lambda_2+\mu)^{i+j+n-r-k+2}}\pi(0,k,2), i,j=1,2,\cdots. \tag{5.12}$$

6 Steady-state Performance Measures

6.1 Steady-state probability Generating Functions

We define

$$\Pi_0^{(2)}(v) = \sum_{j=1}^{\infty}\pi(0,j,2)v^j,$$
$$\Pi^{(1)}(u,v) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\pi(i,j,1)u^iv^j,$$
$$\Pi^{(2)}(u,v) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\pi(i,j,2)u^iv^j.$$

Using (4.44), we get

$$\Pi_0^{(2)}(v) = \frac{\lambda_2 vy(-2\lambda_1 v+y)}{2\{\mu-(\lambda_1+\lambda_2)v\}x(-2\lambda_1+y)}\pi(0,0,0), \tag{6.1.1}$$

where

$$x = \lambda_1+\lambda_2(1-v)+\mu,$$
$$y = x+\sqrt{x^2-4\lambda_1\mu}.$$

Setting $v=1$ in (6.1.1) and simplifying, we get

$$\Pi_0^{(2)}(1) = \frac{\lambda_2}{\lambda_1+\mu}. \tag{6.1.2}$$

Equation (6.1.2) gives the probability that the server is serving a non-priority customer and that there is no priority customer waiting. Consequently, the probability that there is no priority customer in the system is given by

$$\Pi_0^{(2)}(1)+\pi(0,0,0) = \frac{\mu^2-\lambda_1(\lambda_1+\lambda_2)}{\mu(\lambda_1+\mu)}.$$

6.2 Probability that the server is busy with Priority Customer

Let η be the probability that the server is busy with a priority customer in the long run. The favourable states for the event are $(i,j,1), i=1,2,\cdots; j=0,1,2,\cdots$. Consequently, we obtain

$$\eta = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\pi(i,j,1). \tag{6.2.1}$$

By definition, we have

$$\Pi^{(1)}(u,v) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\pi(i,j,1)u^iv^j.$$

Using (4.73), we get

$$\Pi^{(1)}(u,v) = \lim_{\theta\to 0}\theta G^{(1)*}(u,v,\theta) = \frac{\lambda_1 u\pi(0,0,0)+\frac{\lambda_1\mu u}{v}\frac{\{\lambda_1+\lambda_2(1-v)+\mu\}}{[\lambda_1(1-u)+\lambda_2(1-v)+\mu][\lambda_2(1-v)+\mu]}\Pi_0^{(2)}(v)}{(\mu-\lambda_1 u)}. \tag{6.2.2}$$

Setting $u=v=1$ in (6.2.2) and simplifying, we obtain

$$\eta = P^{(1)}(1,1) = \frac{\lambda_1}{\mu}. \tag{6.2.3}$$

6.3 Probability that the server is busy with non-Priority Customer

Let ζ be the probability that the server is busy with a non-priority customer in the long run. The favourable states for the event are $(i,j,2), i=0,1,2,\cdots; j=1,2,\cdots$. Consequently, we obtain

$$\zeta = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\pi(i,j,2). \tag{6.3.1}$$

By definition, we have

$$\Pi^{(2)}(u,v) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\pi(i,j,2)u^i v^j.$$

Using (4.17), we get

$$\Pi^{(2)}(u,v) = \lim_{\theta\to 0}\theta G^{(2)*}(u,v,\theta) = \frac{[\lambda_1+\lambda_2(1-v)+\mu]\Pi_0^{(2)}(v)}{[\lambda_1(1-u)+\lambda_2(1-v)+\mu]}. \tag{6.3.2}$$

Setting $u=v=1$ in (6.3.2) and simplifying, we obtain

$$\zeta = \Pi^{(2)}(1,1) = \frac{\lambda_2}{\mu}. \tag{6.3.3}$$

6.4 Probability that the server is idle

Let ξ be the probability that the server is idle in the long run. The favourable state for the event is $(0,0,0)$. Consequently, we obtain

$$\xi = \pi(0,0,0) = 1-\frac{\lambda_1}{\mu}-\frac{\lambda_2}{\mu}. \tag{6.4.1}$$

6.5 Mean number of priority customers

Let $E(P)$ denote the expected number of priority customers in the long run. Then we obtain

$$E(P) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} i\pi(i,j,1)+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i\pi(i,j,2). \tag{6.5.1}$$

Using the definition, we get

$$E(P) = \left[\frac{d}{du}\{\Pi^{(1)}(u,v)+\Pi^{(2)}(u,v)\}\right]_{u=1,v=1}$$

$$= \left[\frac{d}{du}\{\Pi^{(1)}(u,1)+\Pi^{(2)}(u,1)\}\right]_{u=1}. \tag{6.5.2}$$

From (6.2.2), we get

$$\Pi^{(1)}(u,1) = \frac{\lambda_1 u\pi(0,0,0)+\frac{\lambda_1\lambda_2 u}{\lambda_1(1-u)+\mu}}{\mu-\lambda_1 u}. \tag{6.5.3}$$

From (6.3.2), we get

$$\Pi^{(2)}(u,1) = \frac{\lambda_2}{[\lambda_1(1-u)+\mu]}. \tag{6.5.4}$$

From (6.5.3) and (6.5.4), we get

$$\left[\frac{d}{du}\Pi^{(1)}(u,1)\right]_{u=1} = \frac{\lambda_1\mu^2+\lambda_1^2\lambda_2}{(\mu-\lambda_1)^2}. \tag{6.5.5}$$

From (4.8), we get

$$\left[\frac{d}{du}\Pi^{(2)}(u,1)\right]_{u=1} = \frac{\lambda_1\lambda_2}{\mu^2}. \tag{6.5.6}$$

Substituting (6.5.5) and (6.5.6) in (6.5.2), we get

$$E(P) = \frac{\lambda_1(\lambda_2+\mu)}{\mu(\mu-\lambda_1)}. \tag{6.5.7}$$

6.6 Mean number of non-priority customers

Let $E(N)$ denote the expected number of non-priority customers in the long run. Then we obtain

$$E(N) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} j\pi(i,j,1)+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} j\pi(i,j,2)$$

$$= \frac{d}{dv}\left[\Pi^{(1)}(1,v)+\Pi^{(2)}(1,v)\right]_{v=1}. \tag{6.6.1}$$

From (6.1.1), we get

$$\left[\Pi_0^{(2)'}(v)\right]_{v=1} = \frac{\lambda_2}{\lambda_1+\mu}\left[\frac{\lambda_2}{\lambda_1+\mu}-\left(\frac{\lambda_1+\lambda_2}{\mu-\lambda_1}\right)+\frac{\mu}{\mu-\lambda_1-\lambda_2}\right]. \tag{6.6.2}$$

Using (6.2.2) and (6.3.2), and (6.6.2), we get

$$\left[\frac{d\Pi^{(1)}(1,v)}{dv}\right]_{v=1} = \frac{\lambda_1\lambda_2^2[2\mu^2-\mu(3\lambda_1+\lambda_2)+2\lambda_1(\lambda_1+\lambda_2)]}{\mu^2(\mu-\lambda_1)^2(\mu-\lambda_1-\lambda_2)}, \tag{6.6.3}$$

$$\left[\frac{d\Pi^{(2)}(1,v)}{dv}\right]_{v=1} = \frac{\lambda_2[\lambda_1^2\lambda_2+\lambda_1\lambda_2^2-2\lambda_1\mu^2+\lambda_1^2\mu+\mu^3]}{\mu^2(\mu-\lambda_1)(\mu-\lambda_1-\lambda_2)}. \tag{6.6.4}$$

Substituting (6.6.3) and (6.6.4) in (6.6.1), we get

$$E(N) = \frac{\lambda_2[\mu^4-3\lambda_1\mu^3+\lambda_1\mu^2(2\lambda_2+3\lambda_1^2)-\lambda_1^2\mu(2\lambda_2+\lambda_1)+\lambda_1^2\lambda_2(\lambda_1+\lambda_2)]}{\mu^2(\mu-\lambda_1)^2(\mu-\lambda_1-\lambda_2)}. \tag{6.6.5}$$

7 A Numerical Illustration

We fix $\lambda_1 = 1.2$ and $\lambda_2 = 0.6$ and consider the expressions (6.5.7) and (6.6.5). Subject to the stability condition $\mu > \lambda_1+\lambda_2$, we vary μ from 1.9 to 5.8 and obtain the figures 1 (a) and 1 (b). Fig. 1 (a) exhibits the variation of $E(P)$ as a function of μ and Fig. 1 (b) exhibits the variation of $E(N)$ as a function of μ. In both graphs, we find that the mean values decrease as μ increases. In other words, if the service rate increases, the expected number of priority or non-priority customers in the long run is likely to go down. This has to be true, since as service rate increases, more and more customers (both priority and non-priority) are served, thus confirming that our model behaves as was expected.

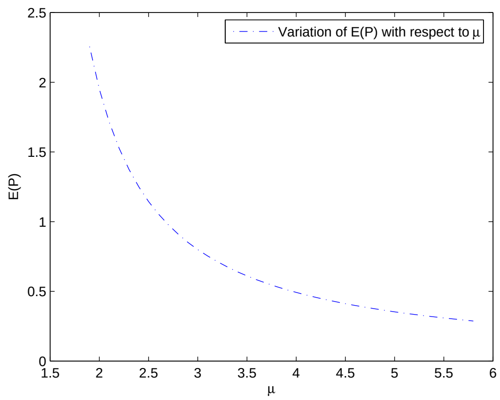

Fig. 1 (a): Variation of $E(P)$ with respect to μ

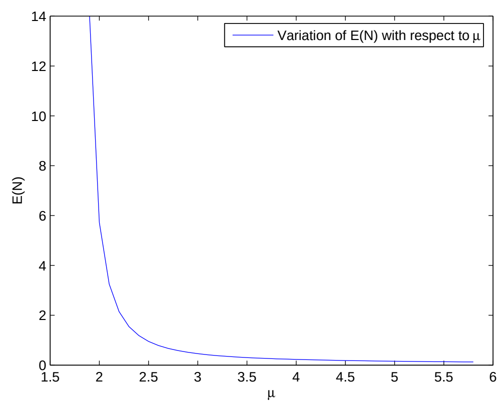

Fig. 1 (b): Variation of $E(N)$ with respect to μ

8 Perspective

In this paper, we have obtained the transient solution for the joint probability distribution of the number of customers of type I priority and the number of customers of type II priority. We have made use of a new technique (through successive differentiation) to obtain explicit expressions for the joint probabilities. We have also obtained some performance measures such as mean number of priority customers and mean number of non-priority customers in the system in the stationary regime.

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Venkata S S Yadavalli is a Professor and Head of Department of Industrial & Systems Engineering, at University of Pretoria, South Africa. He obtained his Ph D from Indian Institute of Technology, Chennai, in 1983. His main areas of interest include, Stochastic modeling towards, Reliability, Queuing, Inventory, Manpower planning systems and Finance. He has published over 150 research papers in these areas. He published articles in various national and international journals including, IEEE Transactions on Reliability, Applied Mathematics and Information Sciences, Computers & Industrial Engineering, Stochastic Analysis and Applications, International Journal of Systems Science, International Journal of Production Economics, International Journal of Production Research, Applied Mathematics & Computation, etc. He was past Vice-President and President of Operations Research Society of South Africa. He is in the editorial board of various national and international journals. He is a Fellow of the South African Statistical Association. He received various awards from various professional bodies like IEOM (Industrial Engineering & Operations Management, USA), South African Institute of Industrial Engineers, South African Institute of Mining and Metallurgy etc. for his contributions towards research.

Krishna Sundar Diatha is a professor of Operations Management at Indian Institute of Management Bangalore, India. His research interests are in the areas of Sustainable Supply Chains, Supply Networks, Operations Strategy, Manufacturing & IT Strategy, Digital Innovation. His research appeared in various international journals (Neuro Computing, Computers & Industrial Engineering, Annals of Operations Research, Information Systems Frontiers etc.,) and presented his work in various international conferences. At present, Krishna Sundar is Chairperson of India Japan Study Centre; Chairperson of Production & Operations Management Area and Founder & Chairperson of Digital Innovation Lab at Indian Institute of Management Bangalore.

S. Udayabaskaran received his MSc and PhD from Indian Institute of Technology, Madras, Chennai, India. He is at present working as Professor of Mathematics in Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, India. He has more than 45 years of teaching and research experience. His research interest includes stochastic analysis of queueing systems, inventory systems and biological systems.

C. T. Dora Pravina received her master degrees MSc and MPhil in mathematics from University of Madras, Tamilnadu, India. She is at present working as Assistant Professor of Mathematics and also working for her PhD in Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, India. She has more than 12 years of teaching. Her research interest is in the analysis of queueing systems, inventory systems and biological systems.