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Modified Generalized Half-Normal Distribution with Application to Lifetimes

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Abstract: In this article we introduce a new distribution which is constructed by taking, as a base, the absolute value of a random variable with alpha skew-normal (ASN) distribution. The density of this distribution and some of its properties such as moments and its skewness and kurtosis coefficients are discussed. We calculate the moment and the maximum likelihood estimators, and carry out a Monte Carlo study. Finally, an example with real data to illustrate the usefulness and applicability of the proposed distribution is presented. It is shown that the obtained result is a distribution with considerable flexibility.

Keywords: Alpha-skew-normal distribution, Generalized half-normal distribution, Asymmetry, Kurtosis, Maximum likelihood estimation

1 Introduction

Azzalini [4] introduced the $\{SN(\lambda), \lambda \in \mathbb{R}\}$ skew-normal distribution, with skewness parameter λ, such that $SN(0)$ is the standard normal distribution. In other words $X \sim SN(\lambda)$ with density function:

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x), \tag{1}$$

where $x, \lambda \in \mathbb{R}$, ϕ and Φ are the density $N(0,1)$ and its cumulative distribution function respectively.

Elal-Olivero [14] introduced ASN distribution, with shape parameter α, such that $ASN(0)$ is the standard normal distribution. In other words $X \sim ASN(\alpha)$ with density function:

$$f_X(x;\alpha) = \left(\frac{(1-\alpha x)^2+1}{2+\alpha^2}\right)\phi(x), \tag{2}$$

where $x, \alpha \in \mathbb{R}$.

One distribution related to ASN distribution is the following density function

$$f_X(x;\alpha) = \left(\frac{2+\alpha^2 x^2}{2+\alpha^2}\right)\phi(x), \tag{3}$$

where $x, \alpha \in \mathbb{R}$ and the random variable X is called symmetric-component-alpha-skew-normal (SCASN) distribution and is denoted as $X \sim SCASN(\alpha)$. The SCASN distribution and its representation are given in the work of Elal-Olivero [14].

The support of the SN and ASN models occurs in all real numbers, and both models are related to the half-normal (HN) distribution. Recent statistical literature has shown growth in the possible applications of extensions of half-normal distribution; for example, Altman [3], Chou and Liu [13], Bland and Altman [8], Lawless [16], Chen and Wang [12], Bland [9], Coffey et al. [10], Barranco-Chamorro et al. [5], Weisstein [21], Cooray and Ananda [11] and Olmos et al. [17,18] present theoretical and applied results related to half-normal distribution. Chapter 2 of the book by Ahsanullah et al. [1] contains some properties of HN distribution.

An extension of the HN model was introduced by Cooray and Ananda [11]. It is called generalized half-normal (GHN) distribution, in other words a random variable X has GHN distribution if its density function is:

$$f_X(x;\beta,\gamma) = \frac{2\gamma z^{\gamma-1}}{\beta^{\gamma}}\phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right),$$

with $\beta > 0$ scale parameter and $\gamma > 0$ shape parameter. We denote this by $X \sim GHN(\beta,\gamma)$. When $\gamma = 1$ we obtain the

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HN distribution as a particular case; the representation of this model is given in the work of Olmos et al. [18].

Following the methodology used by Cooray and Ananda [11] to construct the GHN model (see also Proposition 1 in the work of Olmos et al. [18]), the main purpose of this article is to introduce a new distribution with positive support and two parameters, similar to the GHN model and with considerable flexibility, to model positive unimodal data in the areas of reliability and survival. In particular, we will work with the real data set used in [6],[7],[15], related to the lifetimes in 10-3 cycles of aluminum 6061-T6 pieces cut in parallel angle with the rotation direction, oscillating at the rate of 18 cycles per second at maximum pressure 21000 psi.

The article is organized as follows: Section 2 contains the stochastic representation of this family and we present the new density, its moments, and its skewness and kurtosis coefficients. In Section 3 we make the inference using methods for moment and maximum likelihood estimation, and present a simulation study. Section 4 contains an application with real data. Finally, Section 5 provides discussion of the new model.

2 Density function and properties

In this section we introduce the representation, basic properties, cumulative distribution and hazard rate functions of the new distribution. The following result is important for the construction of the new distribution.

Proposition 1. *Let $X \sim ASN(\alpha)$. Then, the pdf of $Y = |X|$ is given by*

$$f_Y(y;\alpha) = 2\left(\frac{2+\alpha^2y^2}{2+\alpha^2}\right)\phi(y), \qquad (4)$$

where $y > 0$ and $\alpha \geq 0$.

Proof. Using the Jacobian transformation, the density function associated with Y is given by

$$f_Y(y;\alpha) = f_X(y;\alpha) + f_X(-y;\alpha).$$

Using the density function given in (2) we obtain the result. □

Remark. We call the density function given in (4) half-alpha-skew-normal (HASN), and we denote it by $Y \sim HASN(\alpha)$. We observe that when $\alpha = 0$ it corresponds to the HN model as a particular case. We restrict the parametric space to the non-negative values in order to avoid identifiability problems.

2.1 Stochastic Representation

The stochastic representation of this new distribution is

$$Z = \beta Y^{1/\gamma}, \qquad (5)$$

where $\beta > 0$ is the scale parameter, $\gamma > 0$ the shape parameter and Y is a random HASN variable with parameter $\alpha = 5$. We call the distribution of the random variable Z: Modified generalized half-normal (MGHN). The choice of $\alpha = 5$ is adopted to avoid increasing unnecessarily the number of shape parameters, and to include a unimodal distribution as the base, which is not the case when $\alpha = 0$, as used to obtain the GHN model. In applications, the new family of distributions can provide a better fit than other likelihood distributions used previously for this purpose.

2.2 Density function

The following proposition shows the pdf of the MGHN distribution, which is generated using the representation given in (5).

Proposition 2. *Let $Z \sim MGHN(\beta,\gamma)$. Then, the pdf of Z is given by*

$$f_Z(z;\beta,\gamma) = \frac{2\gamma z^{\gamma-1}}{27\beta^{3\gamma}}\left(2\beta^{2\gamma}+25z^{2\gamma}\right)\phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right), \qquad (6)$$

where $z > 0$, $\beta > 0$ and $\gamma > 0$.

Proof. Using the representation given in (5) and the Jacobian transformation, the density function related to Z is given by

$$f_Z(z;\beta,\gamma) = \frac{\gamma}{\beta}\left(\frac{z}{\beta}\right)^{\gamma-1} f_Y\left(\left(\frac{z}{\beta}\right)^{\gamma};5\right).$$

Using the density function given in (4) we obtain the result. □

Figure 1 shows some of the forms taken by the MGHN density for different values of the shape parameter γ, fixing the scale parameter $\beta = 1$.

Proposition 3. *Let $V \sim ASN(\alpha)$ and $W \sim SCASN(\alpha)$. Then $|V| \overset{d}{=} |W| \sim HASN(\alpha)$.*

Proof. Using the Jacobian transformation, we obtain that the density of the random variable $|W|$ is the same as was obtained in (4). □

Remark. In the SN distribution we find that when $X \sim SN(\lambda)$, then $|X| \sim HN(0,1)$ which does not depend on shape parameter λ; while this does not occur when $X \sim ASN(\alpha)$ since the resulting distribution depends on parameter α as we observed in Proposition 1. The methodology used by Cooray and Ananda [11] to construct the GHN model, we now use to construct the MGHN model taking the HASN distribution as the base. The result of Proposition 3 is very important for generating random numbers from the MGHN distribution, as will be seen below in the simulation study.

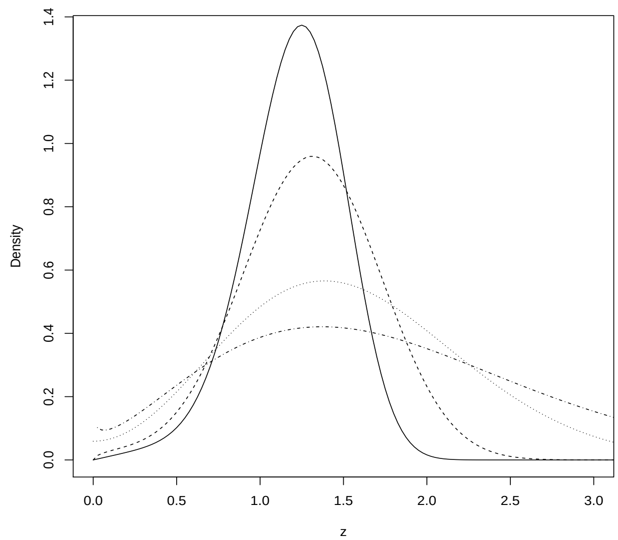

Fig. 1: Density function of Z for $MGHN(1,2)$ (solid line), $MGHN(1,1.5)$ (dashed line), $MGHN(1,1)$ (dotted line) and $MGHN(1,0.8)$ (dashed-dotted line)

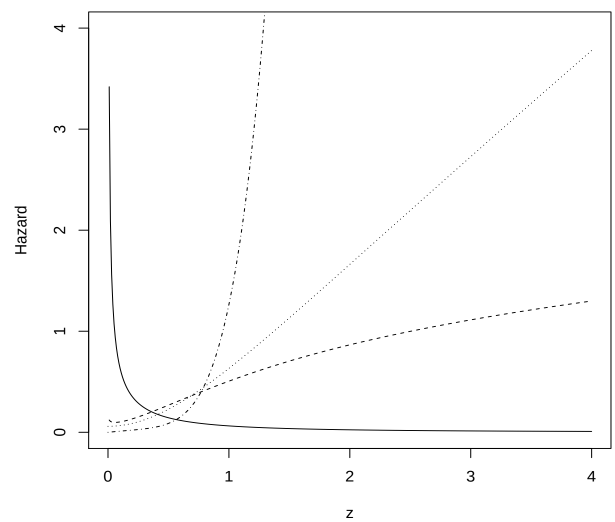

Fig. 2: Hazard function of Z for $MGHN(1,0.1)$ (solid line), $MGHN(1,0.8)$ (dashed line), $MGHN(1,1)$ (dotted line) and $MGHN(1,2)$ (dashed-dashed line)

Proposition 4. *Let $Z \sim MGHN(\beta,\gamma)$, where $\beta > 0$ and $\gamma > 0$. Then, the distribution function of Z is given by*

$$F_Z(z) = 2\Phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right) - \frac{50}{27}\left(\frac{z}{\beta}\right)^{\gamma}\phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right) - 1$$

Proof. Let $Z \sim MGHN(\beta,\gamma)$, then

$$F_Z(z) = \int_0^z f_Z(t;\beta,\gamma)dt = \frac{4\gamma}{27\beta^{\gamma}}\int_0^z t^{\gamma-1}\phi\left(\left(\frac{t}{\beta}\right)^{\gamma}\right)dt + \frac{50\gamma}{27\beta^{3\gamma}}\int_0^z t^{3\gamma-1}\phi\left(\left(\frac{t}{\beta}\right)^{\gamma}\right)dt.$$

Using the change of variable $u = \left(\frac{t}{\beta}\right)^{\gamma}$ we obtain the result. □

Corollary 1. *The hazard rate function for the random variable $Z \sim MGHN(\beta,\gamma)$ is given by*

$$\begin{aligned} h(z) &= \frac{f_Z(z)}{1-F_Z(z)} \\ &= \frac{\gamma z^{\gamma-1}(2\beta^{2\gamma}+25z^{2\gamma})\phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right)}{\beta^{2\gamma}\left[27\beta^{\gamma}+25z^{\gamma}\phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right)-27\beta^{\gamma}\Phi\left(\left(\frac{z}{\beta}\right)^{\gamma}\right)\right]}. \end{aligned}$$

Figure 2 shows the form of the hazard function for some values of the parameter γ.

2.3 Moments

Proposition 5. *Let $Z \sim MGHN(\beta,\gamma)$.Then for $r = 1,2,\ldots$ we have that the r-th moments are*

$$\mu_r = E(Z^r) = \frac{\beta^r(25r+27\gamma)2^{\frac{r}{2\gamma}}\Gamma\left(\frac{r+\gamma}{2\gamma}\right)}{27\sqrt{\pi}\gamma}. \quad (7)$$

Proof. Using the stochastic representation of this distribution given in (3) and $Y \sim HASN(5)$, we have

$$\begin{aligned} \mu_r &= E(Z^r) = E\left(\beta^r Y^{r/\gamma}\right) \\ &= \frac{2\beta^r}{27}\int_0^{\infty}(2y^{r/\gamma}+25y^{r/\gamma+2})\phi(y)dy \\ &= \frac{2\beta^r}{27}\int_0^{\infty}2y^{r/\gamma}\phi(y)dy + \frac{25\beta^r}{27}\int_0^{\infty}2y^{r/\gamma+2}\phi(y)dy. \end{aligned}$$

Using the moments of the HN distribution we obtain the result. □

Corollary 2. *Let $Z \sim MGHN(\beta,\gamma)$. Then, it follows that*

$$\mu_1 = E(Z) = \beta(25+27\gamma)a_1 \quad (8)$$
$$\mu_2 = E(Z^2) = \beta^2(50+27\gamma)a_2 \quad (9)$$
$$\mu_3 = E(Z^3) = \beta^3(75+27\gamma)a_3 \quad (10)$$
$$\mu_4 = E(Z^4) = \beta^4(100+27\gamma)a_4 \quad (11)$$

where $a_r = a_r(\gamma) = \frac{2^{\frac{r}{2\gamma}}\Gamma\left(\frac{r+\gamma}{2\gamma}\right)}{27\sqrt{\pi}\gamma}$.

Proof. It is a direct consequence of Proposition 5. □

Corollary 3. *Let $Z \sim MGHN(\beta,\gamma)$, then the skewness($\sqrt{\beta_1}$) and kurtosis($\beta_2$) coefficients are respectively:*

$$\sqrt{\beta_1} = \frac{\upsilon_3 - 3\upsilon_2\upsilon_1 + 2\upsilon_1^3}{\left(\upsilon_2 - \upsilon_1^2\right)^{3/2}},$$

$$\beta_2 = \frac{\upsilon_4 - 4\upsilon_1\upsilon_3 + 6\upsilon_1^2\upsilon_2 - 3\upsilon_1^4}{\left(\upsilon_2 - \upsilon_1^2\right)^2},$$

where $\upsilon_i = \upsilon_i(\gamma) = (25i + 27\gamma)a_i$, with $i = 1,2,3,4$.

Proof. The result is obtained by using the following definition for skewness and kurtosis coefficients.

$$\sqrt{\beta_1} = \frac{E\left(Z - E(Z)\right)^3}{(Var(Z))^{3/2}} = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}},$$
$$\beta_2 = \frac{E\left(Z - E(Z)\right)^4}{(Var(Z))^2} = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}. \square$$

3 Inference

In this Section, we discuss the moment and maximum likelihood estimators for the MGHN distribution. We also present a simulation study for the maximum likelihood estimators.

3.1 Moment estimators

Proposition 6. *Let $Z_1,\ldots,Z_n$ be a random sample of size n from the $Z \sim MGHN(\beta,\gamma)$ distribution. The moment estimator for β is given by*

$$\widehat{\beta}_M = \frac{\overline{Z}}{\upsilon_1(\widehat{\gamma}_M)}. \quad (12)$$

On the other hand, the moment estimator for γ, denoted by $\widehat{\gamma}_M$, is obtained as a solution of the numerical equation.

$$\overline{Z^2}\upsilon_1^2(\widehat{\gamma}_M) - \overline{Z}^2\upsilon_2(\widehat{\gamma}_M) = 0 \quad (13)$$

Proof. From Proposition 5 and considering the first two equations in the method of moments, we have

$$\overline{Z} = \beta\upsilon_1(\gamma) \quad \text{and} \quad \overline{Z^2} = \beta^2\upsilon_2(\gamma)$$

Solving the first equation above for β yields $\widehat{\beta}_M$. Replacing $\widehat{\beta}_M$ in the second equation above, the result given in (13) is obtained. $\square$

3.2 Maximum likelihood estimators

Given an observed sample $Z_1,\ldots,Z_n$ from the $MGHN(\beta,\gamma)$ distribution, the log-likelihood function for the parameters β and γ, given $\mathbf{Z} = (Z_1,...,Z_n)^\top$, can be written as

$$l(\beta,\gamma) = n\log\left(\frac{2\gamma}{27\sqrt{2\pi}\beta^{3\gamma}}\right) + (\gamma-1)\sum_{i=1}^n \log(z_i) \quad (14)$$
$$+ \sum_{i=1}^n \log(2\beta^{2\gamma} + 25z_i^{2\gamma}) - \frac{1}{2\beta^{2\gamma}}\sum_{i=1}^n z_i^{2\gamma}.$$

The maximum likelihood equations are given by

$$\sum_{i=1}^n \frac{4\beta^{4\gamma}}{2\beta^{2\gamma} + 25z_i^{2\gamma}} + \sum_{i=1}^n z_i^{2\gamma} = 3n\beta^{2\gamma} \quad (15)$$

$$\sum_{i=1}^n \log(z_i) + \sum_{i=1}^n \frac{4\beta^{2\gamma}\log(\beta) + 50z_i^{2\gamma}\log(z_i)}{2\beta^{2\gamma} + 25z_i^{2\gamma}} + \quad (16)$$
$$\frac{n}{\gamma} - \sum_{i=1}^n \left(\frac{z_i}{\beta}\right)^{2\gamma}\log\left(\frac{z_i}{\beta}\right) = 3n\log(\beta).$$

Solutions for equations (15) - (16) can be obtained by using numerical procedures such as the Newton-Raphson algorithm. Likewise, to obtain the maximum likelihood estimator (MLE) which will directly maximize the log-likelihood function given in (14), we can use existing software. For example, in R [19] this can be performed using the optim function. Initial values for the algorithm can be obtained based on the moment estimators given in Proposition 6.

3.3 Simulation study

In this subsection a simulation study was conducted with the main object of assessing the maximum likelihood estimation performance for parameters β and γ under the MGHN model. Below we present Algorithm 1, used to generate samples from $Z \sim MGHN(\beta,\gamma)$.

Algorithm 1

1. Generate $T \sim \chi_3^2$, $V \sim N(0,1)$ and W such that $P(W = 1) = P(W = -1) = \frac{1}{2}$.
2. Compute $R = \sqrt{T}W$
3. Compute $X = \frac{5}{\sqrt{27}}R + \sqrt{\frac{2}{27}}V$
4. Compute $Y = |X|$
5. Compute $Z = \beta Y^{1/\gamma}$

Using Algorithm 1, 10,000 random samples of sizes $n = 50$, 100 and 200 were generated under the MGHN model with different parameter values. A summary of the results from the study is depicted in Table 1. For each sample generated, MLEs were computed numerically using the Newton-Raphson procedure. We report the mean of the estimators, the mean of the estimated standard errors and the root mean squared errors

(RMSEs). We observe that as the sample size increases, the estimates are closer to the true values; moreover standard deviations and RMSEs draw closer as n increases.

Table 1: Simulation study for MGHN model in finite samples.

		$n=50$			$n=100$			$n=200$		
	True	Esti-mate	Mean of s.e.	RMSE	Esti-mate	Mean of s.e.	RMSE	Esti-mate	Mean of s.e.	RMSE
β	1	1.015	0.129	0.129	1.007	0.091	0.093	1.002	0.065	0.065
γ	0.8	0.824	0.097	0.101	0.811	0.067	0.069	0.804	0.047	0.047
β	1	1.011	0.103	0.105	1.005	0.073	0.074	1.002	0.052	0.052
γ	1	1.031	0.121	0.13	1.014	0.084	0.086	1.007	0.059	0.06
β	1	1.003	0.051	0.052	1.002	0.036	0.036	1.001	0.026	0.026
γ	2	2.057	0.242	0.254	2.03	0.168	0.173	2.013	0.118	0.12
β	1	1.002	0.034	0.035	1.001	0.024	0.025	1.001	0.017	0.017
γ	3	3.088	0.363	0.389	3.044	0.252	0.259	3.024	0.177	0.178
β	1	1.001	0.021	0.021	1.001	0.015	0.015	1	0.01	0.01
γ	5	5.15	0.606	0.637	5.081	0.421	0.435	5.038	0.294	0.296
β	5	5.068	0.643	0.672	5.03	0.456	0.46	5.018	0.324	0.324
γ	0.8	0.822	0.097	0.103	0.811	0.067	0.069	0.806	0.047	0.048
β	5	5.063	0.514	0.528	5.031	0.366	0.365	5.01	0.259	0.259
γ	1	1.032	0.121	0.13	1.014	0.084	0.085	1.007	0.059	0.06
β	5	5.021	0.256	0.263	5.011	0.182	0.185	5.005	0.129	0.129
γ	2	2.062	0.242	0.258	2.031	0.168	0.176	2.014	0.118	0.119
β	5	5.013	0.171	0.175	5.006	0.122	0.125	5.002	0.086	0.087
γ	3	3.093	0.363	0.389	3.045	0.252	0.264	3.02	0.176	0.181
β	5	5.005	0.103	0.104	5.003	0.073	0.074	5.002	0.052	0.052
γ	5	5.139	0.604	0.634	5.07	0.42	0.433	5.04	0.295	0.299

4 Application

In this Section, a real data set is used to show that the MGHN distribution can provide a better fit with the data set than the GHN and Birnbaum and Saunders (BS) distributions.

Below we present the results of a real data set analysis using a data set previously analyzed in Birnbaum and Saunders [6,7], related to the lifetimes in 10^{-3} cycles of aluminum $6061-T6$ pieces cut in parallel angle with the rotation direction, oscillating at the rate of 18 cycles per second at maximum pressure $21000\,psi$, with a total sample size of 101 units. The same data were used by Gómez et al. [15], and other studies. Table 2 shows some descriptive statistics from the data set, where b_1 and b_2 are sample skewness and kurtosis coefficients respectively.

Table 2: Descriptive statistics for the data set.

n	$\overline{x}$	s^2	b_1	b_2
101	1400.911	153134.5	0.140	2.766

Computing the moment estimators initially under the MGHN model, we have the following estimates: $\widehat{\beta}_M$ =1130.775 and $\widehat{\gamma}_M$ =1.765. Using the moment estimators as initial values, the maximum likelihood estimates are computed using a numerical method. Table 3 shows the MLEs for the parameters of the MGHN, GHN and BS models. For each model we report the log-likelihood estimate value. For these data, the MGHN model presented the largest value of the estimated log-likelihood function.

Table 3: ML estimates for fitting models on the data set.

Models	Parameters estimated	Standard error	Log-likelihood
MGHN	$\widehat{\beta}$ = 1106.407	47.060	**-746.558**
	$\widehat{\gamma}$ = 1.668	0.131	
GHN	$\widehat{\beta}$ = 1629.517	42.265	-748.598
	$\widehat{\gamma}$ = 2.994	0.235	
BS	$\widehat{\beta}$ = 1336.369	40.765	-751.391
	$\widehat{\gamma}$ = 0.310	0.022	

The AIC and BIC criteria were used to compare the distributions (Akaike, [2]; Schwarz, [20]). It is known that AIC$=2k-2\log lik$ and BIC$=k\log n-2\log lik$ where k is the number of parameters in the model, n is the sample size and $\log lik$ is the maximized value of the likelihood function. Table 4 shows the corresponding AIC and BIC for each model. For these data, the values in the table indicate that the MGHN distribution leads to a better fit than the GHN and BS distributions. Figure 3 presents the histogram for the data with the fitted densities; Figure 4 presents the empirical cdf with estimated cdf for MGHN, GHN and BS models, which also shows the good fit between the MGHN model and the lifetimes data.

Table 4: Akaike and Bayesian information criteria for fitted models.

Criterion	MGHN	GHN	BS
AIC	**1497.116**	1501.196	1506.782
BIC	**1502.346**	1506.426	1512.012

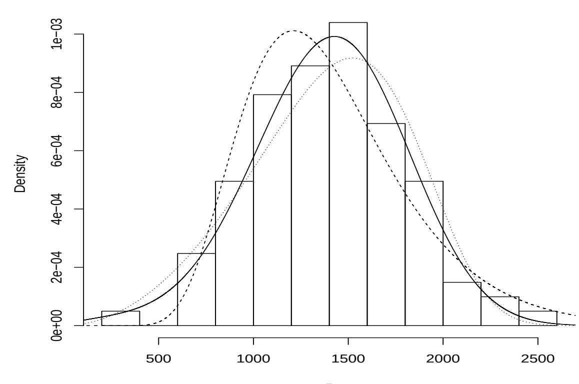

Fig. 3: Data histogram and fitted pdf for MGHN (solid line), GHN (dashed line) and BS (dotted line);

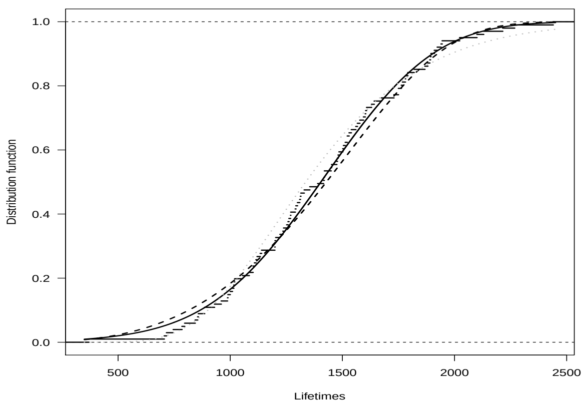

Fig. 4: Empirical cdf with estimated cdf for MGHN (solid line), estimated GHN (dashed line) and estimated BS (dotted line) models.

5 Discussion

In this paper we present a new model with two parameters, called Modified generalized half-normal (MGHN). It was defined following the same principle used to produce the GHN model, but instead of using the HN model as a base, we used a particular case of the HASN model. The new distribution appears to be a viable alternative for fitting reliability and survival data as shown in the application. Properties of MGHN distribution include its moments, skewness, kurtosis and stochastic representation. Parameters are estimated using the moments method and maximum likelihood. We present the results of a simulation study, which shows good parameter recovery for small samples. Some other characteristics of the new model are:

- The proposed model has a closed expression and one of its parts is a function of the GHN model.
- Stochastic representation of the MGHN model is very useful for calculating the most important properties of the model.
- The MGHN model offers a very flexible distribution, for example for modelling lifetime data such as the data set used in the application. The MGHN distribution can accommodate both decreasing and increasing failure rates as shown in Figure 2.
- The moments and the skewness and kurtosis coefficients are closed expressions and are expressed in terms of the gamma function.
- In the application, we used the material fatigue data used in Birnbaum and Saunders (1969b) and two model comparison criteria (AIC and BIC). The two criteria indicate that the model with the best fit for this dataset is the MGHN model.

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