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Stochastic Second-Order Cone Programming: The Equivalent Convex Program

*Mohammad Alabed Alhadi** *and Baha Alzalg*

Department of Mathematics, School of Science, The University of Jordan, Amman 11942, Jordan

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Abstract: The two-stage stochastic second-order cone programming (SSOCP) has been recently introduced in [Appl. Math. Model. 63, 5122–5134 (2012)] to cover a lot of important applications that cannot be captured by the two-stage stochastic linear programming (SLP). Wets [SIAM J. Applied Math. 14, 89–105 (1966)] described and characterized the equivalent convex program of the two-stage SLP. There is no work discussing the equivalent convex program of the two-stage SSOCP. The purpose of this short paper is to describe and characterize the equivalent convex program of the SSOCP problem. We first discuss the properties of the solution set of the SSOCP, and then develop the convex program equivalent to the SSOCP. We show that the objective function of the equivalent convex program is convex and continuous.

Keywords: Linear programming, Stochastic programming, Recourse, Second-order cone programming

1 Introduction

A diverse set of real-world applications can be modeled as stochastic second-order cone programming (SSOCP) problems as Euclidean facility location problem, portfolio optimization with loss risk constraints, optimal covering random ellipsoid problem, and mobile ad hoc networks [1–3]. Since a linear inequality is a special case of a second-order cone inequality, a two-stage SSOCP includes a two-stage stochastic linear program (SLP) as a special case. Several interior-point algorithms have been proposed for solving SSOCP (see for example [4–8]).
Wets [9] described and characterized the equivalent convex program of a two-stage SLP. To our knowledge, there is no work discussing the equivalent convex program of an SSOCP. This paper is devoted to extend the results of Wets [9] for SLP to the case of SSOCP. We also mention that recently, Yang [10, Chapter 2] extended the results of Wets [9] to the case of a two-stage stochastic semidefinite programming.

This paper is organized as follows. In Section 2, we present the standard form of the primal SSOCP. Section 3 discusses the properties of the solution set of an SSOCP. In Section 4, we develop the convex program equivalent to the SSOCP, we then discuss the properties of the objective function of the equivalent convex program and derive a supporting set for this objective function similar to the concept of supporting hyperplane in linear programming.

2 Formulation of primal SSOCP

As mentioned earlier, in this section we present the standard form of the primal SSOCP. To do so, we first introduce notations of the algebra of the second-order cone. These notations follow that of Alizadeh and Goldfarb [11].

2.1 Notations of the algebra of the second-order cone

We use "," for adjoining vectors and matrices in a row, and use ";" for adjoining them in a column. So, for example, if $\boldsymbol{x},\boldsymbol{y}$, and $\boldsymbol{z}$ are vectors, we have:

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = (\boldsymbol{x}^{\mathsf{T}},\boldsymbol{y}^{\mathsf{T}},\boldsymbol{z}^{\mathsf{T}})^{\mathsf{T}} = (\boldsymbol{x};\boldsymbol{y};\boldsymbol{z}).$$

For each vector $\boldsymbol{x} \in \mathbb{R}^n$ whose first entry is indexed with 0, we write $\tilde{\boldsymbol{x}}$ for the subvector consisting of entries 1 through

* Corresponding author e-mail: mhm9140083@fgs.ju.edu.jo

$n-1$; therefore $\boldsymbol{x} = (x_0; \tilde{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. We let $\mathscr{E}^n$ denote the n dimensional real vector space $\mathbb{R} \times \mathbb{R}^{n-1}$ whose first component of element $\boldsymbol{x}$ is indexed with 0.

The *second-order cone* of dimension n is defined as $\mathscr{E}^n_+ := \{\boldsymbol{x} \in \mathscr{E}^n : x_0 \geq \|\bar{\boldsymbol{x}}\|\}$, where $\|\cdot\|$ denotes the standard Euclidean norm.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{E}^n$. We will use the following symbols from the theory of Euclidean Jordan algebras associated with the second-order cone:

$\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y} := \text{trace}(\boldsymbol{x} \circ \boldsymbol{y})$ for the *Frobenius inner product* of $\boldsymbol{x}$ and $\boldsymbol{y}$;
$\|\boldsymbol{x}\|_F := \sqrt{2}\|\boldsymbol{x}\|$ for the *Frobenius norm* of $\boldsymbol{x}$;
$\boldsymbol{e}$ for the *identity element* of $\mathscr{E}^n$;
$\boldsymbol{x} \succeq \boldsymbol{0}$ for $\boldsymbol{x} \in \mathscr{E}^n_+$ (i.e., $\lambda_1, \lambda_2 \geq 0$);
$\boldsymbol{x} \succ \boldsymbol{0}$ for $\boldsymbol{x} \in \text{int}(\mathscr{E}^n_+)$ (i.e., $\lambda_1, \lambda_2 > 0$);
$\boldsymbol{x} \succeq \boldsymbol{y}$ or $\boldsymbol{y} \preceq \boldsymbol{x}$ for $\boldsymbol{x} - \boldsymbol{y} \succeq \boldsymbol{0}$.

The above notions are also used in the block sense as follows. Let $\boldsymbol{x} = (\boldsymbol{x}_1; \boldsymbol{x}_2; \ldots; \boldsymbol{x}_r)$ and $\boldsymbol{y} = (\boldsymbol{y}_1; \boldsymbol{y}_2; \ldots; \boldsymbol{y}_r)$, with $\boldsymbol{x}_i, \boldsymbol{y}_i \in \mathscr{E}^{n_i}$ for $i = 1, 2, \ldots, r$. Then

$$
\begin{aligned}
&\mathscr{E} := \mathscr{E}^{n_1} \times \mathscr{E}^{n_2} \times \cdots \times \mathscr{E}^{n_r};\\
&\mathscr{E}_+ := \mathscr{E}^{n_1}_+ \times \mathscr{E}^{n_2}_+ \times \cdots \times \mathscr{E}^{n_r}_+;\\
&\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y} := \boldsymbol{x}_1{}^{\mathsf{T}}\boldsymbol{y}_1 + \boldsymbol{x}_2{}^{\mathsf{T}}\boldsymbol{y}_2 + \cdots + \boldsymbol{x}_r{}^{\mathsf{T}}\boldsymbol{y}_r;\\
&\|\boldsymbol{x}\|_F^2 := \textstyle\sum_{i=1}^r \|\boldsymbol{x}_i\|_F^2;\\
&\boldsymbol{e} := (\boldsymbol{e}_1; \boldsymbol{e}_2; \ldots; \boldsymbol{e}_r) \text{ is the identity of } \mathscr{E}.
\end{aligned}
$$

We write the multiple-block second-order cone inequality $\boldsymbol{x} \succeq \boldsymbol{0}$ ($\boldsymbol{x} \succ \boldsymbol{0}$) to mean that the vector $\boldsymbol{x} \in \mathscr{E}_+$ ($\boldsymbol{x} \in \text{int}(\mathscr{E}^n_+)$). It is immediately seen that, for every vector $\boldsymbol{x} \in \mathscr{E}$, $\boldsymbol{x} \succeq \boldsymbol{0}$ if and only if $\boldsymbol{x}$ is partitioned conformally as $\boldsymbol{x} = (\boldsymbol{x}_1; \boldsymbol{x}_2; \ldots; \boldsymbol{x}_r)$ with $\boldsymbol{x}_i \succeq \boldsymbol{0}$ for $i = 1, 2, \ldots, r$. We also write $\boldsymbol{x} \succeq \boldsymbol{y}$ or $\boldsymbol{y} \preceq \boldsymbol{x}$ to mean that $\boldsymbol{x} - \boldsymbol{y} \succeq \boldsymbol{0}$.

2.2 Formulation of primal SSOCP

Let $r_1, r_2 \geq 1$ be integers. For each $i = 1, 2, \ldots, r_1$ and $j = 1, 2, \ldots, r_2$, let $m_1, m_2, n_1, n_2, n_{1,i}, n_{2,j}$ be positive integers such that $n_1 = \sum_{i=1}^{r_1} n_{1,i}$ and $n_2 = \sum_{j=1}^{r_2} n_{2,j}$. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}$ and $\boldsymbol{d}$ be vectors, and A, T and W be matrices such that they are partitioned conformally as follows:

$$
\begin{aligned}
\boldsymbol{x} &:= (\boldsymbol{x}_1; \boldsymbol{x}_2; \ldots; \boldsymbol{x}_{r_1}), \text{ where } \boldsymbol{x}_i \in \mathscr{E}^{n_{1,i}};\\
\boldsymbol{y} &:= (\boldsymbol{y}_1; \boldsymbol{y}_2; \ldots; \boldsymbol{y}_{r_2}), \text{ where } \boldsymbol{y}_j \in \mathscr{E}^{n_{2,j}};\\
\boldsymbol{c} &:= (\boldsymbol{c}_1; \boldsymbol{c}_2; \ldots; \boldsymbol{c}_{r_1}), \text{ where } \boldsymbol{c}_i \in \mathscr{E}^{n_{1,i}};\\
\boldsymbol{d} &:= (\boldsymbol{d}_1; \boldsymbol{d}_2; \ldots; \boldsymbol{d}_{r_2}), \text{ where } \boldsymbol{d}_j \in \mathscr{E}^{n_{2,j}};\\
A &:= (A_1, A_2, \ldots, A_{r_1}), \text{ where } A_i \in \mathbb{R}^{m_1 \times n_{1,i}};\\
T &:= (T_1, T_2, \ldots, T_{r_1}), \text{ where } T_i \in \mathbb{R}^{m_2 \times n_{1,j}};\\
W &:= (W_1, W_2, \ldots, W_{r_2}), \text{ where } W_j \in \mathbb{R}^{m_2 \times n_{2,j}}.
\end{aligned}
$$

The standard form of the primal SSOCP (P-SSOCP) [1] is:

$$
\begin{array}{llllr}
\min\ z(\boldsymbol{x}) = & \displaystyle\sum_{i=1}^{r_1} \boldsymbol{c}_i^{\mathsf{T}}\boldsymbol{x}_i & + \ \mathbb{E}_{\boldsymbol{\xi}}\Big\{\displaystyle\sum_{j=1}^{r_2} \boldsymbol{d}_j^{\mathsf{T}}\boldsymbol{y}_j\Big\} & & \\
\text{s.t.} & \displaystyle\sum_{i=1}^{r_1} A_i\boldsymbol{x}_i & & = \boldsymbol{b}, & (1)\\
& \displaystyle\sum_{i=1}^{r_1} T_i\boldsymbol{x}_i & + \displaystyle\sum_{j=1}^{r_2} W_j\boldsymbol{y}_j & = \boldsymbol{\xi}, & \\
& \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r_1}\ , & \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{r_2} & \succeq \boldsymbol{0}, &
\end{array}
$$

where $A \in \mathbb{R}^{m_1 \times n_1}, T \in \mathbb{R}^{m_2 \times n_1}, W \in \mathbb{R}^{m_2 \times n_2}, \boldsymbol{c} \in \mathbb{R}^{n_1}, \boldsymbol{d} \in \mathbb{R}^{n_2}$ and $\boldsymbol{b} \in \mathbb{R}^{m_1}$ are deterministic data. Here $\boldsymbol{\xi} \in \mathbb{R}^{m_2}$ is a random vector defined on the probability space $(\Xi, \mathbb{F}, F)$, where Ξ is a simple space, $\mathbb{F}$ is a set of events, and F is the probability measure. The variable $\boldsymbol{x} \in \mathbb{R}^{n_1}$ is the first-stage decision variable and the variable $\boldsymbol{y} \in \mathbb{R}^{n_2}$ is the second-stage decision variable.

Problem (1) can be compactly rewritten in the following form

$$
\begin{array}{lllr}
\min\ z(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}_{\boldsymbol{\xi}}\{\boldsymbol{d}^{\mathsf{T}}\boldsymbol{y}\} & & & \\
\text{s.t.} \qquad A\boldsymbol{x} & & = \boldsymbol{b}, & \\
\qquad\quad T\boldsymbol{x} \ + & W\boldsymbol{y} & = \boldsymbol{\xi}, & (2)\\
\qquad\quad\ \boldsymbol{x}\ \ , & \boldsymbol{y} & \succeq \boldsymbol{0}. &
\end{array}
$$

Problem (2) is a two-stage second-order cone program under uncertainty. We assume that (2) is solvable. Its objective function in fact can be written as

$$
z(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}_{\boldsymbol{\xi}}\{\min \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} \,|\, \boldsymbol{x}\}, \tag{3}
$$

where "$|\boldsymbol{x}$" means "given $\boldsymbol{x}$".

3 The set of feasible solutions

The focus is on the current (here-and-now) decision. The only decision variable for P-SSOCP is $\boldsymbol{x}$, since once $\boldsymbol{x}$ is taken and $\boldsymbol{\xi}$ is observed, the set of optimal second-stage decisions $\boldsymbol{y}$ is determined uniquely by solving the following second-order cone program

$$
\begin{array}{llr}
\min\ z(\boldsymbol{x}) = \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} & & \\
\text{s.t.} \qquad\quad W\boldsymbol{y} & = \boldsymbol{\xi} - T\boldsymbol{x}, & (4)\\
\qquad\qquad\quad \boldsymbol{y} & \succeq \boldsymbol{0}. &
\end{array}
$$

A feasible solution to P-SSOCP is a vector $\boldsymbol{x}$ that satisfies the first-stage constraints and such that it is always possible to find a feasible solution $\boldsymbol{y}$ to the second-stage problem. We now define

$$
\begin{aligned}
K_1 &:= \{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \succeq \boldsymbol{0}\}, \text{ and}\\
K_2 &:= \{\boldsymbol{x} : \text{for every } \boldsymbol{\xi} \in \Xi, \text{there exists } \boldsymbol{y} \succeq \boldsymbol{0}\\
&\qquad \text{such that } W\boldsymbol{y} = \boldsymbol{\xi} - T\boldsymbol{x}\}
\end{aligned}
$$

In linear programming, a positive spanned closed convex cone from a given matrix W can be defined as pos $(W) := \{W\boldsymbol{y} : \boldsymbol{y} \geq \boldsymbol{0}\}$. Similarly, in second-order cone programming we can define $\overset{+}{\mathrm{soc}}\,(W) := \{W\boldsymbol{y} : \boldsymbol{y} \succeq \boldsymbol{0}\}$. It is easy to see that $\overset{+}{\mathrm{soc}}\,(W)$ is a pointed, closed, convex cone generated positively by the points of $\mathbb{R}^{m_2}$ corresponding to the column of W. Accordingly, we can rewrite K_2 as

$$K_2 = \{\boldsymbol{x} : \boldsymbol{\xi} - T\boldsymbol{x} \in \overset{+}{\mathrm{soc}}\,(W) \text{ for all } \boldsymbol{\xi} \in \Xi\}.$$

It is easy to see that K_1 and K_2 are convex sets. We therefore have the following proposition about the convexity of the set of feasible solutions of (2).

Proposition 1. *The set of feasible solutions of (2), $K := K_1 \cap K_2$, is convex.*

For simplicity, we assume that K has full dimension. Due to the complexity of K_2 when Ξ is a continuum, we examine our problem under the following assumption: **Assumption 1.** We assume that the set Ξ has a finite number of points (card $|\Xi| < \infty$), denoted by $\boldsymbol{\xi}^{(j)}$ for $j = 1, 2, \ldots, k$.

Under Assumption 1, we have

$$K_2 = \{\boldsymbol{x} : \text{for each } j = 1, 2, \ldots, k, \text{there exists a } \boldsymbol{y}^{(j)} \succeq \boldsymbol{0} \text{ such that } T\boldsymbol{x} + W\boldsymbol{y}^{(j)} = \boldsymbol{\xi}^{(j)}\}.$$

We can also conclude three propositions on the feasibility of the second-stage problem (4). First, for $\boldsymbol{x}$ and $\boldsymbol{\xi} \in \Xi$, we define

$$U(\boldsymbol{x}, \boldsymbol{\xi}) := \min\{(\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u} : W^{\mathsf{T}}\boldsymbol{u} \succeq \boldsymbol{0}\}.$$

Next, we need Farkas' lemma for second-order cone programming (see [12, Chapter 14]).

Lemma 1. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $\boldsymbol{b} \in \mathbb{R}^m$. Then either:*

1. there is an $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \succeq \boldsymbol{0}$; or
2. there is a $\boldsymbol{y} \in \mathbb{R}^m$ such that $A^{\mathsf{T}}\boldsymbol{y} \succeq \boldsymbol{0}$ and $\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} < 0$.

Proposition 2. *$\boldsymbol{x} \in K_2$ if and only if for every $\boldsymbol{\xi} \in \Xi$ we have $U(\boldsymbol{x}, \boldsymbol{\xi}) \geq 0$.*

Proof. For a given $\boldsymbol{x}_0$ and for every $\boldsymbol{\xi} \in \Xi$, the inequality $U(\boldsymbol{x}_0, \boldsymbol{\xi}) \geq 0$ implies that the inequalities $(\boldsymbol{\xi} - T\boldsymbol{x}_0)^{\mathsf{T}}\boldsymbol{u} < 0$ and $W^{\mathsf{T}}\boldsymbol{u} \succeq \boldsymbol{0}$ have no solutions. By Farkas' lemma, the system $W\boldsymbol{y} = \boldsymbol{\xi} - T\boldsymbol{x}_0$ has a solution $\boldsymbol{y} \succeq \boldsymbol{0}$ for all $\boldsymbol{\xi} \in \Xi$. Hence $\boldsymbol{x}_0 \in K_2$, vice versa. □

Combining Lemma 1 and Proposition 2, we get the following two propositions.

Proposition 3. *$\boldsymbol{x} \in K$ if and only if $\boldsymbol{x} \in K_1$ and $U(\boldsymbol{x}, \boldsymbol{\xi}) \geq 0$ for every $\boldsymbol{\xi} \in \Xi$.*

Proposition 4. *$\boldsymbol{x} \in K$ if and only if $\boldsymbol{x} \in K_1$ and $U(\boldsymbol{x}, \boldsymbol{\alpha}) \geq 0$, where $\boldsymbol{\alpha}$ is any lower bound of $\boldsymbol{\xi}$.*

However, we note that $U(\boldsymbol{x}, \boldsymbol{\alpha})$ is not necessarily an upper bound as the argument in Wets [9, p.96] shows. Let $\hat{\boldsymbol{u}}$ be an optimal solution to

$$\begin{aligned} &\min\ (\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u} \\ &\text{s.t.}\ \ W^{\mathsf{T}}\boldsymbol{u} \succeq \boldsymbol{0}. \end{aligned}$$

Suppose that we have $\boldsymbol{x}_0$ such that $\boldsymbol{x}_0 \in K_1$ and $U(\boldsymbol{x}_0, \boldsymbol{\alpha}) < 0$ (equivalently $\boldsymbol{\alpha}^{\mathsf{T}}\hat{\boldsymbol{u}} < (T\boldsymbol{x}_0)^{\mathsf{T}}\hat{\boldsymbol{u}}$). By Proposition 4, $\boldsymbol{x}_0 \notin K$. Therefore, every $\boldsymbol{x} \in K$ must satisfy the inequality $(T\boldsymbol{x})^{\mathsf{T}}\hat{\boldsymbol{u}} < \boldsymbol{\alpha}^{\mathsf{T}}\hat{\boldsymbol{u}}$. This constraint can be added to the fixed constraints of K_1, $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \succeq \boldsymbol{0}$, and it cuts off part of the set K_1.

Now, we are ready to discuss an equivalent convex program problem for a SSOCP by extending the result of [9, Section 3].

4 The equivalent convex programming problem

In this section, we show that a P-SSOCP can be expressed, in terms of the first-stage decision variable $\boldsymbol{x}$, as a convex program that it would be called the equivalent convex programming problem. As mentioned earlier, we also derive the properties of the objective function of the equivalent convex program and derive a supporting set for this objective function similar to the concept of supporting hyperplane in linear programming. We then provide a generalized gradient formula for the objective function.

Definition 1. *A programming problem, $\min f(\boldsymbol{x}); \boldsymbol{x} \in K$, is an equivalent convex programming problem to P-SSOCP, if K is the set of feasible solutions to P-SSOCP, if $f(\boldsymbol{x})$ is given explicitly for each $\boldsymbol{x}$, and if an optimal solution to the equivalent programming problem is an optimal solution to P-SSOCP.*

We are now in a position to state and prove the main theorem in this paper.

Theorem 1. *The programming problem:*

$$\begin{aligned} &\min\ z(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}_{\boldsymbol{\xi}}\{\min \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} \,|\, \boldsymbol{x}\} \\ &\text{s.t.}\quad \boldsymbol{x} \in K. \end{aligned} \tag{5}$$

is an equivalent convex program to P-SSOCP.

Proof. Note that K is the set of feasible solutions to P-SSOCP. We need to prove that Program (5) is a convex program. Since the set K is convex (see Proposition (1)), it suffices to show that the objective function $z(\boldsymbol{x})$ is convex in $\boldsymbol{x}$.

First, we show that Program (5) can be written as

$$\begin{aligned} &\min\ z(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathscr{Q}(\boldsymbol{x}) \\ &\text{s.t.}\quad \boldsymbol{x} \in K. \end{aligned} \tag{6}$$

where

$$\mathscr{Q}(\boldsymbol{x}) := \mathbb{E}_{\boldsymbol{\xi}}\{Q(\boldsymbol{x},\boldsymbol{\xi})\} = \int_{\boldsymbol{\xi}\in\Xi} Q(\boldsymbol{x},\boldsymbol{\xi})dF(\boldsymbol{\xi})$$

and, for fixed $\boldsymbol{\xi} \in \Xi$

$$Q(\boldsymbol{x},\boldsymbol{\xi}) := \max\{(\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u} : W^{\mathsf{T}}\boldsymbol{u} \preceq \boldsymbol{d}\}.$$

(Here, for simplicity, we assume that $|Q(\boldsymbol{x},\boldsymbol{\xi})| < \infty$).

Let us consider the second-stage (4) for a fixed $\boldsymbol{\xi} \in \Xi$, as a function of $\boldsymbol{x}$, and define

$$P(\boldsymbol{x},\boldsymbol{\xi}) := \min\{\boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} : W\boldsymbol{y} = \boldsymbol{\xi} - T\boldsymbol{x},\ \boldsymbol{y} \succeq \boldsymbol{0}\}$$

Without loss of generality, we assume here that the primal problem is strictly feasible. So by the strong duality theorem for second-order cone programming [11], we have $P(\boldsymbol{x},\boldsymbol{\xi}) = Q(\boldsymbol{x},\boldsymbol{\xi})$. Therefore, Program (5) can be written as Program (6).

Define also

$$\begin{aligned}\mathscr{T} &:= \{\boldsymbol{x} : T\boldsymbol{x} = \boldsymbol{\xi} - W\boldsymbol{y},\ \text{for some } \boldsymbol{y} \succeq \boldsymbol{0}\}\\ &= \{\boldsymbol{x} : \boldsymbol{\xi} - T\boldsymbol{x} \in \overset{+}{\text{soc}}\,(W)\}.\end{aligned}$$

Next, we show that the function $P(\boldsymbol{x},\boldsymbol{\xi})$ is convex in $\boldsymbol{x}$ on the set $\mathscr{T}$, and in particular on K_2. It is easy to see that $\mathscr{T}$ is convex. Consequently, assume that $\boldsymbol{x}^{(0)},\boldsymbol{x}^{(1)} \in \mathscr{T}$, and $\lambda \in [0,1]$. Then $\boldsymbol{x}^{(\lambda)} := \lambda\boldsymbol{x}^{(0)} + (1-\lambda)\boldsymbol{x}^{(1)} \in \mathscr{T}$. Now, let $\boldsymbol{y}^{(0)},\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(\lambda)}$ be such that

$$\begin{aligned}P(\boldsymbol{x}^{(j)},\boldsymbol{\xi}) &:= \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y}^{(j)}\\ &= \min\{\boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} : W\boldsymbol{y} = \boldsymbol{\xi} - T\boldsymbol{x},\ \boldsymbol{y} \succeq \boldsymbol{0}\},\\ &\quad \text{for } j = 0,1,\lambda.\end{aligned}$$

Then $\tilde{\boldsymbol{y}} = \lambda\boldsymbol{y}^{(0)} + (1-\lambda)\boldsymbol{y}^{(1)}$ is a feasible solution, but in general, not an optimal solution to the second-stage optimization problem (4). Thus, $P(\boldsymbol{x},\boldsymbol{\xi})$ satisfies

$$\begin{aligned}P(\boldsymbol{x}^{(\lambda)},\boldsymbol{\xi}) &= \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y}^{(\lambda)}\\ &\le \boldsymbol{d}^{\mathsf{T}}\tilde{\boldsymbol{y}}\\ &= \lambda\boldsymbol{d}^{\mathsf{T}}\boldsymbol{y}^{(0)} + (1-\lambda)\boldsymbol{d}^{\mathsf{T}}\boldsymbol{y}^{(1)}\\ &= \lambda P(\boldsymbol{x}^{(0)},\boldsymbol{\xi}) + (1-\lambda)P(\boldsymbol{x}^{(1)},\boldsymbol{\xi}).\end{aligned}$$

Thus, $P(\boldsymbol{x},\boldsymbol{\xi})$ is a convex function in $\boldsymbol{x}$ on $\mathscr{T}$. Similar to LP cases in Dantzig [13] and Wets [9], we argue that since $P(\boldsymbol{x},\boldsymbol{\xi})$ is convex on K_2, then $\mathscr{Q}(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{\xi}}\{P(\boldsymbol{x},\boldsymbol{\xi})\}$ is convex on K_2. Thus, $z(\boldsymbol{x})$ is a convex function in $\boldsymbol{x}$. The result is established. □

Observe that we can extract the following corollaries from the proof of our main theorem.

Corollary 1. *The Program (6) is an equivalent convex program to P-SSOCP.*

Corollary 2. *The function $P(\boldsymbol{x},\boldsymbol{\xi})$ is convex in $\boldsymbol{x}$ on the set $\mathscr{T}$, and in particular on K_2.*

Corollary 3. *The function $\mathscr{Q}(\boldsymbol{x})$ is convex on K_2.*

We have shown that the objective function of the equivalent convex program is convex. Proposition 6 shows the continuity of the objective function of the equivalent convex program. We have the following proposition which will be used to prove Proposition 6.

Proposition 5. *The function $Q(\boldsymbol{x},\boldsymbol{\xi})$ is continuous in $\boldsymbol{x}$ on the set $\mathscr{T}$, and in particular on K_2.*

Proof. If there is only one point in set $\mathscr{T}$, then it is trivial to prove. Without loss of generality, we assume that there are at least two points in $\mathscr{T}$. We want to show that for all $\varepsilon > 0$ there exist a $\delta > 0$ such that for all $\boldsymbol{x} \in \mathscr{T}$, and an arbitrary $\boldsymbol{x}^{(0)} \in \mathscr{T}, \|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|_F < \delta \implies |Q(\boldsymbol{x},\boldsymbol{\xi}) - Q(\boldsymbol{x}^{(0)},\boldsymbol{\xi})| < \varepsilon$.

Let $\{W_{(i)}\}$ be a subcollection of the square nonsingular submatrices of W of rank m_2, such that $\{W_{(i)}\}$ are distinct and such that $\overset{+}{\text{soc}}\,(W_{(i)})$ constitute a covering of $\overset{+}{\text{soc}}\,(W)$. Let $\boldsymbol{d}_{(i)}$ be the subvector of $\boldsymbol{d}$ corresponding to the columns of W determining $W_{(i)}$. Then, for every $\boldsymbol{x} \in \mathscr{T}$ and $\boldsymbol{\xi} \in \Xi$, we have $\boldsymbol{\xi} - T\boldsymbol{x} \in \overset{+}{\text{soc}}\,(W_{(j)})$ for some index (j). Consequently, it follows that

$$Q(\boldsymbol{x},\boldsymbol{\xi}) = P(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{d}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{d}_{(j)}^{\mathsf{T}}W_{(j)}^{-1}(\boldsymbol{\xi} - T\boldsymbol{x}).$$

Assume that $\|\boldsymbol{d}_{(j)}\|_F > 0, \|W_{(j)}^{-1}\|_F > 0, \|T\|_F > 0$, and that $\|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|_F < \delta$. Then we have

$$\begin{aligned}|Q(\boldsymbol{x},\boldsymbol{\xi}) - Q(\boldsymbol{x}^{(0)},\boldsymbol{\xi})| &= \left|\boldsymbol{d}_{(j)}^{\mathsf{T}}W_{(j)}^{-1}T(\boldsymbol{x}^{(0)} - \boldsymbol{x})\right|\\ &\le \|\boldsymbol{d}_{(j)}\|_F\|W_{(j)}^{-1}\|_F\|T\|_F\|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|_F\\ &< \delta\ \|\boldsymbol{d}_{(j)}\|_F\|W_{(j)}^{-1}\|_F\|T\|_F\\ &= \delta\, M,\end{aligned} \tag{7}$$

where $M := \|\boldsymbol{d}_{(j)}\|_F\|W_{(j)}^{-1}\|_F\|T\|_F$. Let $\delta := \dfrac{\varepsilon}{M}$. This implies that $|Q(\boldsymbol{x},\boldsymbol{\xi}) - Q(\boldsymbol{x}^{(0)},\boldsymbol{\xi})| < \varepsilon$. The proof is complete. □

Proposition 6. *The function $\mathscr{Q}(\boldsymbol{x})$ is continuous on K_2.*

Proof. For all $\boldsymbol{x} \in K_2$, and an arbitrary $\boldsymbol{x}^{(0)} \in K_2$, assume that $\|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|_F < \delta$. Using (7), we have

$$\begin{aligned}|\mathscr{Q}(\boldsymbol{x}) - \mathscr{Q}(\boldsymbol{x}^{(0)})| &\le \int_{\boldsymbol{\xi}\in\Xi}\left|Q(\boldsymbol{x},\boldsymbol{\xi}) - Q(\boldsymbol{x}^{(0)},\boldsymbol{\xi})\right|dF(\boldsymbol{\xi})\\ &\le \|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|_F\int_{\boldsymbol{\xi}\in\Xi} M\,dF(\boldsymbol{\xi})\\ &< \delta M,\end{aligned}$$

where the last equality follows by noting that $M := \|\boldsymbol{d}_{(j)}\|_F\|W_{(j)}^{-1}\|_F\|T\|_F$ is independent of $\boldsymbol{\xi}$, and from the fact that $\int_{\boldsymbol{\xi}\in\Xi} dF(\boldsymbol{\xi}) = 1$. Having $\delta = \frac{\varepsilon}{M}$, it follows that $|\mathscr{Q}(\boldsymbol{x}) - \mathscr{Q}(\boldsymbol{x}^{(0)})| < \varepsilon$. The proof is complete. □

The dual of Problem (4) can be written as

$$\begin{array}{ll} \min & (\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u} \\ \text{s.t.} & W^{\mathsf{T}}\boldsymbol{u} \preceq \boldsymbol{d}. \end{array} \tag{8}$$

Let $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi})$ be an optimal solution to (8) for fixed $\boldsymbol{x}$ and $\boldsymbol{\xi}$. Strict feasibility guarantees that the optimal solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi})$ and the optimal value $Q(\boldsymbol{x},\boldsymbol{\xi})$ are defined for all $\boldsymbol{x} \in K$ and all $\boldsymbol{\xi} \in \Xi$. Let the vector

$$\boldsymbol{u}(\boldsymbol{x}) := \mathbb{E}_{\boldsymbol{\xi}}\{\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi})\} = \int_{\boldsymbol{\xi}\in\Xi} \boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi})\, dF(\boldsymbol{\xi}). \tag{9}$$

Note that $\boldsymbol{u}(\boldsymbol{x})$ is defined as an expected optimal solution to Problem (8) for a given $\boldsymbol{x}$. Let also the scalar

$$\phi(\boldsymbol{x}) := \mathbb{E}_{\boldsymbol{\xi}}\{\boldsymbol{u}^{\mathsf{T}}(\boldsymbol{x},\boldsymbol{\xi})\,\boldsymbol{\xi}\} = \int_{\boldsymbol{\xi}\in\Xi} \boldsymbol{u}^{\mathsf{T}}(\boldsymbol{x},\boldsymbol{\xi})\,\boldsymbol{\xi}\, dF(\boldsymbol{\xi}).$$

Proposition 7. *The set* $\left\{\boldsymbol{x} : (\boldsymbol{c} - T^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0))^{\mathsf{T}}\boldsymbol{x} = -\phi(\boldsymbol{x}_0)\right\}$ *is a supporting set of* $z(\boldsymbol{x})$ *at* $\boldsymbol{x} = \boldsymbol{x}_0$, *where* $\boldsymbol{x}_0 \in K$.

Proof. (Similar to the proof of [9, Proposition 27]). We need to show that for every $\boldsymbol{x} \in K$

$$0 = (\boldsymbol{c} - T^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0))^{\mathsf{T}}\boldsymbol{x} + \phi(\boldsymbol{x}_0) \leq (\boldsymbol{c} - T^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}))^{\mathsf{T}}\boldsymbol{x} + \phi(\boldsymbol{x}) = z(\boldsymbol{x}).$$

For every $\boldsymbol{x} \in K$ and $\boldsymbol{\xi} \in \Xi$, we have

$$(\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi}) \geq (\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0,\boldsymbol{\xi}).$$

(Since $\boldsymbol{u}(\boldsymbol{x}_0,\boldsymbol{\xi})$ is not a solution to (8), but $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi})$ is).

By integrating both sides with respect to $dF(\boldsymbol{\xi})$ and adding $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ on both sides, we get

$$\begin{aligned} z(\boldsymbol{x}) &= \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \int_{\boldsymbol{\xi}\in\Xi} (\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi})\, dF(\boldsymbol{\xi}) \\ &\geq \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \int_{\boldsymbol{\xi}\in\Xi} (\boldsymbol{\xi} - T\boldsymbol{x})^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0,\boldsymbol{\xi})\, dF(\boldsymbol{\xi}). \ \square \end{aligned}$$

From the definition of $z(\boldsymbol{x})$ and Proposition 7, we conclude the following corollary.

Corollary 4. $(\boldsymbol{c} - T^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0))$ *is a gradient of* $z(\boldsymbol{x})$ *at* $\boldsymbol{x} = \boldsymbol{x}_0$.

In fact, the function $z(\boldsymbol{x})$ is not necessarily differentiable. Hence we also conclude the following corollary.

Corollary 5. *Let* $\boldsymbol{x}_0 \in K$. *Then* $\boldsymbol{x}_0$ *is optimal if and only if there exists* $\boldsymbol{u}(\boldsymbol{x}_0)$ *such that for every* $\boldsymbol{x} \in K, (\boldsymbol{c} - T^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0))^{\mathsf{T}}\boldsymbol{x}_0 \leq (\boldsymbol{c} - T^{\mathsf{T}}\boldsymbol{u}(\boldsymbol{x}_0))^{\mathsf{T}}\boldsymbol{x}$.

5 Conclusion

The two-stage stochastic second-order cone programming is an important class of optimization problems that includes the two-stage stochastic linear programming as a special case. There is no work discussing the equivalent convex program of the stochastic second-order cone programming analogue to that of Wets [9] for the stochastic linear programming. In this short paper, we have discussed the properties of the solution set of the two-stage stochastic second-order cone programming, and have described and characterized its equivalent convex program. We have also shown that the objective function of the equivalent convex program is convex and continuous.

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Mohammad Alabed Alhadi is a PhD student in the Department of Mathematics at the University of Jordan. His research focus in Conic Optimization.

Baha Alzalg is an Associate Professor and the Chairman of Mathematics Department at the University of Jordan. He is an expert in Optimization area. His research focus is mainly in Interior Point Methods for Conic Optimization.