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Some Inequalities via Strongly p-Harmonic Log-Convex Stochastic Processes.

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Abstract: In this paper, we consider and investigate a new class of harmonic convex Stochastic Process, which is called the strongly p-harmonic log-convex Stochastic Process. We establish some new integral inequalities of Hermite-Hadamard type for the product of strongly p-harmonic log-convex Stochastic Process, and with this we obtain particularized results, some results are also obtained for the cases where one of the stochastic processes considered is of type strongly $p-$harmonic $s-$convex, strongly $p-$harmonic $P-$convex and strongly $p-$harmonic $h-$convex. Results obtained in this paper may be starting point for further research.

Keywords: Stochastic Process, Log convex, p-Harmonic.

1 Introduction

It is well known that modern analysis directly or indirectly involves the applications of convexity. Due to its applications and significant importance, the concept of convexity has been extended and generalized in several directions. The concept of convexity and its variant forms have played a fundamental role in the development of various fields. Convex functions are powerful tools for proving a large class of inequalities. They provide an elegant and unified treatment of the most important classical inequalities.

The following definition is well known in the literature as convex function: a function $f: I \subset R \to R$ is said to be convex if

$$f(tx+(1-t)y) \leq t f(x)+(1-t) f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.
The convexity of functions and their generalized forms play an important role in many fields such as Economic Science, Biology, Optimization. In recent years, several extensions, refinements, and generalizations have been considered for classical convexity [2,3,4,12,13,14,23, 28].

The harmonic convex function, was introduced and studied by Anderson et al. [1] and Iscan [5]. Iscan, in [7], introduced the concept of harmonic s-convex function in second sense. Noor et al. in [17,18] considered the strongly harmonic convex functions.

Noor, in [20], introduce and consider a new class of strongly p-harmonic log-convex function with modulus c, and obtain some new integral inequalities for product of this new class with other harmonic p-convex functions.

In the same way, the origins of the stochastic processes study come from the endings of 30's and the stochastic convexity appeared in [16], where Nikodem introduced this notion and some properties of convex stochastic processes were proved based on the definition of additive processes introduced by Nagy in 1974 (see [15]). Other authors like Skrowroński in 1992 obtained some further results in this area (see [26]), and it is possible to find results that involve generalizations of the concept of convexity and their corresponding applications. (see [9,10,24,25]).

Motivated by the work of Noor et. al. in [20], we present some new results that involves Strongly p-Harmonic Log-Convex Stochastic Process.

2 Preliminaries

In [20] we find the following definitions.

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Definition 1. *A set $I=[a,b]\subseteq\mathbb{R}-\{0\}$ is said to be harmonic $p-$convex set, where $p\neq 0$, if*

$$\left(\frac{x^py^p}{tx^p+(1-t)y^p}\right)^{1/p}\in I,\qquad \forall x,y\in I, t\in[0,1].$$

Definition 2. *A function $f:I\subset\mathbb{R}-\{0\}\to\mathbb{R}$ is said to be strongly $p-$harmonic log-convex function on I, if*

$$f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{1/p}\right) \tag{1}$$

$$\leq (f(x))^{1-t}(f(y))^t-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2,$$

for all $x,y\in I$ and $t\in[0,1]$.

Also, it is use to say that f is strongly $p-$harmonic log-concave function if $-f$ is strongly $p-$harmonic log-convex function.

When $t=1/2$ the strongly p-harmonic log-convex function reduces to

$$f\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{1/p}\right)\leq\sqrt{f(x)f(y)}-\frac{c}{4}\left(\frac{x^p-y^p}{x^py^p}\right)^2,$$

for all $x,y\in I$, and it is called Jensen type strongly $p-$harmonic log-convex function.

From (1) we have

$$\begin{aligned} f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{1/p}\right) &\leq (f(x))^{1-t}(f(y))^t-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2\\ &\leq tf(x)+(1-t)f(y)-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2\\ &\leq \max\{f(x),f(y)\}-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2\end{aligned}$$

For $p=1$, Definition 2 reduces to the definition of strongly harmonic log-convex function introduced by Noor et al. [21], and for p=-1, the same definition reduces to the definition of strongly log-convex function, see [29].

The following notions corresponds to Stochastic Process and convex Stochastic Process and its generalizations.

Definition 3. *Let $(\Omega,\mathscr{A},P)$ be an arbitrary probability space. A function $X:\Omega\to\mathbb{R}$ is called a random variable if it is $\mathscr{A}$-measurable. Let $(\Omega,\mathscr{A},P)$ be an arbitrary probability space and let $T\subset\mathbb{R}$ be time. A collection of random variable $X(t,w), t\in T$ with values in $\mathbb{R}$ is called a stochastic processes.*

1. *If $X(t,w)$ takes values in $S=\mathbb{R}^d$ if is called vector-valued stochastic process.*
2. *If the time T can be a discrete subset of $\mathbb{R}$, then $X(t,w)$ is called a discrete time stochastic process.*
3. *If the time T is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a stochastic process with continuous time*

Throughout the paper we restrict our attention stochastic process with continuous time, i.e, index set $T=[0,+\infty)$.

Definition 4. *Set $(\Omega,\mathscr{A},P)$ be a probability space and $I\subset\mathbb{R}$ be an interval. We say that a stochastic process $X:I\times\Omega\to\mathbb{R}$ is*

1. *Convex if*
$$X(\lambda u+(1-\lambda)v,\cdot)\leq\lambda X(u,\cdot)+(1-\lambda)X(v,\cdot) \tag{2}$$
almost everywhere for all $u,v\in T$ and $\lambda\in[0,1]$. This class of stochastic process are denoted by C.
2. *m-convex if*
$$X(tu+m(1-t)v,\cdot)\leq tX(u,\cdot)+m(1-t)X(v,\cdot) \tag{3}$$
almost everywhere for all $u,v\in T$ and $t\in[0,1], m\in(0,1]$.

Definition 5. *Let (Ω,A,P) be a probability space and $T\subset\mathbb{R}$ be an interval. We say that the stochastic process $X:\Omega\to\mathbb{R}$ is called*

1. *Continuous in probability in interval I if for all $t_0\in T$*
$$P-\lim_{t\to t_0}X(t,\cdot)=X(t_0,\cdot)$$
where $P-\lim$ denotes the limit in probability;
2. *Mean-square continuous in the interval I if for all $t_0\in T$*
$$P-\lim_{t\to t_0}\mathbb{E}(X(t,\cdot)-X(t_0,\cdot))=0,$$
where $\mathbb{E}(X(t,\cdot))$ denote the expectation value of the random variable $X(t,\cdot)$;
3. *Increasing (decreasing) if for all $u,v\in I$ such that $t<s$,*
$$X(u,\cdot)\leq X(v,\cdot),(X(u,\cdot)\geq X(v,\cdot))\qquad (a.e.) \tag{4}$$
4. *Monotonic if it's increasing or decreasing;*
5. *Differentiable at a point $t\in I$ if there is a random variable*
$$X'(t,\cdot):T\times\Omega\to\mathbb{R}$$
such that
$$X'(t,\cdot)=P-\lim_{t\to t_0}\frac{X(t,\cdot)-X(t_0,\cdot)}{t-t_0}$$

We say that a stochastic process $X:I\times\Omega\to\mathbb{R}$ is continuous (differentiable) is it is continuous (differentiable) at every point of the interval I (See [11, 26]).

Definition 6. *Let (Ω, A, P) be a probability space $T \subset \mathbb{R}$ be an interval with $E(X(t)^2) < \infty$ for all $t \in T$. Let $[a,b] \subset T, a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a,b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1,2,...,n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[a,b]$ if the following identity holds:*

$$\lim_{n\to\infty} E[X(\theta_k(t_k - t_{k-1}) - Y)^2] = 0$$

Then we can write

$$\int_a^b X(t,\cdot)dt = Y(\cdot) \qquad (a.e.)$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt \leq \int_a^b Z(t,\cdot)dt \qquad (a.e.)$$

where $X(t,\cdot) \leq Z(t,\cdot)$ in $[a,b]$ ([27]).
In throughout paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.
Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes (see [11]).

Theorem 1. *If $X : T \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have*

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{u-v}\int_u^v X(t,\cdot)dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{2} \qquad (a.e.)$$

The following definitions will be the base for our results.

Definition 7. *A Stochastic process $X : I \times \Omega \to R$ is called to be strongly $p-$harmonic log-convex stochastic process on I for $c > 0$ if*

$$X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)$$

$$\leq (X(a,\cdot))^{1-t}(X((b,\cdot))^t - ct(1-t)\left(\frac{a^p+b^p}{a^pb^p}\right)^2,$$

almost everywhere for all $a,b \in I$ and $t \in [0,1]$.

Definition 8. *A Stochastic process $X : I \times \Omega \to R$ is called to be strongly $p-$harmonic $s-$convex stochastic process in second sense on I, for $s \in (0,1]$ and $c > 0$ if*

$$X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)$$

$$\leq t^sX(a,\cdot)+(1-t)^sX((b,\cdot) - ct(1-t)\left(\frac{a^p+b^p}{a^pb^p}\right)^2,$$

almost everywhere for all $a,b \in I$ and $t \in [0,1]$.

Definition 9. *A Stochastic process $X : I \times \Omega \to R$ is called to be strongly $p-$harmonic $P-$convex stochastic process for some $c > 0$ if*

$$X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)$$

$$\leq X(a,\cdot)+X((b,\cdot) - ct(1-t)\left(\frac{a^p+b^p}{a^pb^p}\right)^2,$$

almost everywhere for all $a,b \in I$ and $t \in [0,1]$.

Definition 10. *A Stochastic process $X : I \times \Omega \to R$ is called to be strongly $p-$harmonic $h-$convex stochastic process , for a non-negative and not identically zero function $h : (0,1) \to \mathbb{R}$, and $c > 0$ if*

$$X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)$$

$$\leq h(t)X(a,\cdot)+h(1-t)X((b,\cdot) - ct(1-t)\left(\frac{a^p+b^p}{a^pb^p}\right)^2,$$

almost everywhere for all $a,b \in I$ and $t \in [0,1]$.

3 Main Results

The following results are the objectives of the present work.

Theorem 2. *Let $X, Y : I \times \Omega \to F$ be a strongly $p-$harmonic log-convex stochastic process on $I = [a,b]$, respectively with modulus $c > 0$, and $(X(b,\cdot)/X(a,\cdot)), (Y(b,\cdot)/Y(a,\cdot))$ are positive real numbers for all $w \in \Omega$. If Y^q is a strongly $p-$harmonic log-convex stochastic process for $q \geq 1$then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx \tag{5}$$

$$\leq \left\{\frac{X(b,\cdot)-X(b,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)} - \frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right\}^{1-1/q} \times$$

$$\left\{\frac{X(b,\cdot)Y^q(b,\cdot)-X(a,\cdot)Y^q(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)-q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right.$$

$$-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{X(b,\cdot)-X(b,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)}+\frac{Y^q(b,\cdot)-Y^q(a,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right]$$

$$\left.+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\right\}^{1/q}. \quad (a.e.)$$

Proof. With the change of variables

$$x=\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p}$$

and using power mean inequality, and strongly $p-$harmonic log-convex property of X and Y, we have

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$=\int_0^1 X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)Y\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)dt$$

$$\leq\left[\int_0^1 X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)dt\right]^{1-1/q}$$

$$\times\left[\int_0^1\left(\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)\times\right.\right.$$

$$\left.\left.Y^q\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)\right)dt\right]^{1/q}$$

$$\leq\left\{\int_0^1\left[(X(a,\cdot))^{1-t}(X(b,\cdot))^t-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]dt\right\}^{1-1/q}$$

$$\times\left\{\int_0^1\left[(X(a,\cdot))^{1-t}(X(b,\cdot))^t-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\right.$$

$$\left.\times\left[(Y(a,\cdot))^{q(1-t)}(Y(b,\cdot))^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]dt\right\}^{1/q}$$

$$\leq\left\{X(a,\cdot)\int_0^1\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)^t dt-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1 t(1-t)dt\right\}^{1-1/q}$$

$$\times\left\{X(a,\cdot)Y^q(a,\cdot)\int_0^1\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)^t\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}dt\right.$$

$$-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\times$$

$$\int_0^1\left[X(a,\cdot)\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)^t+Y^q(a,\cdot)\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}\right]t(1-t)dt$$

$$\left.+c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4\int_0^1 t^2(1-t)^2dt\right\}^{1/q}$$

$$\leq\left\{\frac{X(b,\cdot)-X(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right\}^{1-1/q}$$

$$\times\left\{\frac{X(b,\cdot)Y^q(b,\cdot)-X(a,\cdot)Y^q(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)-q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right.$$

$$-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{X(b,\cdot)-X(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)}+\frac{Y^q(b,\cdot)-Y^q(a,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right]$$

$$\left.+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\right\}^{1/q}$$

This complete the proof.

Remark. a) If $q=1$ then (5) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\leq\frac{X(b,\cdot)Y(b,\cdot)-X(a,\cdot)Y(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)-\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}$$

$$-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{X(b,\cdot)-X(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)}+\frac{Y(b,\cdot)-Y(a,\cdot)}{\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right]$$

$$+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\qquad(a.e.)$$

b) If $c=0$ then (5) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\leq\left\{\frac{X(b,\cdot)-X(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)}\right\}^{1-1/q}\times$$

$$\left\{\frac{X(b,\cdot)Y^q(b,\cdot)-X(a,\cdot)Y^q(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)-q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right\}^{1/q}\qquad(a.e.)$$

c) If $q=1$ and $c=0$ then (5) reduces to

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx\leq\frac{X(b,\cdot)Y(b,\cdot)-X(a,\cdot)Y(a,\cdot)}{\ln\left(\frac{X(b,\cdot)}{X(a,\cdot)}\right)-\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}$$

almost everywhere.

Theorem 3. *Let $X,Y:I\times\Omega\to F$ be a $p-$harmonic $h-$convex stochastic process on $I=[a,b]$, and strongly $p-$harmonic log-convex stochastic process , respectively with modulus $c>0$. If Y^q is a strongly $p-$harmonic log-convex stochastic process for $q\geq1$ then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx\tag{6}$$

$$=\left\{(X(a,\cdot)+X(b,\cdot))\int_0^1 h(t)dt\right\}^{1-1/q}$$

$$\times\left\{Y^q(a,\cdot)X(a,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(1-t)dt\right.$$

$$+Y^q(a,\cdot)X(b,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(t)dt$$

$$\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2(X(a,\cdot)+X(b,\cdot))\int_0^1(h(1-t)+h(t))t(1-t)dt\right\}^q$$

almost everywhere.

Proof. With the change of variables

$$x=\left[\frac{a^p b^p}{ta^p+(1-t)b^p}\right]^{1/p}$$

and using power mean inequality, and $p-$harmonic $h-$convex property of X and strongly $p-$harmonic log-convex stochastic process of Y and Y^q, we have

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$=\int_0^1 X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)Y\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p},\cdot\right]^{1/p}\right)dt$$

$$\leq\left\{[\int_0^1 X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)dt\right\}^{1-1/q}$$

$$\times\left\{\int_0^1 X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)\times\right.$$

$$\left.Y^q\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)dt\right\}^{1/q}$$

$$\leq\left\{\int_0^1[h(1-t)X(a,\cdot)+h(t)X(b,\cdot)]dt\right\}^{1-1/q}$$

$$\times\left\{\int_0^1[h(1-t)X(a,\cdot)+h(t)X(b,\cdot)]\times\right.$$

$$\left.\left[(Y(a,\cdot))^{q(1-t)}(Y(b,\cdot))^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]dt\right\}^{1/q}$$

$$\leq\left\{X(a,\cdot)\int_0^1 h(1-t)dt+X(b,\cdot)\int_0^1 h(t)dt\right\}^{1-1/q}\times$$

$$\left\{Y^q(a,\cdot)\int_0^1\left[X(a,\cdot)\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(1-t)\right.\right.$$

$$\left.+X(b,\cdot)\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(t)\right]dt$$

$$\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2(X(a,\cdot)+X(b,\cdot))\int_0^1(h(1-t)+h(t))t(1-t)dt\right\}^q$$

and observing that

$$\int_0^1 h(1-t)dt=\int_0^1 h(t)dt$$

we can write

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\leq\left\{(X(a,\cdot)+X(b,\cdot))\int_0^1 h(t)dt\right\}^{1-1/q}$$

$$\times\left\{Y^q(a,\cdot)\int_0^1\left[X(a,\cdot)\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(1-t)\right.\right.$$

$$\left.+X(b,\cdot)\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(t)\right]dt$$

$$\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2(X(a,\cdot)+X(b,\cdot))\int_0^1(h(1-t)+h(t))t(1-t)dt\right\}^q$$

$$=\left\{(X(a,\cdot)+X(b,\cdot))\int_0^1 h(t)dt\right\}^{1-1/q}$$

$$\times\left\{Y^q(a,\cdot)X(a,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(1-t)dt\right.$$

$$+Y^q(a,\cdot)X(b,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(t)dt$$

$$\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2(X(a,\cdot)+X(b,\cdot))\int_0^1(h(1-t)+h(t))t(1-t)dt\right\}^q.$$

This complete the proof.

Remark. In Theorem 3 we can put $c=0$ and so obtain

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\leq\left\{(X(a,\cdot)+X(b,\cdot))\int_0^1 h(t)dt\right\}^{1-1/q}$$

$$\times\left\{Y^q(a,\cdot)X(a,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(1-t)dt\right.$$

$$\left.+Y^q(a,\cdot)X(b,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{qt}h(t)dt\right\}^{1/q}\qquad(a.e.)$$

and if, additionally, we put $q=1$ then we get

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\leq\left\{Y(a,\cdot)X(a,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{t}h(1-t)dt\right.$$

$$\left.+Y(a,\cdot)X(b,\cdot)\int_0^1\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)^{t}h(t)dt\right\}\qquad(a.e.)$$

Corollary 1. *Let $X,Y:I\times\Omega\to F$ be a $p-$harmonic $s-$convex stochastic process on $I=[a,b]$, and strongly $p-$harmonic log-convex stochastic process , respectively with modulus $c>0$. If Y^q is a strongly $p-$harmonic log-convex stochastic process for $q\geq 1$ then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\leq\left\{\frac{(X(a,\cdot)+X(b,\cdot))}{s+1}\right\}^{1-1/q}$$

$$\times\left\{X(a,\cdot)\left(\frac{Y^q(b,\cdot)-1}{(s+1)(s+2)}+\frac{Y^q(a,\cdot)}{s+1}\right)\right.$$

$$+X(b,\cdot)\left(\frac{Y^q(b,\cdot)-1}{(s+2)}+\frac{Y^q(a,\cdot)}{s+1}\right)$$

$$\left.-2c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left(\frac{X(a,\cdot)+X(b,\cdot)}{(s+2)(s+3)}\right)\right\}^{1/q},$$

almost everywhere.

Proof. If in Theorem 3 we take $h(t)=t^s, s\in(0,1)$ then we get the version for a $p-$harmonic $s-$convex stochastic processes in second sense. Let's see this

$$\int_0^1 h(t)dt=\int_0^1 t^s dt=\frac{1}{s+1},$$

$$\int_0^1 (h(1-t)+h(t))\ t\ (1-t)dt$$
$$=\int_0^1((1-t)^s+t^s)t(1-t)dt$$
$$=\int_0^1(1-t)^{s+1}tdt+\int_0^1(1-t)t^{s+1}dt$$

and since

$$\int_0^1(1-t)^{s+1}tdt=\int_0^1(1-t)t^{s+1}dt=\frac{1}{s+2}-\frac{1}{s+3},$$

we have

$$\int_0^1 (h(1-t)+h(t))t(1-t)dt=\frac{2}{(s+2)(s+3)}$$

Doing $u=(Y(b,\cdot)/Y(a,\cdot))$, we can observe from $u^q\le(u^q-1)t+1$, for all $t\in(0,1)$, that

$$\int_0^1 u^{qt}h(1-t)dt\le\int_0^1((u^q-1)t+1)(1-t)^s dt$$
$$=\frac{u^q-1}{(s+1)(s+2)}+\frac{1}{s+1}$$

and

$$\int_0^1 u^{qt}h(t)dt\le\int_0^1((u^q-1)t+1)t^s dt$$
$$=\frac{u^q-1}{(s+2)}+\frac{1}{s+1}.$$

Making the substitution in the inequality (7)we get, for X $p-$harmonic $s-$ convex stochastic process in second sense on $I=[a,b]$ and Y strongly $p-$harmonic log-convex stochastic process with modulus $c>0$, that

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\le\left\{\frac{(X(a,\cdot)+X(b,\cdot))}{s+1}\right\}^{1-1/q}$$

$$\times\left\{X(a,\cdot)\left(\frac{Y^q(b,\cdot)-1}{(s+1)(s+2)}+\frac{Y^q(a,\cdot)}{s+1}\right)\right.$$

$$+X(b,\cdot)\left(\frac{Y^q(b,\cdot)-1}{(s+2)}+\frac{Y^q(a,\cdot)}{s+1}\right)$$

$$\left.-2c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left(\frac{X(a,\cdot)+X(b,\cdot)}{(s+2)(s+3)}\right)\right\}^{1/q}.$$

The proof is complete.

Remark. If in Corollary 1 we make $c=0$ we have

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\le\left\{\frac{(X(a,\cdot)+X(b,\cdot))}{s+1}\right\}^{1-1/q}$$

$$\times\left\{X(a,\cdot)\left(\frac{Y^q(b,\cdot)-1}{(s+1)(s+2)}+\frac{Y^q(a,\cdot)}{s+1}\right)\right.$$

$$\left.+X(b,\cdot)\left(\frac{Y^q(b,\cdot)-1}{(s+2)}+\frac{Y^q(a,\cdot)}{s+1}\right)\right\}^{1/q}\quad(a.e.),$$

and, if in addition $q=1$ we obtain

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\le\left(\frac{Y(b,\cdot)-1}{(s+1)(s+2)}+\frac{Y(a,\cdot)}{s+1}\right)(X(a,\cdot)+X(b,\cdot))\qquad(a.e.).$$

Corollary 2. *Let $X,Y:I\times\Omega\to F$ be a $p-$harmonic $P-$convex stochastic process on $I=[a,b]$, and strongly $p-$harmonic log-convex stochastic process , respectively with modulus $c>0$. If Y^q is a strongly $p-$harmonic log-convex stochastic process for $q\ge1$ then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\le\{(X(a,\cdot)+X(b,\cdot))\}^{1-1/q}\left\{[X(a,\cdot)+X(b,\cdot)]\frac{Y^q(b,\cdot)-Y^q(a,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right.$$

$$\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left(\frac{X(a,\cdot)+X(b,\cdot)}{3}\right)\right\}^q.\quad(a.e.)$$

*Proof.*In Theorem 3 we make $h(t)=1, t\in(0,1)$ and obtain the desired result.

Remark. If in Corollary 2 we make $c=0$ we get

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\le\{(X(a,\cdot)+X(b,\cdot))\}^{1-1/q}\times$$

$$\left\{[X(a,\cdot)+X(b,\cdot)]\frac{Y^q(b,\cdot)-Y^q(a,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right\}^q\qquad(a.e.),$$

and if in addition $q=1$

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)Y(x,\cdot)}{x^{p+1}}dx$$

$$\le\left\{[X(a,\cdot)+X(b,\cdot)]\frac{Y(b,\cdot)-Y(a,\cdot)}{\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right\}\qquad(a.e.).$$

Theorem 4. *Let $X,Y : I\times\Omega\to F$ be a $p-$harmonic quasi convex stochastic process on $I=[a,b]$, and strongly $p-$harmonic log-convex stochastic process , respectively with modulus $c>0$. If Y^q is a strongly $p-$harmonic log-convex stochastic process for $q\geq 1$ then*

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)\,Y(x,\cdot)}{x^{p+1}}dx$$
$$\leq(\max\{X(a,\cdot),X(b,\cdot)\})\times$$
$$\left(\frac{Y^q(b,\cdot)-Y^q(b,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right)^{1/q}.\quad(a.e.)$$

Proof. With the change of variables

$$x=\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p}$$

and using power mean inequality, and $p-$harmonic quasi convex property of X and strongly $p-$harmonic log-convex stochastic process of Y and Y^q, we have

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)\,Y(x,\cdot)}{x^{p+1}}dx$$
$$=\int_0^1X\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1/p},\cdot\right)Y\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p},\cdot\right]^{1/p}\right)dt$$
$$\leq\left\{\int_0^1\max\{X(a,\cdot),X(b,\cdot)\}\,dt\right\}^{1-1/q}\times$$
$$\left\{\int_0^1[\max\{X(a,\cdot),X(b,\cdot)\}]\times\right.$$
$$\left.\left[(Y(a,\cdot))^{q(1-t)}(Y(b,\cdot))^{qt}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]dt\right\}^{1/q}$$
$$\leq(\max\{X(a,\cdot),X(b,\cdot)\})\times$$
$$\left(\frac{Y^q(b,\cdot)-Y^q(b,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{6}\right)^{1/q}.$$

The proof is complete.

Remark. If in Theorem 4 we make $c=0$ then we have

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)\,Y(x,\cdot)}{x^{p+1}}dx$$
$$\leq(\max\{X(a,\cdot),X(b,\cdot)\})\left(\frac{Y^q(b,\cdot)-Y^q(b,\cdot)}{q\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right)^{1/q},\quad(a.e.),$$

and if in addition $q=1$ we obtain

$$\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{X(x,\cdot)\,Y(x,\cdot)}{x^{p+1}}dx$$
$$\leq(\max\{X(a,\cdot),X(b,\cdot)\})\left(\frac{Y(b,\cdot)-Y(b,\cdot)}{\ln\left(\frac{Y(b,\cdot)}{Y(a,\cdot)}\right)}\right)\quad(a.e.).$$

4 Conclusions

We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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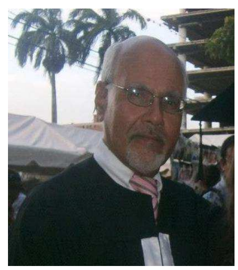

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