Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/120310

Estimation of KIW Parameters in Presence of S-SPALT: Bayesian and Non Bayesian Approach

Rashad M. EL-Sagheer[1,*] *and Nagwa M. Mohamed*[2]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
[2] Department of Mathematics, Faculty of Science, Suez University, Suez, Egypt

Received: 13 Feb. 2018, Revised: 10 Mar. 2018, Accepted: 15 Mar. 2018
Published online: 1 May 2018

Abstract: In this article, based on progressively Type-II censored schemes under step-stress partially accelerated life test model, the maximum likelihood, Bayes, and two parametric bootstrap methods are used for estimating the unknown parameters of the Kumaraswamy inverse Weibull distribution and the acceleration factor. Asymptotic confidence interval estimates of the model parameters and the acceleration factor are also evaluated by using Fisher information matrix. The classical Bayes estimators cannot be obtained in explicit form, so Markov chain Monte Carlo method is used to tackle this problem, which allows us to construct the credible interval of the involved parameters. Finally, analysis of a simulated data set has been also presented for illustrative purposes.

Keywords: Kumaraswamy inverse Weibull, Step-Stress partially accelerated life tests, Bootstrap confidence intervals, Markov chain Monte Carlo approach.

Acronym:

PT-2CS	Progressively Type-II Censored Schemes
ALT	Accelerated Life Test
PALT	Partially Accelerated Life Test
C-SPALT	Constant-Stress Partially Accelerated Life Test
S-SPALT	Step-Stress Partially Accelerated Life Test
MLEs	Maximum Likelihood Estimates
PDF	Probability Density Function
CDF	Cumulative Distribution Function
KIWD	Kumaraswamy Inverse Weibull Distribution
ACIs	Asymptotic Confidence Intervals
MCMC	Markov chain Monte Carlo
SEL	Squared Error Loss
Perc-pCIs	Percentile-p Bootstrap
Stud-tCIs	Studentized-t Bootstrap
CRI	Credible Interval
BPCIs	Bootstrap-P Confidence Intervals
BTCIs	Bootstrap-T Confidence Intervals

1 Introduction

Manufacturers work on producing reliable products which resist the failure under normal enviromental conditions by studying the failure causes, assessing new designs, testing products before marketing. Strong competition among manufacturers and the desire not to lose, leading to test products under severe conditions (stress), such as high temperatures and high voltages to emphasize product quality and reduce test time, such tests called accelerated life testing. ALT can be used to obtain the failure information of a product under accelerated stress conditions in a short time. However, in some situations, the accelerated function is unknown and the ALT is not available. In this case, the products can be tested first under normal conditions until the pre-fixed time and then the surviving products are changed to put in accelerated stress conditions, this case is called PALT.

There are two common types of PALT, C-SPALT and S-SPALT see Nelson [1]. In C-SPALT, the grouped test units are separately put in normal conditions and accelerated stress conditions. For S-SPALT, the surviving units in the experiment are shifted from normal conditions to higher stress conditions after a fixed time or a fixed failure number. Generally, stress is applied until the test unit fails or the test is terminated based on a certain censoring scheme, where the censoring scheme which is used in this article is PT-2CS. For more details on progressive censoring, we refer the reader to Balakrishnan and Aggarwala [2], Balakrishnan [3],

* Corresponding author e-mail: Rashadmath@azhar.edu.eg

Musleh and Helu [4], EL-Sagheer [5,6] and Mahmoud et al. [7,8,9]. The C-SPALT and S-SPALT have been addressed under different censoring schemes for example, see Ismail [10,11], EL-Sagheer [12], Abushal and Soliman [13], Abd-Elfattah et al. [14], Ismail [15], EL-Sagheer and Mohamed [16]. In this paper, we aim to study statistical inference based on PT-2CS in the presence of step-stress partially accelerated life tests.

The remainder of this paper is organized as follows: Section 2 provides a description of KIWD and the tampered random variable model. In Section 3 the MLEs of the parameters under consideration are estimated in addition to the corresponding ACIs. Section 4 concerns with two types of bootstrap confidence intervals. Section 5 is devoted to Bayesian approach that uses the famed MCMC technique. An illustrative example is developed to explain the theoretical results in Section 6. Eventually conclusion is inserted in Section 8.

2 Model description

A brief specification is given in this section about KIWD. Also, the transformed probability density function of KIWD under the tampered random variable model is presented.

2.1 Kumaraswamy inverse Weibull distribution

The KIWD was introduced by Shahbaz et al. [17]. This distribution is an extension of the IWD. The PDF, CDF, reliability function and hazard rate function of the four parameters KIWD are given, for $y>0$, $(a,c,d,b)>0$, respectively, by

$$\begin{aligned} f(y) =& acdby^{-(b+1)}\exp\left\{-dcy^{-b}\right\} \\ &\times\left(1-\exp\left\{-dcy^{-b}\right\}\right)^{a-1}, \end{aligned} \tag{1}$$

$$F(y)=1-\left(1-\exp\left\{-dcy^{-b}\right\}\right)^{a}, \tag{2}$$

$$S(y)=\left(1-\exp\left\{-dcy^{-b}\right\}\right)^{a}, \tag{3}$$

and

$$\begin{aligned} h(y) =& acdby^{-(b+1)}\exp\left\{-dcy^{-b}\right\} \\ &\times\left(1-\exp\left\{-dcy^{-b}\right\}\right)^{-1}. \end{aligned} \tag{4}$$

Special case: If $d=1$ and $b=2$, the resulting distribution is called Kumaraswamy-Inverse Rayleigh distribution see Hussian and Amin [18].

2.2 Basic assumptions

The following assumptions are used throughout the paper:

(1) n identical and independent units are put on the life test and the life time of individual unit has KIWD.
(2) At the beginning each of the units functions under normal use condition. If it does not fail and exceeds a pre-specified time τ, it is put under accelerated condition (stress).
(3) The test is terminated when the mth failure occurs, where m is pre-fixed before $(m\leq n)$.
(4) At the ith failure a random number of the surviving units $R_i=1,2,\ldots,m-1$, are randomly selected and removed from the test. Finally, at the mth failure, the remaining surviving units $R_m=n-m-\sum_{i=1}^{m-1}R_i$ are removed from the test and the test is terminated.
(5) The tampered random variable model is applied. It was designed by Degroot and Goel [19]. According to this model the lifetime of a unit say Y, under S-SPALT can be written as

$$Y=\begin{cases} T, & if\ T\leq\tau, \\ \tau+\frac{1}{e}(T-\tau), & if\ T>\tau, \end{cases} \tag{5}$$

where T is the lifetime of the units under normal condition, τ is the stress change time and e is the acceleration factor, where $e>1$.
(6) Using the transformation variable technique, the PDF of KIWD (a,b,d,c) under S-SPALT is given by

$$f(y)=\begin{cases} 0, & y>0, \\ f_1(y)=f(y), & 0<y\leq\tau, \\ f_2(y), & y>\tau, \end{cases} \tag{6}$$

where $\psi_i(e)=\tau+e(y-\tau)$ and

$$\begin{aligned} f_2(y) =& eacdb\left[\psi_i(e)\right]^{-(b+1)}\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\} \\ &\times\left(1-\exp[-dc\left[\psi_i(e)\right]^{-b}]\right)^{a-1}. \end{aligned}$$

(7) Let n_1 be the number of failures before time τ at the normal condition and $m-n_1$ be the number of failures after time τ at accelerated condition (stress), then the observed progressive censored data are

$$y^{\mathbf{R}}_{1;m,n}<\ldots<y^{\mathbf{R}}_{n_1;m,n}<\tau<y^{\mathbf{R}}_{n_1+1;m,n}<\ldots<y^{\mathbf{R}}_{m;m,n}, \tag{7}$$

where $R=(R_1,R_2,\ldots,R_m)$ and $\sum_{i=1}^{m}R_i=n-m$.

3 Maximum likelihood inference

In this section, the MLEs of the model parameters are obtained. Let $y_i=y^{\mathbf{R}}_{i;m,n}$, $i=1,2,\cdots,m$ be the observed values of the lifetime Y obtained from a PT-2CS under S-SPALT, with censored scheme $R=(R_1,R_2,\cdots,R_m)$. The log-likelihood function $\ell(a,b,d,c,e|\underline{y}) = \log L(a,b,d,c,e|\underline{y})$ of the observations

$y_1 < \cdots < y_{n_1} < \tau < y_{n_1+1} < \cdots < y_m$ without normalized constant is given by

$$
\begin{aligned}
&\ell(a,b,d,c,e|\underline{y}) \\
&= m\log(acdb) + (m-n_1)\log e - (b+1)\sum_{i=1}^{n_1}\log y_i \\
&-\sum_{i=1}^{n_1} dcy_i^{-b} + \sum_{i=1}^{n_1}\left(a\left(R_i+1\right)-1\right)\log\left(1-\exp\left\{-dcy_i^{-b}\right\}\right) \\
&-(b+1)\sum_{i=n_1+1}^{m}\log\psi_i(e) - \sum_{i=n_1+1}^{m} dc\left[\psi_i(e)\right]^{-b} \\
&+\sum_{i=n_1+1}^{m}\left(a\left(R_i+1\right)-1\right)\log\left(1-\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\}\right)
\end{aligned} \tag{8}
$$

Thus, we have the likelihood equations for a,e,c,b and d respectively, as

$$
\begin{aligned}
&\frac{\partial l(a,c,d,b)}{\partial a} \\
&= \frac{m}{a} + \sum_{i=1}^{n_1}(R_i+1)\log\left(1-\exp\left\{-dcy_i^{-b}\right\}\right) \\
&+\sum_{i=n_1+1}^{m}(R_i+1)\log(1-\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\}) \\
&= 0,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&\frac{\partial l(a,c,d,b)}{\partial e} \\
&= \frac{m-n_1}{e} - (b+1)\sum_{i=n_1+1}^{m}\left(\frac{y_i-\tau}{\psi_i(e)}\right) \\
&+\sum_{i=n_1+1}^{m} dcb(y_i-\tau)\left[\psi_i(e)\right]^{-(b+1)} \\
&-\sum_{i=n_1+1}^{m}\frac{dcb\left(a(R_i+1)-1\right)(y_i-\tau)}{(1-\exp(-dc[\psi_i(e)]^{-b}))} \\
&\times\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\}\left[\psi_i(e)\right]^{-(b+1)} \\
&= 0,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\frac{\partial l(a,c,d,b)}{\partial c} \\
&= \frac{m}{c} - \sum_{i=1}^{n_1} dy_i^{-b} \\
&+\sum_{i=1}^{n_1}\frac{dy_i^{-b}\left(a(R_i+1)-1\right)\exp\left\{-dcy_i^{-b}\right\}}{\left(1-\exp\left\{-dcy_i^{-b}\right\}\right)} \\
&-\sum_{i=n_1+1}^{m} d\left[\psi_i(e)\right]^{-b} \\
&+\sum_{i=n_1+1}^{m}\frac{d\left(a(R_i+1)-1\right)\left[\psi_i(e)\right]^{-b}\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\}}{\left(1-\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\}\right)} \\
&= 0,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&\frac{\partial l(a,c,d,b)}{\partial b} \\
&= \frac{m}{b} - \sum_{i=1}^{n_1}\log y_i + \sum_{i=1}^{n_1} dcy_i^{-b}\log y_i \\
&+\sum_{i=n_1+1}^{m} dc\left[\psi_i(e)\right]^{-b}\log\psi_i(e) \\
&-\sum_{i=1}^{n_1}\frac{\left(a(R_i+1)-1\right)\left(dcy_i^{-b}\log y_i\right)\exp\left\{-dcy_i^{-b}\right\}}{\left(1-\exp\left\{-dcy_i^{-b}\right\}\right)} \\
&-\sum_{i=n_1+1}^{m}\log\psi_i(e) - \sum_{i=n_1+1}^{m}\frac{dc\left(a(R_i+1)-1\right)\left[\psi_i(e)\right]^{-b}}{\left(1-\exp\left\{-dc\left[\psi_i(e)\right]^{-b}\right\}\right)} \\
&\times\log\psi_i(e)\exp\left\{-dc[\psi_i(e)]^{-b}\right\} \\
&= 0,
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
&\frac{\partial l(a,c,d,b)}{\partial d} \\
&= \frac{m}{d} - \sum_{i=1}^{n_1} cy_i^{-b} + \sum_{i=1}^{n_1}\frac{cy_i^{-b}\left(a(R_i+1)-1\right)\exp\left\{-dcy_i^{-b}\right\}}{\left(1-\exp\left\{-dcy_i^{-b}\right\}\right)} \\
&-\sum_{i=n_1+1}^{m} c\left[\psi_i(e)\right]^{-b} \\
&+\sum_{i=n_1+1}^{m}\frac{c\left(a(R_i+1)-1\right)\left[\psi_i(e)\right]^{-b}\exp\left\{-dc(\psi_i(e))^{-b}\right\}}{(1-\exp\{-dc[\psi_i(e)]^{-b}\})} \\
&= 0.
\end{aligned} \tag{13}
$$

A system of nonlinear simultaneous equations in five unknowns vaiables a, e, c, b and d is resulted.

It is obvious that an exact solution is not easy to get. To calculate the MLEs $\hat{a}$, $\hat{e}$, $\hat{c}$, $\hat{b}$ and $\hat{d}$ from the nonlinear Equations (9)-(13), we use the Newton-Raphson iterative method, see EL-Sagheer [5].

3.1 Asymptotic confidence intervals

We construct asymptotic confidence intervals of MLEs by using the asymptotic normality theory. The observed Fisher information matrix has second partial derivatives of log-likelihood function given in Eq. (8), with respect to a,b,d,c and e as the entries, which easily can be obtained. Hence, the observed information matrix is given by

$$
\hat{I} = \begin{pmatrix}
-\frac{\partial^2\ell}{\partial a^2} & -\frac{\partial^2\ell}{\partial a\partial b} & -\frac{\partial^2\ell}{\partial a\partial d} & -\frac{\partial^2\ell}{\partial a\partial c} & -\frac{\partial^2\ell}{\partial a\partial e} \\
-\frac{\partial^2\ell}{\partial b\partial a} & -\frac{\partial^2\ell}{\partial b^2} & -\frac{\partial^2\ell}{\partial b\partial d} & -\frac{\partial^2\ell}{\partial b\partial c} & -\frac{\partial^2\ell}{\partial b\partial e} \\
-\frac{\partial^2\ell}{\partial d\partial a} & -\frac{\partial^2\ell}{\partial d\partial b} & -\frac{\partial^2\ell}{\partial d^2} & -\frac{\partial^2\ell}{\partial d\partial c} & -\frac{\partial^2\ell}{\partial d\partial e} \\
-\frac{\partial^2\ell}{\partial c\partial a} & -\frac{\partial^2\ell}{\partial c\partial b} & -\frac{\partial^2\ell}{\partial c\partial d} & -\frac{\partial^2\ell}{\partial c^2} & -\frac{\partial^2\ell}{\partial c\partial e} \\
-\frac{\partial^2\ell}{\partial e\partial a} & -\frac{\partial^2\ell}{\partial e\partial b} & -\frac{\partial^2\ell}{\partial e\partial d} & -\frac{\partial^2\ell}{\partial e\partial c} & -\frac{\partial^2\ell}{\partial e^2}
\end{pmatrix}_A . \tag{14}
$$

where $A = \downarrow(a,b,d,c,e) = (\widehat{a},\widehat{b},\widehat{d},\widehat{c},\widehat{e})$

The asymptotic variance-covariance matrix $\hat{I}^{-1}$ for the MLEs is obtained by inverting the observed information matrix $\hat{I}$ or equivalent

$$\hat{I}^{-1}=\begin{pmatrix} var(\hat{a}) & Cov(\hat{a}\hat{b}) & Cov(\hat{a}\hat{d}) & Cov(\hat{a}\hat{c}) & Cov(\hat{a}\hat{e}) \\ Cov(\hat{b}\hat{a}) & var(\hat{b}) & Cov(\hat{b}\hat{e}) & Cov(\hat{b}\hat{c}) & Cov(\hat{b}\hat{e}) \\ Cov(\hat{d}\hat{a}) & Cov(\hat{d}\hat{b}) & var(\hat{d}) & Cov(\hat{d}\hat{c}) & Cov(\hat{d}\hat{e}) \\ Cov(\hat{c}\hat{a}) & Cov(\hat{c}\hat{b}) & Cov(\hat{c}\hat{d}) & var(\hat{c}) & Cov(\hat{c}\hat{e}) \\ Cov(\hat{e}\hat{a}) & Cov(\hat{e}\hat{b}) & Cov(\hat{e}\hat{d}) & Cov(\hat{e}\hat{c}) & var(\hat{e}) \end{pmatrix}. \tag{15}$$

Then, the $100(1-\delta)\%$ two sided CIs for a, b, d, c and e, can be given by

$$\left.\begin{array}{l} (\hat{a}_L,\hat{a}_U)=\hat{a}\mp z_{\frac{\delta}{2}}\sqrt{var(\hat{a})} \\ (\hat{b}_L,\hat{b}_U)=\hat{b}\mp z_{\frac{\delta}{2}}\sqrt{var(\hat{b})} \\ (\hat{d}_L,\hat{d}_U)=\hat{d}\mp z_{\frac{\delta}{2}}\sqrt{var(\hat{d})} \\ (\hat{c}_L,\hat{c}_U)=\hat{c}\mp z_{\frac{\delta}{2}}\sqrt{var(\hat{c})} \\ (\hat{e}_L,\hat{e}_U)=\hat{e}\mp z_{\frac{\delta}{2}}\sqrt{var(\hat{e})} \end{array}\right\}, \tag{16}$$

where $z_{\frac{\delta}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\delta}{2}$.

4 Bootstrap confidence intervals

In this section, we present the parametric bootstrap method to construct CIs for the unknown parameters a, b, d, c and e. It is very important in estimate CIs and hypothesis tests. So, two parametric bootstrap methods are used: **(i)** Studentized-t (Stud-t) bootstrap CI suggested by Hall [20] and **(ii)** Percentile (Perc-p) bootstrap CI suggested by Efron [21]. Hall [20] showed that the Stud-t CI is better than the Perc-p bootstrap CI from an asymptotic point of view, although the finite sample properties are not yet known. The Bootstrap samples obtained by using the following steps:

(1)Based on the original PT-2C sample, $\underline{y}\equiv y^R_{1;m,n}<...<y^R_{n_1;m,n}<y^R_{n_1+1;m,n}<...<y^R_{m;m,n}$ compute $\hat{a}$, $\hat{b}$, $\hat{d}$, $\hat{c}$ and $\hat{e}$.

(2)Use $\hat{a}$, $\hat{b}$, $\hat{d}$, $\hat{c}$ and $\hat{e}$ to generate a bootstrap sample $\underline{y}^*$ with the same values of R_i, $i=1,2,..,m$.

(3)As in Step 1 based on $\underline{y}^*$ compute the bootstrap sample estimates of $\hat{a}$, $\hat{b}$, $\hat{d}$, $\hat{c}$ and $\hat{e}$ say $\hat{a}^*$, $\hat{b}^*$, $\hat{d}^*$, $\hat{c}^*$ and $\hat{e}^*$.

(4)Repeat the above Steps 2 and 3 N times and arrange all $\hat{a}^*$, $\hat{b}^*$, $\hat{d}^*$, $\hat{c}^*$ and $\hat{e}^*$ in ascending order to obtain the bootstrap sample $(\hat{\vartheta}_k^{*[1]}, \hat{\vartheta}_k^{*[2]}, ..., \hat{\vartheta}_k^{*[N]})$, $k=1,2,3,4,5$ where $\hat{\vartheta}_1^*=\hat{a}^*$, $\hat{\vartheta}_2^*=\hat{b}^*$, $\hat{\vartheta}_3^*=\hat{d}^*$, $\hat{\vartheta}_4^*=,\hat{c}^*$ and $\hat{\vartheta}_5^*=\hat{e}^*$.

4.1 Percentile bootstrap confidence intervals

Let $\Phi(q)=P(\hat{\vartheta}_k^*\leq q)$ be the cdf of $\hat{\vartheta}_k^*$. Define $\hat{\vartheta}^*_{k\,Perc-p}=\Phi^{-1}(q)$ for given q. The approximate Perc-p $100(1-\delta)\%$ CI of $\hat{\vartheta}_k^*$ is given by

$$[\hat{\vartheta}^*_{k\,Perc-p}(\frac{\delta}{2}),\ \ \hat{\vartheta}^*_{k\,Perc-p}(1-\frac{\delta}{2})]. \tag{17}$$

4.2 Studentized-t bootstrap confidence intervals

Let $\widehat{\chi}_k^{*[1]}\leq\widehat{\chi}_k^{*[2]}\leq...\leq\widehat{\chi}_k^{*[N]}$ be the order statistics where

$$\widehat{\chi}_k^{*[j]}=\frac{(\hat{\vartheta}_k^{*[j]}-\hat{\vartheta}_k)\sqrt{N}}{\sqrt{Var(\hat{\vartheta}_k^{*[j]})}},\ j=1,2,...,N;\ k=1,2,3,4,5. \tag{18}$$

Where $\hat{\vartheta}_1=\widehat{a}$, $\hat{\vartheta}_2=\widehat{b}$, $\hat{\vartheta}_3=,\widehat{d}$, $\hat{\vartheta}_4=\widehat{c}$ and $\hat{\vartheta}_5=\widehat{e}$ while $Var(\hat{\vartheta}_k^{*[j]})$ is obtained from inverse of the Fisher information matrix. Let $D(q)=P(\chi_k^*\leq q)$, $k=1,2,3,4,5$ be the cdf of $\widehat{\chi}_k^*$. For a given q, define

$$\hat{\vartheta}^*_{kStud-t}=\hat{\vartheta}_k+N^{-\frac{1}{2}}\sqrt{Var(\hat{\vartheta}_k^*)}D^{-1}(q). \tag{19}$$

Thus, the approximate bootstrap-t $100(1-\delta)\%$ CI of $\hat{\vartheta}_k^*$ is given by

$$[\hat{\vartheta}^*_{kStud-t}(\frac{\delta}{2}),\ \ \hat{\vartheta}^*_{kStud-t}(1-\frac{\delta}{2})]. \tag{20}$$

5 Bayesian estimation

Bayesian estimation deals with a wide variety of problems in many scientific and engineering areas. Bayesian statistic interests in fitting a probability model to a set of data and summarizing the result by a probability distribution on the parameters of the model. The given data comes from the likelihood function and the prior distribution function and the resulting distributions called the posterior distributions. Let us consider independent vague priors parameters a, b, d, c and e, as follows

$$\left.\begin{array}{r} \pi(a)\propto a^{-1},\ a>0, \\ \pi(b)\propto b^{-1},\ b>0, \\ \pi(d)\propto d^{-1},\ d>0, \\ \pi(c)\propto c^{-1},\ c>0, \\ \pi(e)\propto e^{-1}, e>1, \end{array}\right\}. \tag{21}$$

Therefore, the joint prior of the parameters a, b, d, c and e can be written as

$$\pi(a,b,d,c,e)\propto(abdce)^{-1}, a,b,d,c>0, e>1. \tag{22}$$

The joint posterior density function of a, b, d, c and e denoted by $\pi^*(a,b,d,c,e|\underline{y})$ can be written as

$$\pi^*(a,b,d,c,e|\underline{y})=\frac{L(a,b,d,c,e)\times\pi(a,b,d,c,e)}{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(a,b,d,c,e)\times\pi(a,b,d,c,e)\,dadbdddcde}. \tag{23}$$

Therefore, the Bayes estimate of any function of the parameters, say $h(a,b,d,c,e)$, using squared error loss function (SEL) is

$$\hat{h}(a,b,d,c,e) = E_{a,b,d,c,e|\underline{y}}\left[h(a,b,d,c,e)\right]$$
$$= \frac{\int_1^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty h(a,b,d,c,e) \times L(a,b,d,c,e)\,\pi(a,b,d,c,e)\,dadbdddcde}{\int_1^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L(a,b,d,c,e)\,\pi(a,b,d,c,e)\,dadbdddcde}. \tag{24}$$

Generally, the ratio of two integrals given by (24) cannot be obtained in a closed form. In this case, the MCMC technique will be used to generate samples from the posterior distributions and then the Bayes estimates of the parameters a, b, d, c and e will be computed. The main theme of the MCMC technique is to compute an approximate value of integrals given in Eq. (24). An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within Gibbs samplers. The Metropolis algorithm is a random walk that uses an acceptance/rejection rule to converge to the target distribution. The Metropolis algorithm was first proposed in Metropolis et al. [22] and it was then generalized in Hastings [23]. Made into mainstream statistics and engineering via the articles Gelfand and Smith [24] and Gelfand et al. [25] which presented the Gibbs sampler as used in Geman and Geman [26]. From (8), (22) and (23), the joint posterior density function of a, b, d, c and e denoted by $\pi^*(a,b,d,c,e|\underline{y})$ can be written as

$$\begin{aligned}
&\pi^*(a,b,d,c,e|\underline{y}) \\
&\propto a^{m-1} b^{m-1} d^{m-1} c^{m-1} e^{m-n_1-1} \\
&\times \exp\left\{-(b+1)\left(\sum_{i=1}^{n_1}\log[y_i] + \sum_{i=n_1+1}^{m}\log[\psi_i(e)]\right)\right\} \\
&\times \exp\left\{-dc\left(\sum_{i=1}^{n_1} y_i^{-b} + \sum_{i=n_1+1}^{m}[\psi_i(e)]^{-b}\right)\right\} \\
&\times \exp\left\{\sum_{i=1}^{n_1}[a(R_i+1)-1]\log[1-\exp(-dcy_i^{-b})]\right\} \\
&\times \exp\left\{\sum_{i=n_1+1}^{m}[a(R_i+1)-1]\log[1-\exp(-dc[\psi_i(e)]^{-b})]\right\}.
\end{aligned} \tag{25}$$

The conditional posterior densities function of a, b, d, c and e can be given as

$$\begin{aligned}
&\pi_1^*(a|b,d,c,e,\underline{y}) \\
&\propto a^{m-1}\prod_{i=1}^{n_1}\left[[1-\exp\{-dcy_i^{-b}\}]^{a(R_i+1)}\right] \\
&\qquad \times \prod_{i=n_1+1}^{m}[1-\exp\{-dc[\psi_i(e)]^{-b}\}]^{a(R_i+1)},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\pi_2^*(b|a,d,c,e,\underline{y}) \\
&\propto b^{m-1}\prod_{i=1}^{n_1} y_i^{-b-1}\exp\{-dcy_i^{-b}\}[1-\exp\{-dcy_i^{-b}\}]^{a(R_i+1)-1} \\
&\quad \times \prod_{i=n_1+1}^{m}[\psi_i(e)]^{-b-1}[\exp\{-dc[\psi_i(e)]^{-b}\}]^{a(R_i+1)-1} \\
&\quad \times [1-\exp\{-dc[\psi_i(e)]^{-b}\}]^{a(R_i+1)-1},
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\pi_3^*(d|a,b,c,e,\underline{y}) \\
&\propto d^{m-1}\prod_{i=1}^{n_1}[\exp(-dcy_i^{-b})][1-\exp(-dcy_i^{-b})]^{a(R_i+1)-1} \\
&\quad \times \prod_{i=n_1+1}^{m}[\exp(-dc[\psi_i(e)]^{-b})] \\
&\quad \times [1-\exp(-dc[\psi_i(e)]^{-b})]^{a(R_i+1)-1},
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\pi_4^*(c|a,b,c,e,\underline{y}) \\
&\propto c^{m-1}\prod_{i=1}^{n_1}[\exp\{-dcy_i^{-b}\}][1-\exp\{-dcy_i^{-b}\}]^{a(R_i+1)-1} \\
&\quad \times \prod_{i=n_1+1}^{m}[\exp\{-dc[\psi_i(e)]^{-b}\}] \\
&\quad \times [1-\exp\{-dc[\psi_i(e)]^{-b}\}]^{a(R_i+1)-1},
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
&\pi_5^*(e|a,b,d,c,\underline{y}) \\
&\propto e^{m-n_1-1}\prod_{i=n_1+1}^{m}[\psi_i(e)]^{-b-1}[\exp\{-dc[\psi_i(e)]^{-b}\}] \\
&\quad \times [1-\exp\{-dc[\psi_i(e)]^{-b}\}]^{a(R_i+1)-1}.
\end{aligned} \tag{30}$$

Figure 1 shows that all the conditional posterior distributions are almost symmetric and seem to quite skewed. Now, the following steps illustrate the method of the Metropolis-Hastings algorithm within Gibbs sampling to generate the posterior samples as suggested by Tierney [27], and turn in to obtain the Bayes estimates and the corresponding credible intervals:

Step 1. Start with an $a^{(0)} = \hat{a}$, $b^{(0)} = \hat{b}$, $d^{(0)} = \hat{d}$, $c^{(0)} = \hat{c}$ and $e^{(0)} = \hat{e}$.

Step 2. Put $j = 1$.

Step 3. Using the following M-H algorithm, generate $a^{(j)}, b^{(j)}, d^{(j)}$, $c^{(j)}$ and $e^{(j)}$ from (26), (27), (28), (29) and (30) with the normal suggested distribution $N\left(a^{(j-1)}, var(a)\right)$, $N\left(b^{(j-1)}, var(b)\right)$,

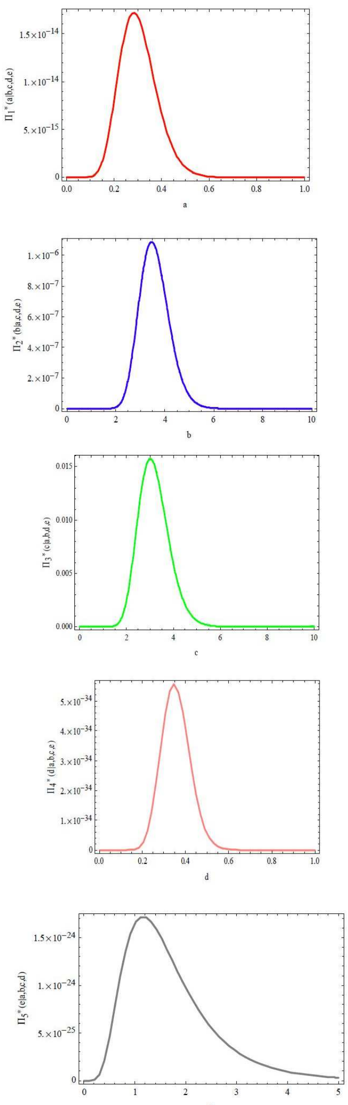

Fig. 1: The conditional posterior density functions.

$N\left(d^{(j-1)}, var\left(d\right)\right)$, $N\left(c^{(j-1)}, var\left(c\right)\right)$ and $N\left(e^{(j-1)}, var\left(e\right)\right)$ respectively, where $var\left(a\right)$, $var(b)$, $var\left(d\right)$, $var\left(c\right)$ and $var\left(e\right)$ can be obtained from the main diagonal in the inverse fisher information matrix (15) compute $a^{(j)}, b^{(j)}, d^{(j)}, c^{(j)}$ and $c^{(j)}$.

(i)Generate a proposal a^{*} from $N\left(a^{(j-1)}, var\left(a\right)\right)$, b^{*} from $N\left(b^{(j-1)}, var\left(b\right)\right)$, d^{*} from $N\left(d^{(j-1)}, var\left(d\right)\right)$, c^{*} from $N(c^{(j-1)}, var\left(c\right))$ and e^{*} from $N\left(e^{(j-1)}, var\left(e\right)\right)$.

(ii)Evaluate the acceptance probabilities

$$\left.\begin{array}{l}\rho_{a}=\min\left[1,\dfrac{\pi_{1}^{*}(a^{*}|b^{(j)},d^{(j-1)},c^{(j-1)},e^{(j-1)},\underline{y})}{\pi_{1}^{*}(a^{(j-1)}|b^{(j)},d^{(j-1)},c^{(j-1)},e^{(j-1)},\underline{y})}\right],\\ \rho_{b}=\min\left[1,\dfrac{\pi_{2}^{*}(b^{*}|a^{(j)},d^{(j)},c^{(j-1)},e^{(j-1)},\underline{y})}{\pi_{2}^{*}(b^{(j-1)}|a^{(j)},d^{(j)},c^{(j-1)},e^{(j-1)},\underline{y})}\right],\\ \rho_{d}=\min\left[1,\dfrac{\pi_{3}^{*}(d^{*}|a^{(j)},b^{(j)},c^{(j)},e^{(j-1)},\underline{y})}{\pi_{3}^{*}(d^{(j-1)}|a^{(j)},b^{(j)},c^{(j)},e^{(j-1)},\underline{y})}\right],\\ \rho_{c}=\min\left[1,\dfrac{\pi_{4}^{*}(c^{*}|a^{(j)},b^{(j)},d^{(j)},e^{(j)},\underline{y})}{\pi_{4}^{*}(c^{(j-1)}|a^{(j)},b^{(j)},d^{(j)},e^{(j)},\underline{y})}\right],\\ \rho_{e}=\min\left[1,\dfrac{\pi_{5}^{*}(e^{*}|a^{(j)},b^{(j)},d^{(j)},c^{(j)},\underline{y})}{\pi_{5}^{*}(e^{(j-1)}|a^{(j)},b^{(j)},d^{(j)},c^{(j)},\underline{y})}\right].\end{array}\right\}.$$

Step 4.Put $j = j+1$.
Step 5.Repeat Steps 3-6 N times.

To guarantee the convergence to remove the influence of the selection of initial values, the first M simulated varieties are ignored. Then the selected samples are $a^{(j)}, b^{(j)}, d^{(j)}$ and $c^{(j)}$, $j = M+1, ..., N$, for sufficiently large N, forms an approximate posterior samples which can be used to obtain the Bayes MCMC point estimates of a, b, d, c and e as

$$\left.\begin{array}{l}\widehat{a}_{MCMC}=\frac{1}{N-M}\sum\limits_{j=M+1}^{N}a^{(j)},\\ \widehat{b}_{MCMC}=\frac{1}{N-M}\sum\limits_{j=M+1}^{N}b^{(j)},\\ \widehat{d}_{MCMC}=\frac{1}{N-M}\sum\limits_{j=M+1}^{N}d^{(j)},\\ \widehat{c}_{MCMC}=\frac{1}{N-M}\sum\limits_{j=M+1}^{N}c^{(j)},\\ \widehat{e}_{MCMC}=\frac{1}{N-M}\sum\limits_{j=M+1}^{N}e^{(j)},\end{array}\right\}.$$

To calculate the credible interval (CRIs) of Ω_k where $\Omega_1 = a, \Omega_2 = b, \Omega_3 = d, \Omega_4 = c$ and $\Omega_5 = e$, we take the quantiles of the sample as the endpoints of the interval. Sort $\left\{\Omega_k^{M+1}, \Omega_k^{M+2}, ..., \Omega_k^{N}\right\}$ as $\left\{\Omega_k^{[1]}, \Omega_k^{[2]}, ..., \Omega_k^{[N-M]}\right\}$. The $100(1-\delta)\%$ symmetric credible interval of Ω_k is

$$\left[\Omega_k(\frac{\delta}{2}(N-M),\ \ \Omega_k((1-\frac{\delta}{2})(N-M))\right]. \tag{31}$$

6 Numerical computations

In this section, a simulation example is presented to assess the estimation procedures. In this example, a PT-2C sample from KIWD under S-SPALT model is generated. The algorithm of generation is performed according to the algorithm described in Balakrishnan and Sandhu [28] as the following:

Table 1. SSPALT simulation data with true values for a, b, d, c and e.

Under normal condition			Under accelerated condition				
0.5305	0.6849	0.7143	1.0835	1.3049	1.5726	1.6547	1.7311
0.7641	0.8370	1.0271	2.2417	2.3382	3.2457	5.9023	

Table 2. Different point estimates for the parameters a, b, d, c and e.

Parameters	$(.)_{ML}$	$(.)_{Perc-p}$	$(.)_{Stud-t}$	$(.)_{MCMC}$
a	0.3487	0.4155	0.4731	0.4839
b	2.5725	2.5496	2.6685	2.4985
d	1.3872	1.6545	1.4352	1.6210
c	0.4624	0.5393	0.5135	0.5280
e	2.4155	2.6663	2.0112	1.9981

Table 3. 95% confidence intervals for a, b, d, c and e.

Method	a	Length	b	Length
ACI	[-0.1449, 0.8423]	0.9873	[-0.2616, 5.4065]	5.6681
Perc-pCIs	[0.1471, 1.3049]	1.1578	[0.3361, 6.2845]	5.9484
Stud-tCIs	[0.2056, 1.1482]	0.9426	[0.2189, 4.7583]	4.5394
CRI	[0.0027, 0.9678]	0.9651	[0.1834, 4.5529]	4.3695
Method	d	Length	c	Length
ACI	[0.1001, 2.6743]	2.5742	[0.0161, 0.9087]	0.8926
Perc-pCIs	[0.2154, 3.5681]	3.3527	[0.1287, 1.5969]	1.4682
Stud-tCIs	[0.1249, 2.3992]	2.2743	[0.0996, 1.2874]	1.1878
CRI	[0.0987, 2.2514]	2.1527	[0.0547, 0.9988]	0.9441
Method	e	Length		
ACI	[2.3809,4.4500]	2.0691		
Perc-pCIs	[1.2348, 3.4571]	2.2223		
Stud-tCIs	[1.8347, 5.0043]	3.1696		
CRI	[2.0875, 4.3872]	2.2997		

(1)Specify the values of n, m and $R_i, i = 1,2,...,m$.
(2)Specify the values of the parameters a, b, d, c and e.
(3)Specify the values of the stress change time τ.
(4)Generate a random sample with size n and censoring size m from the random variable Y given by Eq. (5), the set of data can be considered as

$$y^{\mathbf{R}}_{1;m,n} < ... < y^{\mathbf{R}}_{n_1;m,n} < y^{\mathbf{R}}_{n_1+1;m,n} < ... < y^{\mathbf{R}}_{m;m,n},$$

where $R = (R_1, R_2, ..., R_m)$ and $\sum_{i=1}^{m} R_i = n - m$.
(5)Use the PT-2C sample to compute the MLEs of the model parameters. The Newton–Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters.
(6)Compute the 95% bootstrap confidence intervals for the model parameters, using the steps described in Section 4.
(7)Compute the Baye estimates of the model parameters based on MCMC algorithm described in Section 5.

A simulation data for PT-2C sample under S-SPALT model from KIWD with true values $a = 0.4$, $b = 2.5$, $d = 1.5$, $c = 0.5$ and the acceleration factor $e = 2$, and $\tau = 1.5$, using progressive censoring schemes $n = 30$, $m = 15$ and $R = (3, 2, 1, 2, 1, 0, 2, 0, 2, 0, 1, 0, 1, 0, 0)$ the S-SPALT simulation data has been approximated to four decimal places and it has been presented in Table 1.

In MCMC approach, we run the chain for 25 000 times and discard the first 5000 values as 'burn-in'. The MLEs $(.)_{ML}$, bootstrap $(.)_{Perc-p}, (.)_{Stud-t}$ and Bayes MCMC $(.)_{MCMC}$ point estimates of the parameters are obtained and presented in Table 2. The approximate confidence intervals (ACIs), bootstrap confidence intervals (Perc-pCIs, Stud-tCIs) and credible intervals (CRIs) for the parameters a, b, d, c and e are computed. The results of 95% (ACIs, Perc-pCIs, Stud-tCIs, CRIs) are presented in Table 3.

7 Conclusion

Using PT-2C samples strategy the analysis of the S-SPALT of KIW failure model is performed based on Bayes and non-Bayes methods. The classical Bayes estimates cannot be obtained in explicit form. One can clearly see the scope of MCMC based Bayesian solutions which make every inferential development routinely available. In this paper, we have considered the ML and Bayes estimates for the parameters of the KIWD using PT-2C schemes. Two types of bootstrap confidence intervals are used to obtain 95% CIs for the unknown parameters a, b, d, c and e. It is well known that when all parameters are unknown, the Bayes estimates cannot be obtained in explicit form. We used the MCMC techniques to compute the approximate Bayes estimates and corresponding CRIs. A numerical example using the simulated data set is presented to illustrate how the MCMC and parametric bootstrap methods are worked based on PT-2C data.

Acknowledgment

The authors would like to express their thanks to the editors and the anonymous referees for their valuable comments and suggestions which significantly improved the paper.

References

[1] W. Nelson, Accelerated Testing: Statistical Models, Test Plans and Data Analysis. Wiley, New York.1990.
[2] N. Balakrishnan and R. Aggarwala, Progressive Censoring: Theory, Methods and Applications. Birkhauser, Boston 2000.
[3] N. Balakrishnan, Progressive censoring methodology: An appraisal, Test, Vol. **16**, pp. 211-296 (2007).
[4] R.M. Musleh and A. Helu, Estimation of the inverse Weibull distribution based on progressively censored data: Comparative study, Reliab. Eng. Syst. Safety , Vol. **131**, pp. 216-227 (2014).
[5] R. M. EL-Sagheer. Estimation of parameters of Weibull-gamma distribution based on progressively censored data. Statist Papers 2016; DOI 10.1007/s00362-016-0787-2.
[6] R. M. EL-Sagheer. Assessing the lifetime performance index of extreme value model based on progressive Type-II censored samples, Math. Sci. Lett., Vol. 6, No. 3**, pp.** 279-286 (2017).

[7] M. A. W. Mahmoud, R. M. EL-Sagheer and M. M. M Mansour. Inference on the stress-strength model from Weibull Gamma distribution, Appl. Math. Inf. Sci., Vol. 11, No. 3, pp. 737-744 (2017).
[8] M. A. W. Mahmoud, R. M. EL-Sagheer and . M. M. M. Mansour M. On estimation of Weibull-Gamma parameters based on hybrid type-II censoring scheme, J. Stat. Appl. Pro., Vol. 6, No. 1, pp. 123-131 (2017).
[9] M. A. W. Mahmoud, R. M. EL-Sagheer and W. B. H. Etman. Moments inequalities for NBRUL distributions with hypotheses testing applications, Austrian J. Stat., Vol. 47, **pp.** 95-104 (2018).
[10] A. A. Ismail. Estimating the generalized exponential distribution parameters and the acceleration factor under constant-stress partially accelerated life testing with type-II censoring, Strength of Materials, Vol. **45**, pp. 693-702 (2013).
[11] A. A. Ismail. On designing constant-stress partially accelerated life tests under time-censoring, Strength of Materials, Vol. **46**, pp. 132-139 (2014).
[12] R. M. EL-Sagheer. Inferences in constant-partially accelerated life tests based on progressive type-II censoring, Bull. Malays. Math. Sci. Soc., Vol. 41, pp. 609-626 (2018).
[13] A. T. Abushal and A. A. Soliman. Estimating the Pareto parameters under progressive censoring data for constant-partially accelerated life tests, J Stat Comput Simul., Vol. 85, No. 5, pp. 917–934 (2015).
[14] M. A. Abd-Elfattah, A. Hassan and S. G. Nassr, Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring, Stat Methodol., Vol. 5, No, 6, pp. 502-514 (2008).
[15] A. A. Ismail, Likelihood inference for a step-stress partially accelerated life test model with type-I progressively hybrid censored data from Weibull distribution, J Stat Comput Simul., Vol. **84**, pp. 2486-2494 (2014).
[16] R. M. EL-Sagheer and N. M. Mohamed . On estimation of modified Weibull parameters in presence of accelerated life test, J Stat Theory Practice., DOI 10.1080/15598608.2018.1431575.
[17] Q. M. Shahbaz, S. Shahbaz and N. S. Butt., The Kumaraswamy Inverse Weibull Distribution, Pakistan J Stat Oper Res., Vol. 8, No. 3, pp. 479-489 (2012).
[18] A. M. Hussian and A.E. Amin, Estimation and prediction for the Kumaraswamy inverse Rayleigh distribution based on records, Int J Adv Stat Probab., Vol. 2 **No. 1**, pp. 21-27(2014).
[19] M. H. DeGroot and P. k. Goel, Bayesian and optimal design in partially accelerated life testing, NRLQ, Vol. 16, No. 2, pp. 223-235 (1979).
[20] P. Hall, Theoretical comparison of bootstrap confidence intervals. Annals of Statistics, Vol. **16**, pp. 927-953 (1988).
[21] B. Efron. The jackknife, the bootstrap, and other resampling plans. Soc. Indu. Math., CBMS-NSF Monographs; 1982.
[22] N. Metropolis, AW. Rosenbluth , MN. Rosenbluth , AH. Teller and E. Teller, Equations of state calculations by fast computing machines. J Chem Phys., Vol. **21**, pp. 1087–1091 (1953).
[23] W. K. Hastings, Monte Carlo sampling methods using Markov chains and their applications. Biometrika, Vol. **57**, pp. 97-109 (1970).
[24] A. E. Gelfand and A.F.M. Smith, Sampling-based approaches to calculating marginal densities, J. Amer. Statist. Assoc., Vol. **85**, pp. 398-409 (1990).
[25] A. E. Gelfand and S.E. Hills, A. Racine-Poon and A. F. M. Smith. Illustration of Bayesian inference in normal data models using Gibbs sampling. J. Amer. Statist. Assoc., Vol. **85**, pp. 972-985 (1990).
[26] S. Geman and D. Geman, Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images, IEEE Trans Pattern Anal Mach Intell., Vol. PAMI-**6**, pp. 721-741 (1984).
[27] L. Tierney, Markov chains for exploring posterior distributions, Ann Stat., Vol. **22**, pp. 1701-1728 (1994).
[28] N. Balakrishnan , R. A. Sandhu, A simple simulation algorithm for generating progressively type-II censored samples, Amer Statist, Vol. **49**, pp. 229-230 (1995).

Rashad M. EL-Sagheer is a lecturer of Mathematical Statistics at Mathematics Department Faculty of Science, AL-Azhar University, Cairo, Egypt. He received Ph. D. from Faculty of Science, Sohag University, Egypt in 2014. His areas of research where he has several publications in the international journals include: Statistical inference, Theory of estimation, Bayesian inference, Order statistics, Records, Theory of reliability, censored data, Life testing and distribution theory. He published and Co-authored more than 36 papers in reputed international journals. He supervised for M. Sc. and Ph. D. students.

Nagwa M. Mohamed is a lecturer of Mathematical Statistics at Mathematics Department, Faculty of Science, Suez University, Egypt. She received her Ph. D. from Faculty of Science, Suez University, Egypt in 2016. Her research interests include: Statistical inference, Theory of Reliability, Censored Data, Life testing, Distribution Theory.