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Third-Order Differential Subordination and Differential Superordination Results for Analytic Functions Involving the Srivastava-Attiya Operator

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Abstract: In this article, by making use of the linear operator introduced and studied by Srivastava and Attiya [16], suitable classes of admissible functions are investigated and the dual properties of the third-order differential subordinations are presented. As a consequence, various sandwich-type theorems are established for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform. Relevant connections of the new results presented here with those that were considered in earlier works are pointed out.

Keywords: Analytic functions, univalent functions, differential subordination, differential superordination, srivastava-Attiya operator, sandwich-type theorems, admissible functions

1 Introduction

Let $\mathscr{H}$ be the class of functions which are analytic in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let

$$\mathscr{H}[a,n] \qquad (n \in \mathbb{N} := \{1,2,3,\cdots\};\ a \in \mathbb{C})$$

be the subclass of the analytic function class $\mathscr{H}$ consisting of functions of the following form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (z \in \mathbb{U}).$$

Let $\mathscr{A}\ (\subset \mathscr{H})$ be the class of functions which are analytic in $\mathbb{U}$ and have the *normalized* Taylor-Maclaurin series of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

Suppose that f and g are in $\mathscr{H}$. We say that f is *subordinate* to g (or g is *superordinate* to f), written as follows:

$$f \prec g \quad \text{in} \quad \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a function $\omega \in \mathscr{H}$, satisfying the conditions of the Schwarz lemma, namely

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

It follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if g is *univalent* in $\mathbb{U}$, then the reverse implication also holds true (see, for details, [11]).

The concept of differential subordination is a generalization of various inequalities involving complex

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variables. We recall here some more definitions and terminologies from the theory of differential subordination and differential superordination.

Definition 1. (see [1]) Let $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and suppose that the function $h(z)$ is univalent in $\mathbb{U}$. If the function $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination:

$$\psi\big(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\big) \prec h(z), \qquad (2)$$

then $p(z)$ is called a *solution* of the differential subordination (2). Furthermore, a given univalent function $q(z)$ is called a *dominant* of the solutions of the differential subordination (2) or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2) is said to be the *best dominant*.

Definition 2. (see [23]) Let $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and let the function $h(z)$ be univalent in $\mathbb{U}$. If the function $p(z)$ and

$$\psi\big(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\big)$$

are univalent in $\mathbb{U}$ and satisfy the following third-order differential superordination:

$$h(z) \prec \psi\big(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\big), \qquad (3)$$

then $p(z)$ is called a *solution* of the differential superordination given by (3). An analytic function $q(z)$ is called a *subordinant* of the solutions of the differential superordination given by (3) (or, more simply, a subordinant) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3).

A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (3) is said to be the *best subordinant* of the differential superordination given by (3). We note that both the best dominant and the best subordinant are unique up to rotation of $\mathbb{U}$. The monograph by Miller and Mocanu [11] and the more recent book of Bulboacă [2] provide detailed expositions on the theory of differential subordination and differential superordination.

With a view to defining the Srivastava-Attiya operator, we recall here the general Hurwitz-Lerch Zeta function, which is defined by the following series (see, for example, [17]):

$$\Phi(z,\mu,b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^{\mu}} \qquad (4)$$

$$\big(b \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \mu \in \mathbb{C} \text{ when } z \in \mathbb{U};\ \Re(\mu) > 1 \text{ when } z \in \partial\mathbb{U}\big),$$

where $\mathbb{Z}_0^-$ denotes the set of non-positive integers.

Special cases of the function $\Phi(z,\mu,b)$ include, for example, the Riemann Zeta function given by

$$\zeta(\mu) = \Phi(1,\mu,1),$$

the Hurwitz Zeta function given by

$$\zeta(\mu,b) = \Phi(1,\mu,b),$$

the Lerch Zeta function given by

$$\ell_\mu \vartheta = \Phi\left(e^{2\pi i\vartheta}, \mu, 1\right) \qquad \big(\vartheta \in \mathbb{R};\ \Re(\mu) > 1\big),$$

the Polylogarithm function given by

$$\mathrm{Li}_\mu = z\Phi(z,\mu,1),$$

and so on (see, for further details, [19]).

Srivastava and Attiya [16] considered the following normalized function:

$$\begin{aligned} R_{\mu,b}(z) &= (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \\ &= z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^{\mu} z^n \qquad (z \in \mathbb{U}). \end{aligned} \qquad (5)$$

By making use of $R_{\mu,b}(z)$, they introduced the widely-investigated operator $J_{\mu,b} : \mathscr{A} \to \mathscr{A}$, which is defined in terms of convolution as follows:

$$J_{\mu,b} f(z) = R_{\mu,b}(z) * f(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^{\mu} a_n z^n \qquad (z \in \mathbb{U}). \qquad (6)$$

The operator $J_{\mu,b} f(z)$ is now popularly known in the literature as the Srivastava-Attiya operator. Various applications of the Srivastava-Attiya operator $J_{\mu,b} f(z)$ are found in [4,6,7,8,13,18,20,24] and in the references cited in each of these earlier works.

From (6), it is clear that

$$zJ'_{\mu+1,b} f(z) = (b+1)J_{\mu,b} f(z) - bJ_{\mu+1,b} f(z). \qquad (7)$$

For suitable choices of the parameters involved, the above-defined operator $J_{\mu,b} f(z)$ yields various other linear operators which are introduced in earlier works. For example, we have

1. $J_{0,b} f(z) = f(z)$;
2. $J_{1,0} f(z) = \int_0^z \frac{f(t)}{t} dt =: \mathfrak{A} f(z)$;
3. $J_{1,\eta} f(z) = \frac{1+\eta}{z^\eta} \int_0^z t^{\eta-1} f(t) dt =: \mathfrak{I}_\eta f(z)$ $(\eta > -1)$;
4. $J_{\sigma,1} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\sigma} a_n z^n =: I^\sigma f(z)$ $(\sigma > 0)$,

where $\mathfrak{A}(f)$ and $\mathfrak{I}_\eta$ are the integral operators introduced by Alexander and Bernardi, respectively, and $I^\sigma(f)$ is the Jung-Kim-Srivastava integral operator which is closely related to the multiplier transformation studied by Flett. For further details, we refer the interested reader to the earlier work [13].

Definition 3. (see [1]) Let $\mathbb{Q}$ be the set of all functions q that are analytic and univalent on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{ \xi : \xi \in \partial\mathbb{U} : \lim_{z \to \xi} \{q(z)\} = \infty \right\}, \tag{8}$$

and are such that $\min |q'(\xi)| = \rho > 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathbb{Q}$ for which $q(0) = a$ be denoted by $\mathbb{Q}(a)$ with

$$\mathbb{Q}(0) = \mathbb{Q}_0 \quad \text{and} \quad \mathbb{Q}(1) = \mathbb{Q}_1. \tag{9}$$

The subordination methodology is applied to appropriate classes of admissible functions. The following class of admissible functions is given by Antonino and Miller [1].

Definition 4. (see [1]) Let Ω be a set in $\mathbb{C}$. Also let $q \in \mathbb{Q}$ and $n \in \mathbb{N} \setminus \{1\}$, $\mathbb{N}$ being the set of positive integers. The class $\Psi_n[\Omega, q]$ of admissible functions consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$, which satisfy the following admissibility conditions:

$$\psi(r, s, t, u; z) \notin \Omega$$

whenever

$$r = q(\zeta), \;\; s = k\zeta q'(\zeta), \;\; \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

and

$$\Re\left(\frac{u}{s}\right) \geqq k^2 \Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq n$.

Lemma 1 below is the foundation result in the theory of third-order differential subordination.

Lemma 1. (see [1]) *Let $p \in \mathscr{H}[a, n]$ with $n \geqq 2$ and $q \in \mathbb{Q}(a)$ satisfying the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geqq 0 \quad \textit{and} \quad \left|\frac{zq'(z)}{q'(\zeta)}\right| \leqq k,$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) \subset \Omega,$$

then

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Definition 5. (see [23]) Let Ω be a set in $\mathbb{C}$. Also let $q \in \mathscr{H}[a, n]$ and $q'(z) \neq 0$. The class $\Psi'_n[\Omega, q]$ of admissible functions consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions:

$$\psi(r, s, t, u; \zeta) \in \Omega$$

whenever

$$r = q(z), \;\; s = \frac{zq'(z)}{m}, \;\; \Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \leqq \frac{1}{m^2}\Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq n \geqq 2$.

Lemma 2. (see [23]) *Let $p \in \mathscr{H}[a, n]$ with $\psi \in \Psi'_n[\Omega, q]$. If the function*

$$\psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right)$$

is univalent in $\mathbb{U}$ and $p \in \mathbb{Q}(a)$ satisfying the following conditions:

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0 \quad \textit{and} \quad \left|\frac{zp'(z)}{q'(z)}\right| \leqq m,$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq n \geqq 2$, then

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) : z \in \mathbb{U}\right\},$$

implies that

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

The notion of the third-order differential subordination can be found in the work of Ponnusamy and Juneja [14]. The recent works by Tang *et al.* (see, for example, [22] and [23]; see also [3]) on the third-order differential subordination attracted many researchers in this field. For example, see [5,9,10,12,14,15,21,22,23]. In the present paper, we investigate suitable classes of admissible functions associated with the Srivastava-Attiya operator $J_{\mu,b} f(z)$ and derive sufficient conditions on the normalized analytic function f such that *Sandwich-type subordination* of the following form holds true:

$$h_1(z) \prec \vartheta(f) \prec q_2(z) \qquad (z \in \mathbb{U}),$$

where q_1, q_2 are univalent in $\mathbb{U}$ and ϑ is a suitable operator.

2 Results Related to the Third-Order Subordination

In this section, we start with a given set Ω and a given function q and we determine a set of admissible operators ψ so that (2) holds true. For this purpose, we introduce the following new class of admissible functions which will be required to prove the main third-order differential subordination theorems for the operator $J_{\mu,b} f(z)$ defined by (5).

Definition 6. Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \bigcap \mathscr{H}_0$. The class $\Theta_J[\Omega, q]$ of admissible function consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$$

whenever

$$\alpha = q(\zeta), \quad \beta = \frac{k\zeta q'(\zeta) + bq(\zeta)}{b+1},$$

$$\Re\left(\frac{\gamma(b+1)^2 - b^2\alpha}{(\beta(b+1) - b\alpha)} - 2b\right) \geqq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

and

$$\Re\left(\frac{\delta(b+1)^3 - \gamma(b+1)^2(3b+3) + b^2\alpha(3+2b)}{(b(\beta-\alpha)+\beta)} + 3b^2 + 6b + 2\right)$$
$$\geqq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq 2$.

Theorem 1 *Let $\phi \in \Theta_J[\Omega, q]$. If the function $f \in \mathscr{A}$ and $q \in \mathbb{Q}_0$ satisfy the following conditions*:

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geqq 0, \qquad \left|\frac{J_{\mu,b}f(z)}{q'(\zeta)}\right| \leqq k, \tag{10}$$

and

$$\left\{\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{11}$$

then

$$J_{\mu+1,b}f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Proof. Define the analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) = J_{\mu+1,b}f(z). \tag{12}$$

From equation (7) and (12), we have

$$J_{\mu,b}f(z) = \frac{zp'(z) + bp(z)}{b+1}. \tag{13}$$

By a similar argument, we get

$$J_{\mu-1,b}f(z) = \frac{z^2p''(z) + (2b+1)zp'(z) + b^2p(z)}{(b+1)^2} \tag{14}$$

and

$$J_{\mu-2,b}f(z) = \frac{z^3p'''(z) + (3b+3)z^2p''(z) + (3b^2+3b+1)zp'(z) + b^3p(z)}{(b+1)^3}. \tag{15}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \qquad \beta(r,s,t,u) = \frac{s+br}{b+1},$$

$$\gamma(r,s,t,u) = \frac{t + (2b+1)s + b^2r}{(b+1)^2} \tag{16}$$

and

$$\delta(r,s,t,u) = \frac{u + (3b+3)t + (3b^2+3b+1)s + b^3r}{(b+1)^3}. \tag{17}$$

Let

$$\psi(r,s,t,u) = \phi(\alpha, \beta, \gamma, \delta; z)$$
$$= \phi\left(r, \frac{s+br}{b+1}, \frac{t+(2b+1)s+b^2r}{(b+1)^2}, \frac{u+(3b+3)t+(3b^2+3b+1)s+b^3r}{(b+1)^3}; z\right). \tag{18}$$

The proof will make use of Lemma 1. Using the equations (12) to (15), and from the equation (18), we have

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) = \phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right). \tag{19}$$

Hence, clearly, (11) becomes

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Omega.$$

We note that

$$\frac{t}{s} + 1 = \frac{\gamma(b+1)^2 - b^2\alpha}{(\beta(b+1) - b\alpha)} - 2b$$

and

$$\frac{u}{s} = \frac{\delta(b+1)^3 - \gamma(b+1)^2(3b+3) + b^2\alpha(3+2b)}{(b(\beta-\alpha)+\beta)}.$$

Thus, clearly, the admissibility condition for $\phi \in \Theta_J[\Omega, q]$ in Definition 6 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 4 with $n = 2$. Therefore, by using (10) and Lemma 1, we have

$$J_{\mu+1,b}f(z) \prec q(z).$$

This completes the proof of Theorem 1.

Our next result is a consequence of Theorem 1 for the case when the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 1 *Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\phi \in \Theta_J[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathscr{A}$ and q_ρ satisfies the following conditions*:

$$\Re\left(\frac{\zeta q_\rho''(\zeta)}{q_\rho'(\zeta)}\right) \geqq 0, \qquad \left|\frac{J_{\mu,b}f(z)}{q_\rho'(\zeta)}\right| \leqq k \qquad (z \in \mathbb{U};\ k \geqq 2;\ \zeta \in \partial\mathbb{U} \setminus E(q_\rho))$$

and

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) \in \Omega,$$

then

$$J_{\mu+1,b}f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Proof. By applying Theorem 1, we get

$$J_{\mu+1,b}f(z) \prec q_\rho(z) \qquad (z \in \mathbb{U}).$$

The result asserted by Corollary 1 is now deduced from the following subordination property

$$q_\rho(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

This completes the proof of Corollary 1.

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Theta_J[h(\mathbb{U}),q]$ is written as $\Theta_J[h,q]$. This leads to the following immediate consequence of Theorem 1.

Theorem 2 *Let $\phi \in \Theta_J[h,q]$. If the function $f \in \mathscr{A}$ and $q \in \mathbb{Q}_0$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geqq 0, \quad \left|\frac{J_{\mu,b}f(z)}{q'(\zeta)}\right| \leqq k, \tag{20}$$

and

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) \prec h(z), \tag{21}$$

then

$$J_{\mu+1,b}f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

The next result is an immediate consequence of Corollary 1.

Corollary 2 *Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in $\mathbb{U}$ with $q(0) = 0$. Also let $\phi \in \Theta_J[h,q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathscr{A}$ and q_ρ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''_\rho(\zeta)}{q'_\rho(\zeta)}\right) \geqq 0, \quad \left|\frac{J_{\mu,b}f(z)}{q'_\rho(\zeta)}\right| \leqq k \qquad (z \in \mathbb{U};\ k \geqq 2;\ \zeta \in \partial\mathbb{U} \setminus E(q_\rho)),$$

and

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) \prec h(z),$$

then

$$J_{\mu+1,b}f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

The following result yields the best dominant of the differential subordination (21).

Theorem 3 *Let the function h be univalent in $\mathbb{U}$. Also let $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ and ψ be given by (18). Suppose that the following differential equation:*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z), \tag{22}$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies the condition (10). If the function $f \in \mathscr{A}$ satisfies the condition (21) and if

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right)$$

is analytic in $\mathbb{U}$, then

$$J_{\mu+1,b}f(z) \prec q(z) \qquad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.

Proof. From Theorem 1, we see that q is a dominant of (21). Since q satisfies (22), it is also a solution of (21). Therefore, q will be dominated by all dominants. Hence q is the best dominant. This completes the proof of Theorem 3.

In view of Definition 6, and in the special case when $q(z) = Mz \quad (M > 0)$, the class $\Theta_J[\Omega,q]$ of admissible functions, denoted by $\Theta_J[\Omega,M]$, is expressed as follows.

Definition 7. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class $\Theta_J[\Omega,M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ such that

$$\phi\Bigg(Me^{i\theta}, \frac{(k+b)Me^{i\theta}}{b.+1}, \frac{L+[(2b+1)k+b^2]Me^{i\theta}}{(b+1)^2}, \frac{N+(3b+3)L+[(3b^2+3b+1)k+b^3]Me^{i\theta}}{(b+1)^3}; z\Bigg) \notin \Omega \tag{23}$$

whenever $z \in \mathbb{U}$,

$$\Re(Le^{-i\theta}) \geqq (k-1)kM$$

and

$$\Re\left(Ne^{-i\theta}\right) \geqq 0 \qquad \forall\ \theta \in \mathbb{R};\ k \geqq 2).$$

Corollary 3 *Let $\phi \in \Theta_J[\Omega,M]$. If the function $f \in \mathscr{A}$ satisfies the following conditions:*

$$\left|J_{\mu,b}f(z)\right| \leqq kM \qquad (z \in \mathbb{U};\ k \geqq 2;\ M > 0)$$

and

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) \in \Omega,$$

then

$$\left|J_{\mu+1,b}f(z)\right| < M.$$

In the special case when $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Theta_J[\Omega,M]$ is simply denoted by $\Theta_J[M]$. Corollary 3 can now be rewritten in the following form.

Corollary 4 *Let $\phi \in \Theta_J[M]$. If the function $f \in \mathscr{A}$ satisfies the following conditions:*

$$\left|J_{\mu,b}f(z)\right| \leqq kM \qquad (z \in \mathbb{U};\ k \geqq 2;\ M > 0)$$

and

$$\left|J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right| < M,$$

then

$$\left|J_{\mu+1,b}f(z)\right| < M.$$

Corollary 5 *Let $k \geqq 2$, $0 \neq \mu \in \mathbb{C}$ and $M > 0$. If the function $f \in \mathscr{A}$ satisfies the following conditions:*

$$|J_{\mu,b}f(z)| \leqq kM$$

and

$$\left|J_{\mu,b}f(z) - J_{\mu+1,b}f(z)\right| < \frac{M}{|b+1|},$$

then

$$|J_{\mu+1,b}f(z)| < M.$$

Proof. Let

$$\phi(\alpha,\beta,\gamma,\delta;z)=\beta-\alpha \quad \text{and} \quad \Omega=h(\mathbb{U}),$$

where

$$h(z)=\frac{Mz}{|b+1|} \qquad (M>0).$$

Use Corollary 3, we need to show that $\phi\in\Theta_J[\Omega,M]$, that is, that the admissibility condition (23) is satisfied. This follows readily, since it is seen that

$$|\phi(v,w,x,y;z)|=\left|\frac{(k-1)Me^{i\theta}}{b+1}\right|\geqq\frac{M}{|b+1|}$$

whenever $z\in\mathbb{U}$, $\theta\in\mathbb{R}$ and $k\geqq 2$. The required result now follows from Corollary 3. This completes the proof of Corollary 5.

Definition 8. Let Ω be a set in $\mathbb{C}$ and $q\in\mathbb{Q}_1\cap\mathscr{H}_1$. The class $\Theta_{J,1}[\Omega,q]$ of admissible functions consists of those functions $\phi:\mathbb{C}^4\times\mathbb{U}\longrightarrow\mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\alpha,\beta,\gamma,\delta;z)\notin\Omega$$

whenever

$$\alpha=q(\zeta), \qquad \beta=\frac{k\zeta q'(\zeta)+(b+1)q(\zeta)}{b+1},$$

$$\Re\left(\frac{(b+1)(\gamma-\alpha)}{\beta-\alpha}-2(1+b)\right)\geqq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right)$$

and

$$\Re\left(\frac{\delta(1+b)^2-3\gamma(b+2)(b+1)+3\alpha(b+2)(b+1)-(1+b)^2\alpha}{\beta-\alpha}+3b^2+12b+11\right)$$
$$\geqq k^2\Re\left(\frac{\zeta^2q'''(\zeta)}{q'(\zeta)}\right),$$

where $z\in\mathbb{U}$, $\zeta\in\partial\mathbb{U}\setminus E(q)$ and $k\geqq 2$.

Theorem 4 *Let $\phi\in\Theta_{J,1}[\Omega,q]$. If the function $f\in\mathscr{A}$ and $q\in\mathbb{Q}_1$ satisfy the following conditions*:

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)\geqq 0, \qquad \left|\frac{J_{\mu,b}f(z)}{zq'(\zeta)}\right|\leqq k \tag{24}$$

and

$$\left\{\phi\left(\frac{J_{\mu+1,b}f(z)}{z},\frac{J_{\mu,b}f(z)}{z},\frac{J_{\mu-1,b}f(z)}{z},\frac{J_{\mu-2,b}f(z)}{z};z\right):z\in\mathbb{U}\right\}\subset\Omega, \tag{25}$$

then

$$\frac{J_{\mu+1,b}f(z)}{z}\prec q(z) \qquad (z\in\mathbb{U}).$$

Proof. Define the analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z)=\frac{J_{\mu+1,b}f(z)}{z}. \tag{26}$$

From the equations (7) and (26), we have

$$\frac{J_{\mu,b}f(z)}{z}=\frac{zp'(z)+(b+1)p(z)}{b+1}. \tag{27}$$

By a similar argument, we get

$$\frac{J_{\mu-1,b}f(z)}{z}=\frac{z^2p''(z)+zp'(z)(3+2b)+p(z)(1+b)^2}{(b+1)^2} \tag{28}$$

and

$$\frac{J_{\mu-2,b}f(z)}{z}=\frac{z^3p'''(z)+3(b+2)z^2p''(z)+(3b^2+9b+7)zp'(z)+p(z)(b+1)^3}{(b+1)^3}. \tag{29}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u)=r, \qquad \beta(r,s,t,u)=\frac{s+(b+1)r}{(b+1)},$$

$$\gamma(r,s,t,u)=\frac{t+(3+2b)s+(b+1)^2r}{(b+1)^2} \tag{30}$$

and

$$\delta(r,s,t,u)=\frac{u+3(b+2)t+(3b^2+9b+7)s+(b+1)^3r}{(b+1)^3}. \tag{31}$$

Let

$$\psi(r,s,t,u)=\phi(\alpha,\beta,\gamma,\delta;z)=\phi\left(r,\frac{s+(1+b)r}{(1+b)},\frac{t+(3+2b)s+(b+1)^2r}{(b+1)^2},\right.$$
$$\left.\frac{u+3(b+2)t+(3b^2+9b+7)s+(b+1)^3r}{(b+1)^3};z\right) \tag{32}$$

The proof will make use of Lemma 1. Using the equations (26) to (29), and from (32), we have

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=\phi\left(\frac{J_{\mu+1,b}f(z)}{z},\frac{J_{\mu,b}f(z)}{z},\frac{J_{\mu-1,b}f(z)}{z},\frac{J_{\mu-2,b}f(z)}{z};z\right). \tag{33}$$

Hence the equation (25) becomes

$$\psi\left(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\right)\in\Omega.$$

We also note that

$$\frac{t}{s}+1=\frac{(b+1)(\gamma-\alpha)}{\beta-\alpha}-2(1+b)$$

and

$$\frac{u}{s}=\frac{\delta(1+b)^2-3\gamma(b+2)(b+1)+3\alpha(b+2)(b+1)-(1+b)^2\alpha}{\beta-\alpha}$$
$$+3b^2+12b+11.$$

Thus, clearly, the admissibility condition for $\phi\in\Theta_{J,1}[\Omega,q]$ in Definition 8 is equivalent to the admissibility condition for $\psi\in\Psi_2[\Omega,q]$ as given in Definition 4 with $n=2$. Therefore, by using (24) and Lemma 1, we have

$$\frac{J_{\mu+1,b}f(z)}{z}\prec q(z).$$

This completes the proof of Theorem 4.

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Theta_{J,1}[h(\mathbb{U}),q]$ is written as $\Theta_{J,1}[h,q]$. An immediate consequence of Theorem 4 is stated below.

Theorem 5 *Let $\phi \in \Theta_{J,1}[h,q]$. If the function $f \in \mathscr{A}$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geqq 0, \quad \left|\frac{J_{\mu,b}f(z)}{zq'(\zeta)}\right| \leqq k \tag{34}$$

and

$$\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right) \prec h(z), \tag{35}$$

then

$$\frac{J_{\mu+1,b}f(z)}{z} \prec q(z) \qquad (z \in \mathbb{U}).$$

In view of Definition 8, and in the special case when $q(z) = Mz \quad (M > 0)$, the class $\Theta_{J,1}[\Omega,q]$ of admissible functions, denoted by $\Theta_{J,1}[\Omega,M]$, is expressed as follows.

Definition 9. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class $\Theta_{J,1}[\Omega,M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{(k+1+b)Me^{i\theta}}{1+b}, \frac{L+[(3+2b)k+(b+1)^2]Me^{i\theta}}{(b+1)^2}, \frac{N+3(b+2)L+[(3b^2+9b+7)k+(b+1)^3]Me^{i\theta}}{(b+1)^3}; z\right) \notin \Omega. \tag{36}$$

whenever $z \in \mathbb{U}$,

$$\Re\left(Le^{-i\theta}\right) \geqq (k-1)kMm$$

and

$$\Re\left(Ne^{-i\theta}\right) \geqq 0 \qquad (\forall\, \theta \in \mathbb{R};\ k \geqq 2).$$

Corollary 6 *Let $\phi \in \Theta_{J,1}[\Omega,M]$. If the function $f \in \mathscr{A}$ satisfies the following conditions:*

$$\left|\frac{J_{\mu,b}f(z)}{z}\right| \leqq kM \qquad (z \in \mathbb{U};\ k \geqq 2;\ M > 0)$$

and

$$\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right) \in \Omega,$$

then

$$\left|\frac{J_{\mu+1,b}f(z)}{z}\right| < M.$$

In the special case when $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Theta_{J,1}[\Omega,M]$ is simply denoted by $\Theta_{J,1}[M]$. Corollary 6 can now be rewritten in the following form.

Corollary 7 *Let $\phi \in \Theta_{J,1}[M]$. If the function $f \in \mathscr{A}$ satisfies the following conditions:*

$$\left|\frac{J_{\mu,b}f(z)}{z}\right| \leqq kM \qquad (z \in \mathbb{U};\ k \geqq 2;\ M > 0)$$

and

$$\left|\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}\right); z\right| < M,$$

then

$$\left|\frac{J_{\mu+1,b}f(z)}{z}\right| < M.$$

Definition 10. Let Ω be a set in $\mathbb{C}$ and let $q \in \mathbb{Q}_1 \cap \mathscr{H}_1$. The class $\Theta_{J,2}[\Omega,q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\alpha,\beta,\gamma,\delta;z) \notin \Omega$$

whenever

$$\alpha = q(\zeta), \quad \beta = \frac{1}{(b+1)}\left(\frac{k\zeta q'(\zeta)}{q(\zeta)} + (b+1)q(\zeta)\right),$$

$$\Re\left(\frac{(1+b)(\beta\gamma+2\alpha^2-3\alpha\beta)}{(\beta-\alpha)}\right) \geqq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right)$$

and

$$\Re\Big((\delta-\gamma)(1+b)^2\beta\gamma-(1+b)^2(\gamma-\beta)\beta(1-\beta-\gamma+3\alpha)-3(b+1)(\gamma-\beta)\beta+2(\beta-\alpha)+3(1+b)\alpha(\beta-\alpha)$$
$$+(\beta-\alpha)^2(1+b)((\beta-\alpha)(1+b)-3-4(1+b)\alpha)+\alpha^2(1+b)^2(\beta-\alpha)\Big)\cdot(\beta-\alpha)^{-1} \geqq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq 2$.

Theorem 6 *Let $\phi \in \Theta_{J,2}[\Omega,q]$. If the function $f \in \mathscr{A}$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geqq 0, \quad \left|\frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)q'(\zeta)}\right| \leqq k \tag{37}$$

and

$$\left\{\phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{38}$$

then

$$\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \prec q(z) \qquad (z \in \mathbb{U}).$$

Proof. Define the analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) = \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}. \tag{39}$$

From the equations (7) and (39), we have

$$\frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)} = \frac{1}{(b+1)}\left[\frac{zp'(z)}{p(z)} + (b+1)p(z)\right] := \frac{A}{b+1}. \tag{40}$$

By a similar argument, we get

$$\frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)} := \frac{B}{b+1} \tag{41}$$

and

$$\frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)} = \frac{1}{b+1}\left[B+B^{-1}(C+A^{-1}D-A^{-2}C^2)\right], \tag{42}$$

where

$$B := (b+1)p(z)+\frac{zp'(z)}{p(z)}+\frac{\frac{z^2p''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+(b+1)zp'(z)}{\frac{zp'(z)}{p(z)}+(b+1)p(z)},$$

$$C := \frac{z^2p''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+(b+1)zp'(z)$$

and

$$D := \frac{3z^2p''(z)}{p(z)}+\frac{z^3p'''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-3\left(\frac{zp'(z)}{p(z)}\right)^2-\frac{3z^3p'(z)p''(z)}{p^2(z)}+2\left(\frac{zp'(z)}{p(z)}\right)^3 + (b+1)zp'(z)+(b+1)z^2p''(z).$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \qquad \beta(r,s,t,u) = \frac{1}{b+1}\left[\frac{s}{r}+(b+1)r\right] := \frac{E}{b+1},$$

$$\gamma(r,s,t,u) = \frac{1}{b+1}\left[\frac{s}{r}+(b+1)r+\frac{\frac{t}{r}+\frac{s}{r}-(\frac{s}{r})^2+(b+1)s}{\frac{s}{r}+(b+1)r}\right] := \frac{F}{b+1} \tag{43}$$

and

$$\delta(r,s,t,u) = \frac{1}{b+1}\left[F+F^{-1}(L+E^{-1}H-E^{-2}L^2)\right], \tag{44}$$

where

$$L := (1+b)s+\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2$$

and

$$H := \frac{3t}{r}+\frac{u}{r}+\frac{s}{r}-3\left(\frac{s}{r}\right)^2-3\left(\frac{st}{r^2}\right) + 2\left(\frac{s}{r}\right)^3+(1+b)s+(1+b)t.$$

Let

$$\psi(r,s,t,u) = \phi(\alpha,\beta,\gamma,\delta;z) = \phi\left(r,\frac{E}{b+1},\frac{F}{b+1},\frac{F+F^{-1}(L+E^{-1}H-E^{-2}L^2)}{b+1}\right). \tag{45}$$

The proof will make use of Lemma 1. Using the equations (39) to (42), and from (45), we have

$$\psi\left(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\right) = \phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)},\frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)},\frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)},\frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)};z\right). \tag{46}$$

Hence the (38) becomes

$$\psi\left(p(z),zp'(z),z^2p''(z),z^3p'''(z);z\right) \in \Omega.$$

We note that

$$\frac{t}{s}+1 = \left(\frac{(1+b)(\beta\gamma+2\alpha^2-3\alpha\beta)}{(\beta-\alpha)}\right)$$

and

$$\frac{u}{s} = \Big[(\delta-\gamma)(1+b)^2\beta\gamma-(1+b)^2(\gamma-\beta)\beta(1-\beta-\gamma+3\alpha) -3(b+1)(\gamma-\beta)\beta+2(\beta-\alpha)+3(1+b)\alpha(\beta-\alpha) +(\beta-\alpha)^2(1+b)((\beta-\alpha)(1+b)-3-4(1+b)\alpha) +\alpha^2(1+b)^2(\beta-\alpha)\Big]\cdot(\beta-\alpha)^{-1}.$$

Thus, clearly, the admissibility condition for $\phi \in \Theta_{J,2}[\Omega,q]$ in Definition 10 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega,q]$ as given in Definition 4 with $n=2$. Therefore, by using (37) and Lemma 1, we have

$$\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \prec q(z) \qquad (z \in \mathbb{U}). \tag{47}$$

This completes the proof of Theorem 6.

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Theta_{J,1}[h(\mathbb{U}),q]$ is written simply as $\Theta_{J,2}[h,q]$. An immediate consequence of Theorem 6 is now stated below without proof.

Theorem 7 *Let $\phi \in \Theta_{J,2}[h,q]$. If the function $f \in \mathscr{A}$ and $q \in \mathbb{Q}_1$ satisfy the conditions (37) and*

$$\phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)},\frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)},\frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)},\frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)};z\right) \prec h(z), \tag{48}$$

then

$$\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \prec q(z) \qquad (z \in \mathbb{U}).$$

3 Results Related to the Third-Order Superordination

In this section, we investigate and prove several theorems involving the third-order differential superordination for the operator $J_{\mu,b}f(z)$ defined in (6). For the purpose, we consider the following class of admissible functions.

Definition 11. Let Ω be a set in $\mathbb{C}$ and let $q \in \mathscr{H}_0$ with $q'(z) \neq 0$. The class $\Theta'_J[\Omega,q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathscr{U}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(\alpha,\beta,\gamma,\delta;\zeta) \in \Omega$$

whenever

$$\alpha = q(z), \beta = \frac{zq'(z) + mbq(z)}{m(b+1)},$$

$$\Re\left(\frac{\gamma(b+1)^2 - b^2\alpha}{(\beta(b+1) - b\alpha)} - 2b\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\Re\left(\frac{\delta(b+1)^3 - \gamma(b+1)^2(3b+3) + b^2\alpha(3+2b)}{(b(\beta-\alpha)+\beta)} + 3b^2 + 6b + 2\right) \leqq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq 2$.

Theorem 8 *Let $\phi \in \Theta'_J[\Omega, q]$. If the function $f \in \mathscr{A}$, with $J_{\mu+1,b}f(z) \in \mathbb{Q}_0$, and if $q \in \mathscr{H}_0$ with $q'(z) \neq 0$, satisfying the following conditions:*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0, \quad \left|\frac{J_{\mu,b}f(z)}{q'(z)}\right| \leqq m \tag{49}$$

and the function

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right),$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) : z \in \mathbb{U}\right\}, \tag{50}$$

implies that

$$q(z) \prec J_{\mu+1,b}f(z) \qquad (z \in \mathbb{U}).$$

Proof. Let the function $p(z)$ be defined by (12) and ψ by (18). Since $\phi \in \Theta'_J[\Omega, q]$, from (19) and (50), we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in \mathbb{U}\right\}.$$

From (16) and (17), we see that the admissibility condition for $\phi \in \Theta'_J[\Omega, q]$ in Definition 11 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 5 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega, q]$ and, by using (50) and Lemma 2, we find that

$$q(z) \prec J_{\mu+1,b}f(z) \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 8.

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Theta'_J[h(\mathbb{U}), q]$ is written simply as $\Theta'_J[h, q]$. Theorem 9 follows as an immediate consequence of Theorem 8.

Theorem 9 *Let $\phi \in \Theta'_J[h, q]$ and let h be analytic in $\mathbb{U}$. If the function $f \in \mathscr{A}$ and $J_{\mu+1,b}f(z) \in \mathbb{Q}_0$, and if $q \in \mathscr{H}_0$ with $q'(z) \neq 0$, satisfying the conditions (49) and the function*

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right),$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) \tag{51}$$

implies that

$$q(z) \prec J_{\mu+1,b}f(z) \qquad (z \in \mathbb{U}).$$

Theorems 8 and 9 can only be used to obtain subordination for the third-order differential superordination of the form (50) or (51). The following theorem gives the existence of the best subordinant of (51) for a suitable ϕ.

Theorem 10 *Let the function h be univalent in $\mathbb{U}$ and let $\phi : \mathbb{C}^4 \times \overline{\mathscr{U}} \longrightarrow \mathbb{C}$ and let ψ be given by (18). Suppose that the following differential equation:*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z) \tag{52}$$

has a solution $q(z) \in \mathbb{Q}_0$. If the function $f \in \mathscr{A}$, with $J_{\mu+1,b}f(z) \in \mathbb{Q}_0$ and if $q \in \mathscr{H}_0$ with $q'(z) \neq 0$, satisfying the conditions (49) and

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right),$$

is analytic in $\mathbb{U}$, then

$$h(z) \prec \phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right)$$

implies that

$$q(z) \prec J_{\mu+1,b}f(z) \qquad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.

Proof. By applying Theorem 8 and Theorem 9, we deduce that q is a subordinant of (51). Since q satisfies (52), it is also a solution of (51) and, therefore, q will be subordinated by all subordinants. Hence q is the best subordinant. This completes the proof of Theorem 10.

Definition 12. Let Ω be a set in $\mathbb{C}$ and let $q \in \mathscr{H}_1$ with $q'(z) \neq 0$. The class $\Theta'_{J,1}[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathscr{U}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\alpha, \beta, \gamma, \delta; \zeta) \in \Omega$$

whenever

$$\alpha = q(z), \quad \beta = \frac{zq'(z) + (1+b)mq(z)}{(1+b)m},$$

$$\Re\left(\frac{(b+1)(\gamma-\alpha)}{\beta-\alpha} - 2(1+b)\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\Re\left(\frac{\delta(1+b)^2 - 3\gamma(b+2)(b+1) + 3\alpha(b+2)(b+1) - (1+b)^2\alpha}{\beta-\alpha} + 3b^2 + 12b + 11\right)$$
$$\leqq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq 2$.

Theorem 11 *Let $\phi \in \Theta'_{J,1}[\Omega,q]$. If the function $f \in \mathscr{A}$, with $\frac{J_{\mu,b}f(z)}{z} \in \mathbb{Q}_1$, and if $q \in \mathscr{H}_1$ with $q'(z) \neq 0$, satisfying the following conditions*:

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0, \left|\frac{J_{\mu,b}f(z)}{zq'(z)}\right| \leqq m \tag{53}$$

and the function

$$\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right),$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right) : z \in \mathbb{U}\right\} \tag{54}$$

implies that

$$q(z) \prec \frac{J_{\mu+1,b}f(z)}{z} \qquad (z \in \mathbb{U}).$$

Proof. Let the function $p(z)$ be defined by (26) and ψ by (32). Since $\phi \in \Theta'_{J,1}[\Omega,q]$, we find from (33) and (54) that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in \mathbb{U}\right\}.$$

From the equations (30) and (31), we see that the admissible condition for $\phi \in \Theta'_{J,1}[\Omega,q]$ in Definition 12 is equivalent to the admissible condition for ψ as given in Definition 5 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega,q]$ and, by using (53) and Lemma 2, we have

$$q(z) \prec \frac{J_{\mu+1,b}f(z)}{z} \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 11.

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Theta'_{J,1}[h(\mathbb{U}),q]$ is written simply as $\Theta'_{J,1}[h,q]$. Theorem 12 follows as an immediate consequence of Theorem 11.

Theorem 12 *Let $\phi \in \Theta'_{J,1}[h,q]$ and let h be analytic in $\mathbb{U}$. If the function $f \in \mathscr{A}$, with $q \in \mathscr{H}_1$ and $q'(z) \neq 0$, satisfying the conditions (53) and the function*

$$\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right),$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right),$$

implies that

$$q(z) \prec \frac{J_{\mu+1,b}f(z)}{z} \qquad (z \in \mathbb{U}).$$

Definition 13. Let Ω be a set in $\mathbb{C}$ and $q \in \mathscr{H}_1$ with $q'(z) \neq 0$. The class $\Theta'_{J,2}[\Omega,q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathscr{U}} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(\alpha,\beta,\gamma,\delta;\zeta) \in \Omega$$

whenever

$$\alpha = q(z), \quad \beta = \frac{1}{b+1}\left(\frac{zq'(z)}{mq(z)} + (b+1)q(z)\right),$$

$$\Re\left(\frac{(1+b)(\beta\gamma+2\alpha^2-3\alpha\beta)}{(\beta-\alpha)}\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)}+1\right)$$

and

$$\begin{aligned}\Re\Big(&(\delta-\gamma)(1+b)^2\beta\gamma-(1+b)^2(\gamma-\beta)\beta(1-\beta-\gamma+3\alpha)\\ &-3(b+1)(\gamma-\beta)\beta+2(\beta-\alpha)+3(1+b)\alpha(\beta-\alpha)\\ &+(\beta-\alpha)^2(1+b)((\beta-\alpha)(1+b)-3-4(1+b)\alpha)\\ &+\alpha^2(1+b)^2(\beta-\alpha)\Big)\cdot(\beta-\alpha)^{-1} \leqq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),\end{aligned}$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq 2$.

Theorem 13 *Let $\phi \in \Theta'_{J,2}[\Omega,q]$. If the function $f \in \mathscr{A}$, with $\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \in \mathbb{Q}_1$ and if $q \in \mathscr{H}_1$ with $q'(z) \neq 0$, satisfying the following conditions*:

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0, \qquad \left|\frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)q'(z)}\right| \leqq m \tag{55}$$

and the function

$$\phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right),$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right) : z \in \mathbb{U}\right\} \tag{56}$$

implies that

$$q(z) \prec \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \qquad (z \in \mathbb{U}).$$

Proof. Let the function $p(z)$ be defined by (39) and ψ by (45). Since $\phi \in \Theta'_{J,2}[\Omega,q]$, we find from (46) and (56) that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in \mathbb{U}\right\}.$$

From the equations (43) and (44), we see that the admissible condition for $\phi \in \Theta'_{J,2}[\Omega,q]$ in Definition 13 is equivalent to the admissible condition for ψ as given in Definition 5 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega,q]$ and, by using (55) and Lemma 2, we have

$$q(z) \prec \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 13.

Theorem 14 *Let $\phi \in \Theta'_{J,2}[h,q]$. If the function $f \in \mathscr{A}$ and $\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \in \mathbb{Q}_1$, with $q \in \mathscr{H}_1$ and $q'(z) \neq 0$, satisfying the conditions (55) and the function*

$$\phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right),$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right) \tag{57}$$

implies that

$$q(z) \prec \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \qquad (z \in \mathbb{U}).$$

4 A Set of Sandwich-Type Results

By combining Theorems 2 and 9, we obtain the following sandwich-type theorem.

Theorem 15 *Let h_1 and q_1 be analytic functions in $\mathbb{U}$. Also let h_2 be univalent function in $\mathbb{U}$ and $q_2 \in \mathbb{Q}_0$ with*

$$q_1(0) = q_2(0) = 0 \qquad \text{and} \qquad \phi \in \Theta_J[h_2,q_2] \cap \Theta'_J[h_1,q_1].$$

If the function $f \in \mathscr{A}$, with $J_{\mu+1,b}f(z) \in \mathbb{Q}_0 \cap \mathscr{H}_0$ and the function

$$\phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right),$$

is univalent in $\mathbb{U}$, and if the conditions (10) and (49) are satisfied, then

$$h_1(z) \prec \phi\left(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec J_{\mu+1,b}f(z) \prec q_2(z) \qquad (z \in \mathbb{U}). \tag{58}$$

If, on the other hand, we combine Theorems 5 and 12, we obtain the following sandwich-type theorem.

Theorem 16 *Let h_1 and q_1 be analytic functions in $\mathbb{U}$. Also let h_2 be univalent function in $\mathbb{U}$ and $q_2 \in \mathbb{Q}_1$ with*

$$q_1(0) = q_2(0) = 1 \qquad \text{and} \qquad \phi \in \Theta_{J,1}[h_2,q_2] \cap \Theta'_{J,1}[h_1,q_1].$$

If the function $f \in \mathscr{A}$, with $\frac{J_{\mu+1,b}f(z)}{z} \in \mathbb{Q}_1 \cap \mathscr{H}_1$ and the function

$$\phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right),$$

is univalent in $\mathbb{U}$, and the conditions (24) and (53) are satisfied, then

$$h_1(z) \prec \phi\left(\frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{J_{\mu+1,b}f(z)}{z} \prec q_2(z) \qquad (z \in \mathbb{U}). \tag{59}$$

Finally, by combining Theorems 7 and 14, we obtain the following sandwich-type theorem.

Theorem 17 *Let h_1 and q_1 be analytic functions in $\mathbb{U}$. Also let h_2 be univalent functions in $\mathbb{U}$ and $q_2 \in \mathbb{Q}_1$ with*

$$q_1(0) = q_2(0) = 1 \qquad \text{and} \qquad \phi \in \Theta_{J,2}[h_2,q_2] \cap \Theta'_{J,2}[h_1,q_1].$$

If the function $f \in \mathscr{A}$, with $\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \in \mathbb{Q}_1 \cap \mathscr{H}_1$ and the function

$$\phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right),$$

is univalent in $\mathbb{U}$, and the conditions (37) and (55) are satisfied, then

$$h(z) \prec \phi\left(\frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \prec q_2(z) \qquad (z \in \mathbb{U}). \tag{60}$$

5 Perspective

In our present investigation, we have made use of the linear operator introduced and studied by Srivastava and Attiya [16], to systematically investigate several suitable classes of admissible functions. We have presented the dual properties of the third-order differential subordinations. As consequences of some of our main results, various sandwich-type theorems are established for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform. We have also indicated relevant connections of the new results presented in this article with those that were considered in earlier works.

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