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Bayesian Prediction Based on Unified Hybrid Censored Data from the Exponentiated Rayleigh Distribution

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Abstract: In this article, we studied one- and two-sample Bayesian prediction intervals based on unified hybrid censored data from the exponentiated Rayleigh distribution. We use Markov chain Monte Carlo (MCMC) samples to obtain the approximate predictive survival function, since one- and two- sample Bayesian predictive survival function can not be computed in closed-form. Finally, one- and two-sample Bayesian prediction intervals are explored based on a real data set as illustrative example.

Keywords: Exponentiated Rayleigh distribution, Unified hybrid censoring scheme, Maximum likelihood estimators, Bayesian Prediction, MCMC method.

1 Introduction

Epstein [1] considered a hybrid censored scheme (HCS), which is a mixture of Type-I and Type-II censoring schemes, these schemes have been used in practice. However, these censoring schemes have some disadvantages. To avoid these disadvantages, Chandrasekar et al.[2] proposed two new schemes which are called generalized Type-I and Type-II HCS. In generalized Type-I HCS, fix $k,r \in (1,2,...,n)$ and $T \in (0,\infty)$ such that $k<r<n$. If the k^{th} failure occurs before time T, the experiment is terminated at $\min\{X_{r:n},T\}$. If the k^{th} failure occurs after time T, the experiment is terminated at $X_{k:n}$, so, it is clear that this HCS modifies the Type-I HCS by allowing the experiment to continue after time T if very few failures had observed until that time. In generalized Type-II HCS, fix $r \in (1,2,...,n)$ and $T_1,T_2 \in (0,\infty)$ such that $T_2>T_1$. If the r^{th} failure occurs before time T_1, the experiment is terminated at T_1. If the r^{th} failure occurs between T_1 and T_2, the experiment is terminated at $X_{r:n}$. If the r^{th} failure occurs after T_2, the experiment is terminated at T_2. This hybrid censoring scheme guarantees that the experiment time will not exceed T_2. Although generalized hybrid censoring schemes are improvements over Type I and Type II hybrid censoring schemes but they have some drawbacks. To avoid the drawbacks in these schemes, Balakrishnan et al.[3] introduced a mixture of generalized Type-I and Type-II HCS which is called the unified hybrid censoring scheme (UHCS), which can be described as follows, fix $r,k \in \{1,....,n\}$ where $k<r<n$ and $T_1,T_2 \in (0,\infty)$ where $T_2>T_1$. If the k^{th} failure occurs before time T_1, the experiment is terminated at $\min\{\max\{X_{r:n},T_1\},T_2\}$. If the k^{th} failure occurs between T_1 and T_2, the experiment is terminated at $\min\{X_{r:n},T_2\}$ and if the k^{th} failure occurs after time T_2, the experiment is terminated at $X_{k:n}$. Under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures.

The exponentiated Rayleigh (ER) distribution has been used for the lifetime modelling in reliability analysis, life testing problems and acceptance sampling plans. The ER distribution is obtained by generalization of the Rayleigh distribution. It is also called the two parameter (scale and shape) Burr type X distribution. The ER distribution was studied by Sartawi and Abu-Salih[4], Jaheen[5,6], Ahmad et al.[7], Raqab[8] and Surles and Padgett[9].

The cumulative distribution function (CDF) is given by

$$F(x;\alpha,\beta)=(1-e^{-\beta x^2})^{\alpha}, \qquad x>0, \ \ (\alpha,\ \beta>0). \tag{1}$$

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The probability density function (PDF) is

$$f(x;\alpha,\beta)=2\alpha\beta\ x\ e^{-\beta x^2}(1-e^{-\beta x^2})^{\alpha-1},\qquad x>0,\ \ (\alpha,\ \beta>0). \tag{2}$$

Here α and β are the shape and scale parameters, respectively.
Recently, Surles and Padgett[10] introduced two parameter Burr Type X distribution and correctly named as the ER distribution. The two parameter ER distribution is a special case of the Weibull distribution originally suggested by Mudholkar and Srivastava[11]. See also, Kundu and Raqab[12], Raqab and Madi[13], Abd-Elfattah[14], Raqab and Madi[15] and Mahmoud and Ghazal[16].
Prediction is one of the most important issues in statistical inference and it is equally important and useful as statistical estimation. Meteorology, medicine, economics, finance, engineering, politics and education are applied disciplines in which prediction is essential and is therefore of greet interest. Prediction of future observation comes up quite naturally in many life-testing experiments. Several researchers have considered Bayesian prediction for future observations based on different forms of observed data, see, for example; Geisser[17], Dunsmore[18], AL-Hussaini[19], AL-Hussaini and Ahmad[20], Shafay and Balakrishnan[21] and Balakrishnan and Shafay [22]. Recently, Shafay[23,24] developed procedures for determining one- and two-sample Bayesian prediction intervals based on generalized Type-II HCS and generalized Type-I HCS, respectively and Mohie El-Din and Shafay [25] discussed the statistical inference under unified hybrid censoring scheme
In this paper, we discuss the same problem based on UHCS which involves some additional complications. Let $X_{1:n}<X_{2:n}<...<X_{n:n}$ be the order statistics from a random sample of size n from an absolutely continuous. Let D_j denote the number of $X_{i:n}$'s that are at most T_j, $j=1,2$. Then, D_j is a discrete random variable has the binomial distribution $B(n,F(T_j))$, $j=1,2$, with support $\{0,1,...,n\}$. Therefor, under the UHCS, described above, we have the following six cases:
Case I: $0<x_{k:n}<x_{r:n}<T_1<T_2$, the experiment is terminated at T_1.
Case II: $0<x_{k:n}<T_1<x_{r:n}<T_2$, the experiment is terminated at $x_{r:n}$.
Case III:$0<x_{k:n}<T_1<T_2<x_{r:n}$, the experiment is terminated at T_2.
Case IV: $0<T_1<x_{k:n}<x_{r:n}<T_2$, the experiment is terminated at $x_{r:n}$.
Case V: $0<T_1<x_{k:n}<T_2<x_{r:n}$, the experiment is terminated at T_2.
Case VI: $0<T_1<T_2<x_{k:n}<x_{r:n}$, the experiment is terminated at $x_{k:n}$.
Thus, the likelihood function of the unified hybrid censored sample $\underline{X}=(X_{1:n}<X_{2:n}<...<X_{R:n})$ is as follows:

$$L(\underline{x},\theta)=\frac{n!}{(n-R)!}\left[\prod_{i=1}^{R}f(x_i)\right]\left[1-F(C)\right]^{n-R}, \tag{3}$$

$$(R,C)=\begin{cases}(D_1,T_1), & \text{for Case I,}\\ (r,x_{r:n}), & \text{for Case II and Case IV,}\\ (D_2,T_2), & \text{for Case III and for Case V,}\\ (k,x_{k:n}), & \text{for Case VI,}\end{cases} \tag{4}$$

where R indicates the number of the total failures in experiment up to time C (the stopping time point) and D_1 and D_2 indicate the number of failures that occur before time points T_1 and T_2, respectively,
In the Bayesian approach, the unknown parameter is regarded as a realization of a random variable, which has some prior distribution. We assume that α and β are independent and have the following gamma prior distributions

$$\pi_1(\alpha)\propto\alpha^{a_1-1}e^{-b_1\alpha},\qquad \alpha>0, \tag{5}$$

$$\pi_2(\beta)\propto\beta^{a_2-1}e^{-b_2\beta},\qquad \beta>0. \tag{6}$$

Here all the hyper parameters a_1,a_2,b_1 and b_2 are assumed to be known and non-negative.
The joint prior distribution for α and β is

$$\pi(\alpha,\beta)\propto\alpha^{a_1-1}\beta^{a_2-1}e^{-(b_1\alpha+b_2\beta)}. \tag{7}$$

From (3) and (7) we obtain the joint posterior density function

$$\begin{aligned}\pi^*(\alpha,\beta|\underline{x})\propto&\,\alpha^{a_1+R-1}\beta^{a_2+R-1}e^{-\beta(b_2+\sum_{i=1}^{R}x_i^2)}e^{-\alpha[b_1-\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})]}\\ &e^{-\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})}\left[1-(1-e^{-\beta C^2})^{\alpha}\right]^{n-R}.\end{aligned} \tag{8}$$

The rest of the paper is organized as follows: In Section 2, we discuss the Bayesian prediction intervals based on UHCS from the ER distribution. In Section 3, we apply MCMC technique to obtaine the Bayesian prediction intervals. Real data set has been analyzed for illustrative purposes in Section 4. Finally, conclusions are given in Section 5.

2 Bayesian Prediction Intervals

2.1 One-sample Bayesian prediction

In this section, based on the observed UHCS $\underline{X} = (X_{1:n} < X_{2:n} < \ldots < X_{R:n})$, we develop a general procedure for deriving the interval predictions for the s^{th} future order statistic $X_{s:n}$ for ER distribution, where $R < s \leq n$. For more details about Bayesian prediction, see for example, Shafay [23,24] and Mohie El-Din and Shafay [25] the conditional density function of $X_{s:n}$, based on UHCS $\underline{X} = (X_{1:n} < X_{2:n} < \ldots < X_{R:n})$, is as follows:

$$f(x_s|\underline{\mathbf{x}}) = \begin{cases} f_1(x_s|\underline{\mathbf{x}}) & \text{if } (R,C) = (D_1,T_1), & \text{for Case I,} \\ f_2(x_s|\underline{\mathbf{x}}) & \text{if } (R,C) = (r,x_{r:n}), & \text{for Case II and Case IV,} \\ f_3(x_s|\underline{\mathbf{x}}) & \text{if } (R,C) = (D_2,T_2), & \text{for Case III and for Case V,} \\ f_4(x_s|\underline{\mathbf{x}}) & \text{if } (R,C) = (k,x_{k:n}), & \text{for Case VI,} \end{cases} \tag{9}$$

where

$$\begin{aligned} f_1(x_s|\underline{\mathbf{x}}) &= \frac{1}{P(r \leq D_1 \leq s-1)} \sum_{d=r}^{s-1} f(x_s|,\underline{\mathbf{x}}, D_1 = d) P(D_1 = d), \\ &= \sum_{d=r}^{s-1} \frac{(n-d)!\phi_d(T_1)}{(s-d-1)!(n-s)!} \\ &\quad \times \frac{[F(x_s) - F(T_1)]^{s-d-1}[1-F(x_s)]^{n-s} f(x_s)}{[1-F(T_1)]^{n-d}}, \end{aligned}$$

with $\underline{\mathbf{x}} = (x_1, \ldots x_{D_1})$, $x_s > T_1$ and $\phi_d(T_1) = \frac{P(D_1=d)}{\sum_{j=r}^{s-1} P(D_1=j)}$,

from (2.1), we get

$$f_1(x_s|\underline{\mathbf{x}}) = \sum_{d=r}^{s-1} \sum_{\omega=0}^{s-d-1} \sum_{q=0}^{n-s} A_1 [F(x_s)]^{s-d-\omega+q-1} [F(T_1)]^{\omega+d} f(x_s) \psi_j(T_1), \tag{10}$$

where

$$A_1 = \frac{(-1)^{\omega+q}(n-d)! \binom{n}{d} \binom{s-d-1}{\omega} \binom{n-s}{q}}{(s-d-1)!(n-s)!},$$

and

$$\psi_j(T_1) = \frac{1}{\sum_{j=r}^{s-1} \binom{n}{j} \left[F(T_1)\right]^j \left[1-F(T_1)\right]^{(n-j)}}.$$

And, for $x_s > x_r$, we get

$$\begin{aligned} f_2(x_s|\underline{\mathbf{x}}) = f_2(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!} \\ &\quad \times \frac{[F(x_s) - F(x_r)]^{s-r-1}[1-F(x_s)]^{n-s} f(x_s)}{[1-F(x_r)]^{n-r}}, \end{aligned}$$

with $\underline{\mathbf{x}} = (x_1, \ldots, x_r)$, so, we can get

$$f_2(x_s|x_r) = \sum_{\omega=0}^{s-r-1} \sum_{q=0}^{n-s} \frac{A_2 [F(x_s)]^{s-r-\omega+q-1} [F(x_r)]^{\omega} f(x_s)}{[1-F(x_r)]^{n-r}}, \tag{11}$$

with

$$A_2 = \frac{(-1)^{\omega+q}(n-r)! \binom{s-r-1}{\omega} \binom{n-s}{q}}{(s-r-1)!(n-s)!}.$$

Also, for $x_s > T_2$, we have

$$f_3(x_s|\underline{\mathbf{x}}) = \frac{1}{P(k \le D_2 \le r^*-1)} \sum_{d=k}^{r^*-1} f(x_s|,\underline{\mathbf{x}}, D_2 = d) P(D_2 = d),$$

$$= \sum_{d=k}^{r^*-1} \frac{(n-d)!\phi_d(T_2)}{(s-d-1)!(n-s)!} \times \frac{[F(x_s)-F(T_2)]^{s-d-1}[1-F(x_s)]^{n-s} f(x_s)}{[1-F(T_2)]^{n-d}},$$

with $\underline{\mathbf{x}} = (x_1,\dots,x_{D_2})$, $\phi_d(T_2) = \frac{P(D_2=d)}{\sum_{j=k}^{r^*-1} P(D_2=j)}$ and $r^* = \min(r,s)$. So, for $x_s > T_2$, we get

$$f_3(x_s|\underline{\mathbf{x}}) = \sum_{d=k}^{r^*-1} \sum_{\omega=0}^{s-d-1} \sum_{q=0}^{n-s} A_3 [F(x_s)]^{s-d-\omega+q-1} [F(T_2)]^{\omega+d} f(x_s) \psi_j(T_2), \tag{12}$$

where

$$A_3 = \frac{(-1)^{\omega+q}(n-d)! \binom{n}{d} \binom{s-d-1}{\omega} \binom{n-s}{q}}{(s-d-1)!(n-s)!},$$

and

$$\psi_j(T_2) = \frac{1}{\sum_{j=k}^{r^*-1} \binom{n}{j} \left[F(T_2)\right]^j \left[1-F(T_2)\right]^{(n-j)}}.$$

Finally, for $x_s > x_k$, we have

$$f_4(x_s|\underline{\mathbf{x}}) = f(x_s|x_k) = \frac{(n-k)!}{(s-k-1)!(n-s)!} \times \frac{[F(x_s)-F(x_k)]^{s-k-1}[1-F(x_s)]^{n-s} f(x_s)}{[1-F(x_k)]^{n-k}},$$

with $\underline{\mathbf{x}} = (x_1,\dots,x_r)$, so, we can get

$$f_4(x_s|x_k) = \sum_{\omega=0}^{s-k-1} \sum_{q=0}^{n-s} \frac{A_4 [F(x_s)]^{s-k-\omega+q-1} [F(x_k)]^{\omega} f(x_s)}{[1-F(x_k)]^{n-k}}, \tag{13}$$

where

$$A_4 = \frac{(-1)^{\omega+q}(n-k)! \binom{s-k-1}{\omega} \binom{n-s}{q}}{(s-k-1)!(n-s)!}.$$

Upon substituting (1) and (2) in (10), (11), (12) and (13), we obtain the conditional density functions of $X_{s:n}$, given the UHCS, as follows

$$f_1(x_s|\underline{\mathbf{x}}) = \sum_{d=r}^{s-1} \sum_{\omega=0}^{s-d-1} \sum_{q=0}^{n-s} A_1 2\alpha\beta \; x_s \; e^{-\beta x_s^2} [1-e^{-\beta x_s^2}]^{\alpha(s-d-\omega+q)-1} [1-e^{-\beta T_1^2}]^{\alpha(\omega+d)} \psi_j(T_1), \tag{14}$$

where

$$\psi_j(T_1) = \frac{1}{\sum_{j=r}^{s-1} \binom{n}{j} \left[1-e^{-\beta T_1^2}\right]^{\alpha d} \left[1-(1-e^{-\beta T_1^2})^{\alpha}\right]^{(n-d)}},$$

$$f_2(x_s|x_r) = \sum_{\omega=0}^{s-r-1} \sum_{q=0}^{n-s} \frac{A_2 2\alpha\beta \; x_s \; e^{-\beta x_s^2} [1-e^{-\beta x_s^2}]^{\alpha(s-r-\omega+q)-1} [1-e^{-\beta x_r^2}]^{\alpha\omega}}{[1-(1-e^{-\beta x_r^2})^{\alpha}]^{n-r}}, \tag{15}$$

$$f_3(x_s|\underline{\mathbf{x}}) = \sum_{d=k}^{r^*-1}\sum_{\omega=0}^{s-d-1}\sum_{q=0}^{n-s} A_3 2\alpha\beta\ x_s\ e^{-\beta x_s^2}[1-e^{-\beta x_s^2}]^{\alpha(s-d-\omega+q)-1}[1-e^{-\beta T_2^2}]^{\alpha(\omega+d)}\psi_j(T_2), \tag{16}$$

where

$$\psi_j(T_2) = \frac{1}{\sum_{j=k}^{r^*-1}\binom{n}{j}[1-e^{-\beta T_2^2}]^{\alpha d}[1-(1-e^{-\beta T_2^2})^{\alpha}]^{(n-j)}},$$

and

$$f_4(x_s|x_k) = \sum_{\omega=0}^{s-k-1}\sum_{q=0}^{n-s}\frac{A_4 2\alpha\beta\ x_s\ e^{-\beta x_s^2}[1-e^{-\beta x_s^2}]^{\alpha(s-k-\omega+q)-1}[1-e^{-\beta x_k^2}]^{\alpha\omega}}{[1-(1-e^{-\beta x_k^2})^{\alpha}]^{n-k}}. \tag{17}$$

From (8) and (9), we obtain the Bayesian predictive density function of $X_{s:n}$, given the UHCS as follows:

$$f^*(x_s|\underline{\mathbf{x}}) = \begin{cases} f_1^*(x_s|\underline{\mathbf{x}}) & \text{if } (R,C)=(D_1,T_1), & \text{for Case I,}\\ f_2^*(x_s|\underline{\mathbf{x}}) & \text{if } (R,C)=(r,x_{r:n}), & \text{for Case II and Case IV,}\\ f_3^*(x_s|\underline{\mathbf{x}}) & \text{if } (R,C)=(D_2,T_2), & \text{for Case III and for Case V,}\\ f_4^*(x_s|\underline{\mathbf{x}}) & \text{if } (R,C)=(k,x_{k:n}), & \text{for Case VI,}\end{cases} \tag{18}$$

where, for $x_s > T_1$

$$\begin{aligned} f_1^*(x_s|\underline{\mathbf{x}}) &= \int_0^\infty\int_0^\infty f_1(x_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta\\ &= \sum_{d=r}^{s-1}\sum_{\omega=0}^{s-d-1}\sum_{q=0}^{n-s}\int_0^\infty\int_0^\infty A_1 2\alpha\beta\ x_s\ e^{-\beta x_s^2}[1-e^{-\beta x_s^2}]^{\alpha(s-d-\omega+q)-1}[1-e^{-\beta T_1^2}]^{\alpha(\omega+d)}\psi_j(T_1)\\ &\quad\times\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta, \end{aligned} \tag{19}$$

with $\underline{\mathbf{x}} = (x_1,\ldots x_{D_1})$. For $x_s > x_r$,

$$\begin{aligned} f_2^*(x_s|\underline{\mathbf{x}}) &= \int_0^\infty\int_0^\infty f_2(x_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta\\ &= \sum_{\omega=0}^{s-r-1}\sum_{q=0}^{n-s}\int_0^\infty\int_0^\infty \frac{A_2 2\alpha\beta\ x_s\ e^{-\beta x_s^2}[1-e^{-\beta x_s^2}]^{\alpha(s-r-\omega+q)-1}[1-e^{-\beta x_r^2}]^{\alpha\omega}}{[1-(1-e^{-\beta x_r^2})^{\alpha}]^{n-r}}\\ &\quad\times\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta, \end{aligned} \tag{20}$$

with $\underline{\mathbf{x}} = (x_1,\ldots,x_r)$. For $x_s > T_2$,

$$\begin{aligned} f_3^*(x_s|\underline{\mathbf{x}}) &= \int_0^\infty\int_0^\infty f_3(x_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta\\ &= \sum_{d=k}^{r-1}\sum_{\omega=0}^{s-d-1}\sum_{q=0}^{n-s}\int_0^\infty\int_0^\infty A_3 2\alpha\beta\ x_s\ e^{-\beta x_s^2}[1-e^{-\beta x_s^2}]^{\alpha(s-d-\omega+q)-1}[1-e^{-\beta T_2^2}]^{\alpha(\omega+d)}\psi_j(T_2)\\ &\quad\times\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta, \end{aligned} \tag{21}$$

with $\underline{\mathbf{x}} = (x_1,\ldots,x_{D_2})$. And for $x_s > x_k$,

$$\begin{aligned} f_4^*(x_s|\underline{\mathbf{x}}) &= \int_0^\infty\int_0^\infty f_4(x_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta\\ &= \sum_{\omega=0}^{s-k-1}\sum_{q=0}^{n-s}\int_0^\infty\int_0^\infty \frac{A_4 2\alpha\beta\ x_s\ e^{-\beta x_s^2}[1-e^{-\beta x_s^2}]^{\alpha(s-k-\omega+q)-1}[1-e^{-\beta x_k^2}]^{\alpha\omega}}{[1-(1-e^{-\beta x_k^2})^{\alpha}]^{n-k}}\\ &\quad\times\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta, \end{aligned} \tag{22}$$

with $\underline{\mathbf{x}} = (x_1, \ldots, x_k)$, for $x_s > x_k$.
From (18), for the s^{th} future UHCS we obtain the predictive survival function $P(X_s > t|\underline{\mathbf{x}}) = \bar{F}^*(t|\underline{\mathbf{x}})$, for $t \geq 0$, as follows

$$\bar{F}^*(t|\underline{\mathbf{x}}) = \begin{cases} \bar{F}_1^*(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (D_1, T_1), & \text{for Case I,} \\ \bar{F}_2^*(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (r, x_{r:n}), & \text{for Case II and Case IV,} \\ \bar{F}_3^*(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (D_2, T_2), & \text{for Case III and for Case V,} \\ \bar{F}_4^*(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (k, x_{k:n}), & \text{for Case VI,} \end{cases} \tag{23}$$

where

$$\bar{F}_i^*(t|\underline{\mathbf{x}}) = \int_0^\infty \int_0^\infty h(x_s|\underline{\mathbf{x}}) \pi^*(\alpha, \beta|\underline{\mathbf{x}}) d\alpha d\beta, \qquad i = 1,\, 2,\, 3,\, 4, \tag{24}$$

where

$$h(u|\underline{\mathbf{x}}) = \begin{cases} h_1(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (D_1, T_1), & \text{for Case I,} \\ h_2(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (r, x_{r:n}), & \text{for Case II and Case IV,} \\ h_3(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (D_2, T_2), & \text{for Case III and for Case V,} \\ h_4(t|\underline{\mathbf{x}}) & \text{if } (R,C) = (k, x_{k:n}), & \text{for Case VI,} \end{cases} \tag{25}$$

where

$$h_i(x_s|\underline{\mathbf{x}}) = \int_t^\infty f_i(x_s|\underline{\mathbf{x}}) dx_s, \qquad i = 1,\, 2,\, 3,\, 4,$$

$$h_1(x_s|\underline{\mathbf{x}}) = \sum_{d=r}^{s-1} \sum_{\omega=0}^{s-d-1} \sum_{q=0}^{n-s} \frac{A_1 \left[1 - \left(1 - e^{-\beta t^2}\right)^{\alpha(s-d-\omega+q)}\right] [1 - e^{-\beta T_1^2}]^{\alpha(\omega+d)} \psi_j(T_1)}{(s-d-\omega+q)}, \tag{26}$$

$$h_2(x_s|x_r) = \sum_{\omega=0}^{s-r-1} \sum_{q=0}^{n-s} \frac{A_2 \left[1 - \left(1 - e^{-\beta t^2}\right)^{\alpha(s-r-\omega+q)}\right] [1 - e^{-\beta x_r^2}]^{\alpha\omega}}{[1 - (1 - e^{-\beta x_r^2})^\alpha]^{n-r} (s-r-\omega+q)}, \tag{27}$$

$$h_3(x_s|\underline{\mathbf{x}}) = \sum_{d=k}^{r^*-1} \sum_{\omega=0}^{s-d-1} \sum_{q=0}^{n-s} \frac{A_3 \left[1 - \left(1 - e^{-\beta t^2}\right)^{\alpha(s-d-\omega+q)}\right] [1 - e^{-\beta T_2^2}]^{\alpha(\omega+d)} \psi_j(T_2)}{(s-d-\omega+q)}, \tag{28}$$

and

$$h_4(x_s|x_k) = \sum_{\omega=0}^{s-k-1} \sum_{q=0}^{n-s} \frac{A_4 \left[1 - \left(1 - e^{-\beta t^2}\right)^{\alpha(s-k-\omega+q)}\right] [1 - e^{-\beta x_k^2}]^{\alpha\omega}}{[1 - (1 - e^{-\beta x_k^2})^\alpha]^{n-k} (s-k-\omega+q)}. \tag{29}$$

Then, the Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for $X_{s:n}$, $R < s \leq n$, can be obtained by solving the following two equations:

$$\bar{F}^*(L_{X_{s:n}}|\underline{\mathbf{x}}) = 1 - \frac{\gamma}{2} \quad \text{and} \quad \bar{F}^*(U_{X_{s:n}}|\underline{\mathbf{x}}) = \frac{\gamma}{2}, \tag{30}$$

where $\bar{F}^*(t|\underline{\mathbf{x}})$ is given as in (23), and $L_{X_{s:n}}$ and $U_{X_{s:n}}$ indicate the lower and upper bounds, respectively.

2.2 Two-sample Bayesian prediction

Let $Y_{1:m} \leq Y_{2:m} \leq \ldots \leq Y_{m:m}$ be the order statistics from a future random sample of size m from the same population. We develop in this section a general procedure for deriving the interval predictions for $Y_{s:m}$, $1 \leq s \leq m$ for ER distribution based on UHCS. It is well known that the marginal density function of the s^{th} order statistic from a sample of size m from a continuous distribution with CDF $F(x)$ and PDF $f(x)$ is given by

$$f_{Y_{s:m}}(y_s|\theta) = \frac{m!}{(s-1)!(m-s)!} [F(y_s)]^{s-1} [1 - F(y_s)]^{m-s} f(y_s),$$

$$= \sum_{q=0}^{m-s} \frac{(-1)^q \binom{m-s}{q} m!}{(s-1)!(m-s)!} [F(y_s)]^{s+q-1} f(y_s), \tag{31}$$

where $y_s > 0$ and $1 \leq s \leq m$, see Arnold et al. [26].
Upon substituting (1) and (2) in (31), the marginal density function of $Y_{s:m}$ becomes

$$f_{Y_{s:m}}(y_s|\alpha,\beta) = \sum_{q=0}^{m-s} \frac{(-1)^q \binom{m-s}{q} m!}{(s-1)!(m-s)!} [1-e^{-\beta y_s^2}]^{\alpha(s+q-1)} 2\alpha\beta\ y_s\ e^{-\beta y_s^2} [1-e^{-\beta y_s^2}]^{\alpha-1},$$

$$f_{Y_{s:m}}(y_s|\alpha,\beta) = \sum_{q=0}^{m-s} \frac{(-1)^q \binom{m-s}{q} m!}{(s-1)!(m-s)!} 2\alpha\beta\ y_s\ e^{-\beta y_s^2} [1-e^{-\beta y_s^2}]^{\alpha(s+q)-1}, \tag{32}$$

From (8) and (32), we obtain the Bayesian predictive density function of $Y_{s:m}$, given the UHCS as follows:

$$f^*(y_s|\underline{\mathbf{x}}) = \int_0^\infty \int_0^\infty f(y_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{x})d\alpha d\beta,$$

$$\begin{aligned} f^*(y_s|\underline{\mathbf{x}}) = \sum_{q=0}^{m-s} \frac{(-1)^q \binom{m-s}{q} m!}{(s-1)!(m-s)!} &\int_0^\infty \int_0^\infty 2\alpha\beta\ y_s\ e^{-\beta y_s^2} [1-e^{-\beta y_s^2}]^{\alpha(s+q)-1} \\ &\times \pi^*(\alpha,\beta|\underline{x})d\alpha d\beta. \end{aligned} \tag{33}$$

The predictive survival function $\bar{F}^*_{Y_{s:m}}(t|\underline{\mathbf{x}})$, for $t \geq 0$

$$\bar{F}^*(t|\underline{\mathbf{x}}) = \int_0^\infty \int_0^\infty h(y_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{\mathbf{x}})d\alpha d\beta, \tag{34}$$

where

$$h(y_s|\underline{\mathbf{x}}) = \int_t^\infty f(y_s|\underline{\mathbf{x}})dy_s,$$

$$h(y_s|\underline{\mathbf{x}}) = \sum_{q=0}^{m-s} \frac{(-1)^q \binom{m-s}{q} m!}{(s-1)!(m-s)!(s+q)} \left[1-\left(1-e^{-\beta t^2}\right)^{\alpha(s+q)}\right], \tag{35}$$

Then, the Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for $y_{s:n}$, $1 \leq s \leq m$, can be obtained by solving the following two equations:

$$\bar{F}^*(L_{Y_{s:n}}|\underline{\mathbf{x}}) = 1-\frac{\gamma}{2} \quad \text{and} \quad \bar{F}^*(U_{Y_{s:n}}|\underline{\mathbf{x}}) = \frac{\gamma}{2}, \tag{36}$$

where $F^*_{Y_{s:m}}(t|\underline{\mathbf{x}})$ is given as in (34), and $L_{Y_{s:n}}$ and $U_{Y_{s:n}}$ indicate the lower and upper bounds, respectively.
It is evident that is not possible to compute (24) and (34) analytically. Then, we suggested using MCMC method for constructing the Bayesian prediction intervals.

3 MCMC Method

We suggested using MCMC to generate (α,β) from the posterior density function (8). The Metropolis-Hastings-within-Gibbs sampling is given as follow:

Algorithm 3.1

1.Take some initial guess of α and β, say $\alpha^{(0)}$ and $\beta^{(0)}$ respectively, M = burn-in.
2.Set $j = 1$.
3.Generate $\alpha^{(j)}$ from Gamma$(a_1+R, b_1-\sum_{i=1}^{R}\ln(1-e^{-\beta^{(j-1)}x_i^2}))$.

4.Using Metropolis-Hastings see Metropolis et al. [27], generate $\beta^{(j)}$ from $\pi_2^*(\beta|\alpha,\underline{x})$ with the $N(\beta^{(j-1)},\sigma^2)$ proposal distribution where σ^2 is the variance of β obtained using variance-covariance matrix.
(i) Calculate the acceptance probability

$$r = min\left[1, \frac{\pi_2^*(\beta^*|\alpha^j,\underline{x})}{\pi_2^*(\beta^{j-1}|\alpha^j,\underline{x})}\right]. \tag{37}$$

(ii) Generate u from a Uniform $(0,1)$ distribution.
(iii) If $u \le r$, accept the proposal and set $\beta^i = \beta^*$, else set $\beta^i = \beta^{i-1}$.
5.Compute $\alpha^{(j)}$ and $\beta^{(j)}$.
6.Set $j = j+1$.
7.Repeat steps $3-6$ N times and obtain $\alpha^{(j)}$ and $\beta^{(j)}$ where $j = M+1,...,N$.
8.In case of one-sample Bayesian prediction, the approximate value of $\int_0^\infty\int_0^\infty h(x_s|\mathbf{\underline{x}})\pi^*(\alpha,\beta|\mathbf{\underline{x}})d\alpha d\beta$ is obtained as

$$\int_0^\infty\int_0^\infty h(x_s|\mathbf{\underline{x}})\pi^*(\alpha,\beta|\mathbf{\underline{x}})d\alpha d\beta = \frac{1}{N-M}\sum_{i=M+1}^{N} h(x_s|\mathbf{\underline{x}}),$$

where

$$h(t|\mathbf{\underline{x}}) = \begin{cases} h_1(t|\mathbf{\underline{x}}), & \text{for Case I,} \\ h_2(t|\mathbf{\underline{x}}), & \text{for Case II and Case IV,} \\ h_3(t|\mathbf{\underline{x}}), & \text{for Case III and for Case V,} \\ h_4(t|\mathbf{\underline{x}}), & \text{for case VI.} \end{cases}$$

9.In case of two-sample Bayesian prediction, the approximate value of $\int_0^\infty\int_0^\infty h(y_s|\mathbf{\underline{x}})\pi^*(\alpha,\beta|\mathbf{\underline{x}})d\alpha d\beta$ is obtained as

$$\int_0^\infty\int_0^\infty h(y_s|\mathbf{\underline{x}})\pi^*(\alpha,\beta|\mathbf{\underline{x}})d\alpha d\beta = \frac{1}{N-M}\sum_{i=M+1}^{N} h(y_s|\mathbf{\underline{x}}),$$

where

$$h(y_s|\mathbf{\underline{x}}) = \sum_{q=0}^{m-s} \frac{(-1)^q\binom{m-s}{q}m!}{(s-1)!(m-s)!(s+q)}\left[1-\left(1-e^{-\beta t^2}\right)^{\alpha(s+q)}\right].$$

4 Real Life Data

We have taken the daily average wind speeds from 1 / 3 / 2015 to 30 / 3 / 2015 for Cairo city as follows:

4.3	5.2	3.7	4.7	5.9	6.8	3.1	3.2	4.4	7.6	11.3	5.0	3.2	4.5	6.2
5.7	6.5	5.6	4.0	3.4	5.2	3.4	6	7.8	8.8	8.5	4.1	4.6	5.3	5.2.

This data was produced by the national climatic data center (NCDC) in Asheville in the United States of America. Now, one of the most important subjects is type of distribution of any set of data will be known during statistical tests which are called the goodness of fit. We depended on Kolmogorov-Smirnov (K-S) test to fit whether the data distribution as ER distribution or not. The calculated value of the K-S test is 0.102956 for the ER distribution and this value is smaller than their corresponding values expected at 5% significance level, which is 0.24175 at $n = 30$. We have just plotted the empirical $S(t)$ and the fitted $S(t)$ in Figure (1). Observe that the ER distribution can be a good model fitting this data. In Figure (2) we present the P-P plot for this data. This plot shows a strong relationship supporting the appropriateness of the ER distribution. So, it can be seen that the ER distribution fits the data very well. P-value = 0.876157, therefore, the high p-value indicates that ER distribution can be used to analyze this data set.

Now, we consider the case when the data are censored. We have six cases as following:
Case I: $T_1 = 5.25, T_2 = 5.8, k = 15, r = 16$. In this case: $R = 17, C = T_1 = 5.25$.
Case II: $T_1 = 5.25, T_2 = 5.8, k = 16, r = 18$. In this case: $R = 18, C = x_{r:n} = 5.5$.
Case III: $T_1 = 5.65, T_2 = 5.8, k = 18, r = 21$. In this case: $R = 20, C = T_2 = 5.8$.
Case IV: $T_1 = 5.75, T_2 = 6.7, k = 21, r = 22$. In this case: $R = 22, C = x_{r:n} = 6.1$.

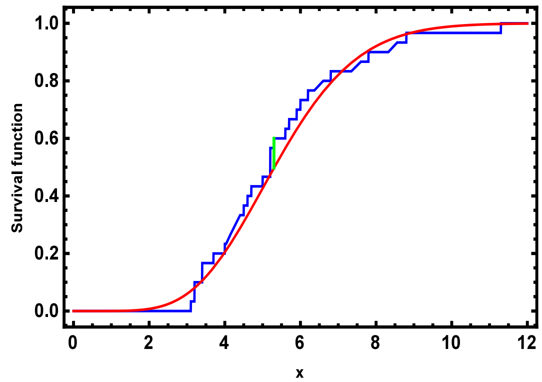

Fig. 1: Empirical and fitted survival functions.

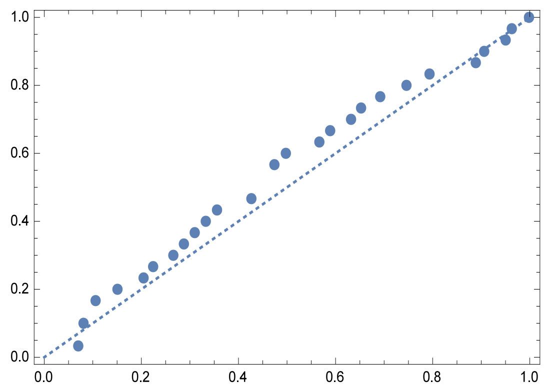

Fig. 2: $P-P$ plot compare data to a specific distribution.

Case V: $T_1 = 5.95, T_2 = 6.4, k = 22, r = 24$. In this case: $R = 23, C = T_2 = 6.4$.
Case VI: $T_1 = 5.95, T_2 = 6.4, k = 25, r = 26$. In this case: $R = 25, C = x_{k:n} = 7$.

Based on the above six UHCS, we used the results presented earlier in Section 2 to construct 95% one-sample Bayesian prediction intervals for future order statistics $X_{s:n}$, from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:m}$, where $s = 1, 2, \ldots, 20$, from a future unobserved sample with size $m = 20$. To examine the sensitivity of the Bayesian prediction intervals with respect to the hyper-parameters (a_1, a_2, b_1, b_2), we used two different choices of the hyper-parameters Prior 1 $(a_1, a_2, b_1, b_2) = (0, 0, 0, 0)$, Prior 2 $(a_1, a_2, b_1, b_2) = (0.9, 0.4, 4, 5)$. Tables 1, 2, 3, 4, 5 and 6 presents the results for one-sample predictions, for two choices of the hyper-parameters and Tables 7, 8, 9, 10, 11 and 12 presents the results for two-sample predictions, for two choices of the hyper-parameters. In all cases $\alpha = 3.0746$ and $\beta = 0.0567$ are considered.

Table 1: 95% One-sample Bayesian prediction bounds for $X_{s:n}$, $s = 18, \ldots, 30$ from the ER distribution in Case I of UHCS for $C = 5.25$ and $R = 17$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
18	5.2583	6.7420	1.4837	5.2597	6.9283	1.6686
19	5.2688	7.3699	2.1011	5.2711	7.5346	2.2634
20	5.2876	8.2001	2.9124	5.2897	8.5011	3.2114
21	5.3178	9.2626	3.9448	5.3246	9.7108	4.3861
22	5.3674	10.1791	4.8116	5.3777	10.752	5.3748
23	5.4614	11.5279	6.0664	5.4746	11.9939	6.5192
24	5.6195	12.345	6.7255	5.6400	13.2272	7.5871
25	5.8347	13.4844	7.6496	5.8809	14.5485	8.6675
26	6.1227	15.9364	9.8136	6.2222	16.6770	10.4548
27	6.5059	16.9762	10.4702	6.7299	18.7970	12.0671
28	6.9712	19.0909	12.1197	7.2480	20.6904	13.4423
29	7.4873	22.1440	14.6567	7.8949	24.2879	16.3930
30	8.4439	29.3684	20.9245	8.9541	33.1461	24.1920

Table 2: 95% One-sample Bayesian prediction bounds for $X_{s:n}$, $s = 19, \ldots, 30$ from the ER distribution in Case II of UHCS for $C = 5.5$ and $R = 18$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
19	5.5078	6.8626	1.3547	5.5092	7.0546	1.5454
20	5.5753	7.7084	2.1331	5.5899	8.0472	2.4572
21	5.6864	8.5074	2.8210	5.7325	9.0336	3.3011
22	5.8361	9.2688	3.4327	5.9142	9.9399	4.0257
23	6.0033	10.3284	4.3250	6.1294	11.2708	5.1413
24	6.1987	11.149	4.9502	6.3603	12.1335	5.7731
25	6.4379	12.3422	5.9043	6.6491	13.5354	6.8862
26	6.6984	13.6546	6.9560	6.9337	14.5700	7.6362
27	6.9858	14.9344	7.9485	7.3050	16.8767	9.5716
28	7.4393	17.9919	10.5525	7.7752	18.3696	10.5944
29	7.8649	19.1721	11.3071	8.4182	22.5021	14.0838
30	8.5891	25.1613	16.5722	9.3513	27.6087	18.2574

Table 3: 95% One-sample Bayesian prediction bounds for $X_{s:n}$, $s = s = 21, \ldots, 30$ from the ER distribution in Case III of UHCS for $C = 5.8$ and $R = 20$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
21	5.2449	7.8110	2.5661	5.7065	8.1813	2.4747
22	5.9905	8.3729	2.3824	6.0404	8.8870	2.8465
23	6.1352	9.0416	2.9064	6.2106	9.6706	3.4600
24	6.2937	9.7166	3.4229	6.4077	10.4150	4.0072
25	6.4693	10.2314	3.7619	6.6601	11.4072	4.7471
26	6.6725	11.381	4.7084	6.8926	12.227	5.3348
27	6.9541	12.0946	5.1403	7.2595	13.7240	6.4644
28	7.2638	13.6558	6.3920	7.6520	15.3159	7.6638
29	7.6517	15.4312	7.7794	8.1007	17.3353	9.2345
30	8.2676	18.8062	10.5386	8.8844	21.1293	12.2448

Table 4: 95% One-sample Bayesian prediction bounds for $X_{s:n},\ s = 23,\ldots,30$ from the ER distribution in Case IV of UHCS for $C = 6.1$ and $R = 22$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
23	6.1068	7.1587	1.0519	6.1089	7.4516	1.3427
24	6.1672	7.8078	1.6405	6.1890	8.3082	2.1191
25	6.2772	8.4225	2.1453	6.3321	9.0594	2.7272
26	6.4316	9.1828	2.7511	6.5345	9.9564	3.4218
27	6.6316	9.9974	3.3657	6.7944	11.0657	4.2711
28	6.8780	11.0606	4.1825	7.1092	12.2021	5.0927
29	7.2102	12.4691	5.2588	7.5622	14.1063	6.5440
30	7.7321	15.1425	7.4103	8.2822	17.7015	9.4192

Table 5: 95% One-sample Bayesian prediction bounds for $X_{s:n},\ s = 24,\ldots,30$ from the ER distribution in Case V of UHCS for $C = 6.4$ and $R = 23$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
24	6.4612	7.9386	1.4773	6.3306	8.3850	2.0543
25	6.5614	8.5094	1.9480	6.6173	9.1825	2.5651
26	6.7007	9.1668	2.4661	6.7966	9.8963	3.0997
27	6.8818	9.9120	3.0302	7.0317	10.8972	3.8653
28	7.1177	10.9309	3.8132	7.3343	12.0055	4.6712
29	7.4088	12.1345	4.7256	7.7568	13.7806	6.0237
30	7.9421	14.8530	6.91079	8.4071	17.1061	8.6989

Table 6: 95% One-sample Bayesian prediction bounds for $X_{s:n},\ s = 26,\ldots,30$ from the ER distribution in Case VI of UHCS for $C = 7$ and $R = 25$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
26	7.0079	8.1991	1.1912	7.0107	8.5847	1.5740
27	7.0825	8.9853	1.9028	7.1106	9.5881	2.4775
28	7.2311	9.8379	2.6068	7.3158	10.7799	3.4640
29	7.4641	11.0639	3.5996	7.6372	12.2854	4.64812
30	7.8835	13.6134	5.7298	8.1856	15.2973	7.1116

Table 7: 95% Two-sample Bayesian prediction bounds for $Y_{s:m}$, $s=1,\ldots,20$ from the ER distribution in Case I of UHCS for $C=5.25$ and $R=17$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
1	0.0047	2.7823	2.7775	0.0034	2.2532	2.2497
2	0.0943	3.4682	3.3739	0.0470	3.0289	2.9819
3	0.2517	4.0598	3.8080	0.1435	3.6829	3.5394
4	0.4080	4.5832	4.1751	0.2901	4.3263	4.0362
5	0.7321	5.0964	4.3642	0.5065	4.9353	4.4287
6	1.1015	5.5837	4.4822	0.7512	5.5848	4.8335
7	1.4089	6.2402	4.8313	1.0200	6.3202	5.3002
8	1.7798	6.8334	5.0536	1.3828	6.9364	5.5535
9	2.1378	7.5092	5.3714	1.6952	7.8231	6.1278
10	2.5392	8.3284	5.7892	2.1461	8.5926	6.4465
11	2.9822	8.9162	5.9340	2.5777	9.5093	6.9315
12	3.3911	10.2106	6.8194	3.0177	10.3971	7.3794
13	3.8008	11.4846	7.6837	3.4719	11.3975	7.9255
14	4.2480	12.6501	8.4020	3.9764	12.9553	8.9788
15	4.7236	13.7418	9.0181	4.5015	14.5857	10.0842
16	5.1912	14.8206	9.6294	5.0134	15.8043	10.7909
17	5.6756	18.3652	12.6896	5.6404	17.7703	12.1299
18	6.2409	19.1960	12.955	6.3639	21.0906	14.7267
19	6.9423	22.6758	15.7334	7.1269	25.6580	18.5310
20	7.8467	28.5914	20.7446	8.3482	30.4278	22.0795

Table 8: 95% Two-sample Bayesian prediction bounds for $Y_{s:m}$, $s=1,\ldots,20$ from the ER distribution in Case II of UHCS for $C=5.5$ and $R=18$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
1	0.0219	2.8943	2.8724	0.0064	2.3619	2.3554
2	0.1517	3.5479	3.3962	0.0697	3.1071	3.0373
3	0.3948	4.1010	3.7062	0.2046	3.7296	3.5249
4	0.6132	4.5978	3.9846	0.3968	4.3346	3.9378
5	0.8952	5.0764	4.1811	0.6346	4.8993	4.2647
6	1.2430	5.5586	4.3156	0.8966	5.4639	4.5672
7	1.6165	6.0555	4.4390	1.2181	6.0637	4.8455
8	1.9622	6.5984	4.6362	1.5414	6.6995	5.1581
9	2.3633	7.1676	4.8043	1.9439	7.3426	5.3987
10	2.7050	7.8464	5.1414	2.2903	8.0986	5.8082
11	3.0846	8.6862	5.6015	2.7206	8.8536	6.1330
12	3.5044	9.1350	5.6305	3.1601	9.7162	6.5561
13	3.9189	10.0689	6.1499	3.5765	10.5035	6.9269
14	4.3321	10.7794	6.4472	4.0803	11.7723	7.6920
15	4.7708	12.3505	7.5796	4.5209	13.3308	8.8098
16	5.1907	13.0916	7.9008	5.0423	14.1968	9.1544
17	5.6865	14.2786	8.5920	5.5992	15.8504	10.2511
18	6.2113	16.7492	10.5379	6.2725	17.8564	11.5839
19	6.8266	19.2227	12.3961	6.9966	20.8534	13.8568
20	7.7594	23.8182	16.0588	8.2385	27.2911	19.0525

Table 9: 95% Two-sample Bayesian prediction bounds for $Y_{s:m},\ s = 1,\ldots,20$ from the ER distribution in Case III of UHCS for $C = 5.8$ and $R = 20$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
1	0.1088	3.1312	3.0224	0.0330	2.5665	2.5334
2	0.4058	3.7306	3.3248	0.1939	3.2673	3.0734
3	0.7136	4.1851	3.4715	0.4061	3.8140	3.4079
4	1.0805	4.6026	3.5221	0.6722	4.3183	3.6460
5	1.3967	5.0000	3.6033	0.9710	4.7896	3.8186
6	1.7639	5.3712	3.6072	1.2692	5.2826	4.0133
7	2.0900	5.7849	3.6949	1.5881	5.7461	4.1579
8	2.3934	6.1904	3.7969	1.9368	6.2483	4.3114
9	2.7925	6.5972	3.8046	2.2729	6.7552	4.4822
10	3.1013	7.0491	3.9477	2.6255	7.2798	4.6543
11	3.4066	7.6401	4.2335	2.9659	7.9730	5.0071
12	3.7650	8.1504	4.3854	3.3500	8.5213	5.1712
13	4.1000	8.5667	4.4666	3.7457	9.0178	5.2721
14	4.4469	9.1987	4.7518	4.1286	10.1122	5.9835
15	4.8046	10.0141	5.2094	4.5456	10.8188	6.2731
16	5.1599	10.9094	5.74947	5.0187	11.9029	6.8842
17	5.5439	11.7243	6.1803	5.4879	13.0679	7.5799
18	5.9772	12.6954	6.7182	6.0633	15.0606	8.9972
19	6.5780	14.5732	7.9952	6.7372	16.8791	10.1419
20	7.2556	17.9675	10.7119	7.7747	20.9054	13.1307

Table 10: 95% Two-sample Bayesian prediction bounds for $Y_{s:m},\ s = 1,\ldots,20$ from the ER distribution in Case IV of UHCS for $C = 6.1$ and $R = 22$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
1	0.2746	3.3610	3.0863	0.08142	2.7572	2.6757
2	0.7673	3.8809	3.1136	0.3472	3.3948	3.0475
3	1.1544	4.2752	3.1208	0.6728	3.8960	3.2231
4	1.51208	4.63518	3.1231	0.9608	4.3400	3.3791
5	1.8689	4.9598	3.0908	1.2958	4.7463	3.4505
6	2.2360	5.2782	3.0422	1.5871	5.1580	3.5709
7	2.5176	5.5865	3.0688	1.9004	5.5619	3.6615
8	2.8301	5.9019	3.0718	2.2432	5.9870	3.7437
9	3.1291	6.2749	3.1458	2.5508	6.4184	3.8675
10	3.4209	6.6192	3.1983	2.8730	6.9029	4.0299
11	3.6865	7.0051	3.3185	3.2024	7.3317	4.1293
12	3.9645	7.4670	3.5025	3.5274	7.8424	4.3149
13	4.2596	7.8439	3.5842	3.8797	8.3816	4.5018
14	4.5372	8.2571	3.7199	4.2300	8.9940	4.7639
15	4.8338	8.8411	4.0073	4.5996	9.6926	5.0930
16	5.1392	9.5823	4.4431	4.9733	10.4600	5.4866
17	5.4791	10.2492	4.7700	5.4374	11.3847	5.9472
18	5.8727	11.1925	5.3198	5.9062	12.6783	6.7719
19	6.3370	12.7138	6.3767	6.5415	14.2753	7.7337
20	7.0154	15.1784	8.1630	7.3999	17.7655	10.3656

Table 11: 95% Two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1,\ldots,20$ from the ER distribution in Case V of UHCS for $C = 6.4$ and $R = 23$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
1	0.3710	3.4431	3.0720	0.1130	2.8377	2.7247
2	0.8856	3.9499	3.0643	0.43709	3.4643	3.0272
3	1.3199	4.3234	3.0035	0.7548	3.9408	3.1860
4	1.7200	4.6635	2.9434	1.0806	4.3740	3.2934
5	2.0484	4.9711	2.9226	1.3964	4.7750	3.3785
6	2.3724	5.2828	2.9103	1.7484	5.1605	3.4120
7	2.6719	5.5981	2.9261	2.0154	5.5651	3.5496
8	2.9325	5.9296	2.9971	2.3499	5.9619	3.6119
9	3.1970	6.2686	3.0716	2.6710	6.3472	3.6762
10	3.5109	6.5690	3.0581	2.9820	6.7342	3.7522
11	3.7713	6.8873	3.1160	3.2828	7.2376	3.9548
12	4.0564	7.2944	3.2379	3.6162	7.6657	4.0495
13	4.3109	7.7044	3.3934	3.9549	8.2125	4.2576
14	4.5857	8.1957	3.6099	4.2866	8.8544	4.5677
15	4.8666	8.5364	3.6698	4.6291	9.3400	4.7109
16	5.1851	9.2926	4.1075	5.0097	10.1861	5.1763
17	5.4844	9.9110	4.4265	5.4172	10.9506	5.5333
18	5.8770	10.8206	4.9436	5.8811	12.0961	6.2150
19	6.3348	12.2402	5.9052	6.4967	13.7538	7.2571
20	6.9785	14.5531	7.5745	7.3661	16.9324	9.5663

Table 12: 95% Two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1,\ldots,20$ from the ER distribution in Case VI of UHCS for $C = 7$ and $R = 25$.

	Prior 1			Prior 2		
s	Lower	Upper	Length	Lower	Upper	Length
1	0.5922	3.5937	3.0015	0.1969	2.9956	2.7987
2	1.2016	4.0698	2.8683	0.6017	3.5832	2.9814
3	1.6310	4.4113	2.7802	0.9785	4.0366	3.0581
4	2.0469	4.7313	2.6844	1.3310	4.4188	3.0877
5	2.3623	5.0166	2.6543	1.6662	4.8032	3.1369
6	2.6377	5.3135	2.6758	1.9735	5.1521	3.1786
7	2.9158	5.5782	2.6624	2.2746	5.5179	3.2432
8	3.2096	5.8618	2.6521	2.5922	5.8564	3.2642
9	3.4600	6.1374	2.6773	2.8724	6.2224	3.3499
10	3.7113	6.4094	2.6980	3.1753	6.5925	3.4171
11	3.9357	6.7696	2.8338	3.4726	7.0084	3.5357
12	4.2030	7.0554	2.8523	3.7720	7.4056	3.6335
13	4.4330	7.4358	3.0028	4.0689	7.8494	3.7805
14	4.6873	7.8827	3.1953	4.3806	8.2952	3.9145
15	4.9552	8.2799	3.3247	4.7046	8.9168	4.2121
16	5.2168	8.7480	3.5312	5.0626	9.5966	4.5340
17	5.5151	9.3342	3.8190	5.4338	10.3184	4.8846
18	5.8473	10.1979	4.3505	5.8722	11.2973	5.4250
19	6.2772	11.2341	4.9568	6.4446	12.8177	6.3729
20	6.8876	13.4857	6.5980	7.1927	15.4572	8.2644

5 Conclusion

1. From Tables 1−6 and 7−12, we notic that, when s increase the lower and upper increase.
2. It is clear from Tables 1, 2, 3, 4, 5 and 6 that, the lower bounds are relatively insensitive to the specification of the hyper-parameters (a_1, a_2, b_1, b_2) while the upper bounds are somewhat sensitive.
3. It is clear from Tables 7, 8, 9, 10, 11 and 12 that, the lower and upper bounds are sensitive to the specification of the hyper-parameters (a_1, a_2, b_1, b_2).

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