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Analysis of Hybrid Censored Competing Risks Model with Bathtub-Shaped Failure Time Distribution

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Abstract: The mixture of type-I and type-II censoring schemes, called the hybrid censoring scheme is quite common in life-testing or reliability experiments. In this study, we consider the estimation problem of competing risk model when the data obtained from the experiments are hybrid censored. It is assumed that the latent cause of failure have independent Chen distributions with common shape parameter. Maximum likelihood and Bayes estimates of the model parameters are obtained. MCMC techniques like Metropolis-Hastings algorithm and Gibbs sampler has been utilized to obtain Bayes estimates. At the end, a real data study is attempted to establish the applicability of the proposed model.

Keywords: Chen distribution, bathtub failure rate, competing risk, MCMC techniques, hybrid censoring

1 Introduction

In life time data analysis, it is quite common that more than one cause of failure may direct to an object (system) at the same time. It is often interesting that an investigator needs to estimate a specific risk in presence of other risk factors. In statistical literature, this process is known as competing risk model. People get sick or die or hardware fails due to one of the several possible causes. In competing risk model, all these causes have given probabilities of occurrence over time and compete to be the first to occur and thus cause the event. Examples in medicine include the analysis of cause to death data, the analysis of relapse and death in remission in cancer studies, or right random censoring. In engineering applications, competing risk model arises for analyzing series systems. Classical competing risk [1] deals with the modeling of probability of failure in the observed system (crude probabilities) or in system with some causes of failure removed (net or partial crude probabilities).

Miyawaka [2] obtained the maximum likelihood estimators (MLEs) and the uniformly minimum variance unbiased estimators (UMVUEs) of the failure rates of different failure distributions under the assuming the failure time distribution as exponential. Kundu and Basu [3] considered the same model of Miayawaka [2] by assuming that every member of a certain target population either dies of a particular cause, say cancer, or by other causes. Recently, Bhattacharya et al [4] consider the analysis of hybrid censored competing risks data, based on Coxs latent failure time model assumptions.

Although widespread literature is available on the inference procedures for the competing risks models under complete samples, but much attention has not been paid when the data are hybrid censored. Censored data means that some items were put on test, and some/all of them may not have failed at the time of termination of experiment hence lifetimes of such items/units could not be observed. The conventional Type-I and Type-II censoring schemes are two most popular censoring schemes used for the analysis of such data. In the Type-I censoring scheme, the termination time for the experiment is pre-fixed but efficiency level cannot be controlled because one cannot predict the observed number of failures in such experiment. Although, the Type-II censoring scheme ensures that m failures take place during the experiment so the level of efficiency is guaranteed, but the termination time cannot be controlled in this case as the exact time of pre-fixed number of failures i.e. m failures is uncertain [5]. So, both Type-I and Type-II censoring schemes have their respective, merits and demerits. For these reasons, a mixture of Type-I and Type-II censoring schemes, known as hybrid censoring scheme has been introduced by Epstein [6] and used for the first time in reliability acceptance test MIL-STD-781C [7]. It is interesting to note that all the three above-mentioned situations namely the complete sample situation, Type-I and Type-II censoring

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schemes can be obtained as special cases of the hybrid censoring scheme. For a comprehensive review of hybrid censoring scheme, we refer to Balakrishnan and Kundu [8].

A class of life distributions which has acknowledged significant attention is the class of bathtub shaped failure rate life distributions. A organized version of such distributions was provided by Rajarshi and Rajarshi [9]. A lifetime model is said to have bathtub shaped failure rate if its failure rate functions decreases at first and then remains constant for a period and finally it increases with time. In other words, the failure rate function has a bathtub shape. This corresponds to the three distinct phases of a component or a system: early life, useful life and wear-out. During the early life period, failures tend to be caused by manufacturing defects or birth defects in the case of human beings. Failures in the useful life period can be called chance failures. The wear-out region has an increasing failure rate with time because of the older the unit the more likely it is to fail.

The lifetime distributions like Chen distribution having bathtub shaped hazard rate have attracted the attention of many researchers as the lifetimes of various industrial items including electrical and mechanical products, as well as survival times of various biological entities exhibit such characteristics (see for instance [9,10]). The probability density function (pdf) of the two parameter Chen distribution $CH(\alpha,\beta)$ where$\alpha$is the shape parameter and$\beta$is the scale parameter (Chen [11]) is given by

$$f(x)=\alpha\beta x^{\beta-1}\exp\left\{\alpha\left(1-e^{x^{\beta}}\right)+x^{\beta}\right\};\ \ x,\alpha,\beta>0 \tag{1}$$

with the corresponding survival and failure rate function given by

$$S(x)=\exp\left\{\alpha\left(1-e^{x^{\beta}}\right)\right\};\ \ x,\alpha,\beta>0 \tag{2}$$

$$h(x)=\alpha\beta x^{\beta-1}\exp\left(x^{\beta}\right);\ \ x,\alpha,\beta>0 \tag{3}$$

For $\beta<1$, $h(x)$ has a bathtub-shape and reaches a minimum at $x=\left\{\frac{1-\beta}{\beta}\right\}^{1/\beta}$ while for $\beta\geq 1$, it is increasing. The case $\alpha=1$ corresponds to the exponential power distribution. Chen (2000) has discussed in detail the hazard rate behavior of the Chen distribution.

In the last decade, many authors considered Chen distribution as a lifetime model in different contexts. Wu et al. [12] discussed the estimation procedure for the shape parameter of the Chen distribution. Wu [10] obtained MLEs of parameters of Chen distribution using progressively Type-II censoring and also provided exact confidence intervals and confidence regions for these parameters. Sarhan et al. [13] derived MLE and Bayes estimators of the unknown parameters using a complete sample for Chen distribution while Ahmad [14] obtained maximum likelihood and Bayes estimates of the model parameters, reliability and hazard functions for the same distribution when sample is available from progressive Type-II censoring scheme. Rastogi and Tripathi [15] considered the problem of estimating unknown parameters of a two-parameter Chen distribution using hybrid censored sample. Recently, Pundir and Gupta [16] consider the estimation of m-component load-sharing parameters by assuming the failure time distribution of components as Chen. To the best of our knowledge, Chen distribution has yet not been considered for competing risks modeling when the data are hybrid censored.

In view of the above, we present the estimation problem of competing risks model by assuming independent Chen distribution with common shape parameter. The data obtained from the experiment are assumed to be hybrid censored. In section 2, we describe the model and notations of the terminology used. Maximum likelihood (ML) and Bayes estimate of parameters under hybrid censoring scheme is discussed in section 3. Also the asymptotic distribution of the model parameters is used to construct the approximate confidence interval (CI). Markov Chain Monte Carlo (MCMC) techniques such as Metropolis-Hastings algorithm and Gibbs sampler have been utilized to generate simulated draws from the posterior density of the model parameter. In section 4, a real data study has been done to explore the applicability of the proposed theory.

2 Model Description and Notations

Here, we assume that there are only two independent cause of failure, although all the methods proposed in the study can be easily extended for more than two cases. We assume the following notations:

X_i: lifetime of system i

X_{ji}: lifetime of the ith individual under causes j, j=1,2

d_j : number of failure due to causes j, j=1,2

$F(.)$: cumulative distribution function of X_i

$F_j(.)$: cumulative distribution function of X_{ji}
$S_j(.)$: survival function of X_{ji}
δ_i: indicator variable denoting the cause of the failure of the ith individual
$Ch(\alpha,\beta)$: Chen random variable with p.d.f defined in equation (1)
Suppose $\{T_{1i},T_{2i};i=1,2,.....n\}$ are n independently and identically distributed random variables. Also, the lifetime distribution of $\{T_{1i},T_{2i};i=1,2,.....n\}$ follows $Ch(\alpha_1,\beta)$ and $Ch(\alpha_2,\beta)$ respectively and both T_{1i} and T_{2i} are independent to each other.

3 Estimation Procedure

3.1 Maximum Likelihood Estimation and Asymptotic Confidence Interval

Based on the observed data discussed in section 2, the likelihood function of $(\alpha_1,\alpha_2,\beta)$ is given by

$$L(data|\alpha_1,\alpha_2,\beta)\propto\alpha_1^{d_1}\alpha_1^{d_2}\beta^d\prod_{i=1}^{d}t_i^{\beta-1}e^{\sum_{i=1}^{d}t_i^\beta}\exp\{-(\alpha_1+\alpha_2)\xi(\beta)\}\tag{4}$$

where,

$$\xi(\beta)=\sum_{i=1}^{d}\left(e^{t_i^\beta}-1\right)+(n-d)\left(e^{z_*^\beta}-1\right);\ \ d=d_1+d_2\tag{5}$$

The corresponding log-likelihood function (say $l(\alpha_1,\alpha_2,\beta)$) for equation in (4) without the additive constants is given by

$$l(\alpha_1,\alpha_2,\beta)\propto d_1\ln\alpha_1+d_2\ln\alpha_2+d\ln\beta+(\beta-1)\sum_i\ln t_i+\sum_i t_i^\beta-(\alpha_1+\alpha_2)\xi(\beta)\tag{6}$$

The MLEs of α_1, $\alpha_2$and β say $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\beta}$ can be obtained as the solutions of the following equations:

$$\frac{\partial\ell}{\partial\alpha_1}=\frac{d_1}{\alpha_1}-\xi(\beta)=0\tag{7}$$

$$\frac{\partial\ell}{\partial\alpha_2}=\frac{d_2}{\alpha_2}-\xi(\beta)=0\tag{8}$$

$$\frac{\partial\ell}{\partial\beta}=\frac{d}{\beta}+\sum_i\ln t_i\left(1+t_i^\beta\right)-(\alpha_1+\alpha_2)\xi'(\beta)\tag{9}$$

where, $\xi'(\beta)=\left\{\sum_i e^{t_i^\beta}t_i^\beta\ln t_i+(n-d)e^{Z_*^\beta}Z_*^\beta\ln Z_*^\beta\right\}$
From equation (7) and (8), we get the MLEs $(\hat{\alpha}_1,\hat{\alpha}_2)$of (α_1,α_2)as a function of β

$$\hat{\alpha}_1(\beta)=\frac{d_1}{\xi(\beta)}\tag{10}$$

$$\hat{\alpha}_2(\beta)=\frac{d_2}{\xi(\beta)}\tag{11}$$

Replacing the expression of $\hat{\alpha}_1(\beta)$and $\hat{\alpha}_2(\beta)$ in log-likelihood function, we obtain the profile log-likelihood of βwithout the additive constant is

$$p(\beta)\propto d\ln\beta-d\ln\xi(\beta)+\beta\sum_i\ln t_i+\sum_i t_i^\beta\tag{12}$$

Now, the MLE of β can be obtained by solving the equation $p^{'}(\beta)=0$. Therefore, if$\hat{\beta}$ is the MLE of β , then it can be obtained as the fixed point solution of the following

$$\beta = h(\beta) = \frac{d\xi(\beta)}{d\xi'(\beta) - \xi(\beta)\sum_i \left(1+t_i^{\beta}\right)\ln t_i} \tag{13}$$

The equation in (13) can be solved for $\hat{\beta}$ by using the following simple iterative procedure as $\hat{\beta}$ is a fixed point solution of this non-linear equation

$$h\left(\beta_{(k)}\right) = \beta_{(k+1)}$$

Where $\beta_{(k)}$is the k^{th}iteration of$\hat{\beta}$. The iteration procedure should be stopped when $\left|\beta_{(k)}-\beta_{(k+1)}\right|$is sufficiently small. However, one can also use the uniroot() function of R software to solve equation in (9). Now, using$\hat{\beta}$, we can obtain $\hat{\alpha}_1$and $\hat{\alpha}_2$ from equation (7) and (8) as $\hat{\alpha}_1=\hat{\alpha}_1\left(\hat{\beta}\right)and\hat{\alpha}_2=\hat{\alpha}_2\left(\hat{\beta}\right)$.

Using large sample theory of MLEs, the asymptotic sampling distribution of $\left(\hat{\alpha}_1-\alpha_1,\,\hat{\alpha}_2-\alpha_2,\,\hat{\beta}-\beta\right)'$ is $N_3\left(0,\Delta^{-1}\right)$where$\Delta$the observed Fisher information matrix is. The elements of Δ are given by
$\Delta_{11}=-\frac{\partial^2\ell}{\partial\alpha_1^2}\Big|_{\alpha_1=\hat{\alpha}_1}$; $\Delta_{22}=-\frac{\partial^2\ell}{\partial\alpha_2^2}\Big|_{\alpha_2=\hat{\alpha}_2}$; $\Delta_{33}=-\frac{\partial^2\ell}{\partial\beta^2}\Big|_{\beta=\hat{\beta}}$; $\Delta_{12}=\Delta_{21}=0$
$\Delta_{13}=\Delta_{31}=-\frac{\partial^2\ell}{\partial\alpha_1\partial\beta}\Big|_{\alpha_1=\hat{\alpha}_1,\,\beta=\hat{\beta}}$ and $\Delta_{23}=\Delta_{32}=-\frac{\partial^2\ell}{\partial\alpha_2\partial\beta}\Big|_{\alpha_2=\hat{\alpha}_2,\,\beta=\hat{\beta}}$
Where, $\frac{\partial^2\ell}{\partial\alpha_2^2}=-\frac{d_2}{\alpha_2^2}$
$\frac{\delta^2\ell}{\delta\beta^2}=-\frac{d}{\beta^2}+\sum_i t_i{}^{\beta}\{\log(t_i)\}^2-(\alpha_1+\alpha_2)\,\xi''(\beta)$
$\frac{\partial^2\ell}{\partial\alpha_1\partial\beta}=\frac{\partial^2\ell}{\partial\beta\partial\alpha_1}=-\xi''(\beta)$; $\frac{\partial^2\ell}{\partial\alpha_2\partial\beta}=\frac{\partial^2\ell}{\partial\beta\partial\alpha_2}=-\xi''(\beta)$
Where, $\xi''(\beta)=\sum_i\{\log(t_i)\}^2 e^{t_i^{\beta}}t_i^{\beta}\left(1+t_i^{\beta}\right)+(n-d)\{\log(Z_*)\}^2 e^{Z_*^{\beta}}Z_*^{\beta}\left(1+Z_*^{\beta}\right)$
The asymptotic $(1-\gamma)\times 100\%$ confidence intervals (C.I.) for $\Omega=(\alpha_1,\alpha_2,\beta)is\hat{\Omega}\pm z_{\gamma/2}\sqrt{Var\left(\hat{\Omega}\right)}$ Here $Var\left(\hat{\Omega}\right)$is the variance of $\hat{\Omega}$ obtained from $\Delta$and $z_{\gamma/2}$ is the upper$100\times(\gamma/2)^{th}$ percentile of a standard normal distribution.

4 Bayesian Estimation and HPD Interval of R

In many practical situations, it is observed that the behavior of the parameters representing the various model characteristics cannot be treated as fixed constant throughout the life testing period. Therefore, it would be reasonable to assume the parameters involved in the life time model as random variables. While thinking for Bayesian analysis, first thing to contract with the problem of constructing prior for the unknown parameters involved in the model. We all know that the choice of a wrong prior may lead to misleading information, so the selection of prior becomes more and more important part of the Bayesian analysis. In practice, informative and non-informative priors are used to represent uncertainties about the model parameters. Berger [17] pointed out that when there is no information or very difficult to gather regarding the prior variations in the parameters, it is better to use non-informative prior distribution. However, non-informative priors generally lack invariance property under one-to-one transformation, thereby leading to incoherent analysis. On the other hand, informative priors are based on the investigators experience about the random behavior of the process under consideration. In lieu of this, we consider the Bayesian method of estimation with both informative and non-informative priors. First, we assume that the prior distributions of α_1, $\alpha_2$and $\beta$are assumed to be gamma with respective pdf's as

$$g_1(\alpha_1)\propto\alpha_1{}^{\lambda_1-1}\exp(-\theta_1\alpha_1)\,;\;\;(\alpha_1,\,\theta_1,\,\lambda_1\,>0) \tag{14}$$
$$g_2(\alpha_2)\propto\alpha_2{}^{\lambda_2-1}\exp(-\theta_2\alpha_2)\,;\;\;(\alpha_2,\,\theta_2,\,\lambda_2\,>0) \tag{15}$$

and

$$g_3(\beta)\propto\beta^{\lambda_3-1}\exp(-\theta_3\beta)\,;\;\;(\beta,\,\theta_3,\,\lambda_3,>0) \tag{16}$$

Using the likelihood function in (4) and prior distributions in (14)-(16), the joint distribution of $\alpha_1,\ \alpha_2, \beta$ given the data is

$$\prod(\alpha_1,\ \alpha_2, \beta\,|data) = \frac{L(data\,|\,\alpha_1,\ \alpha_2, \beta\,)\,g_1(\alpha_1)\,g_2(\alpha_2)\,g_3(\beta)}{\int\limits_{\alpha_1}\int\limits_{\alpha_2}\int\limits_{\beta} L(data\,|\,\alpha_1,\ \alpha_2, \beta\,)\,g_1(\alpha_1)\,g_2(\alpha_2)\,g_3(\beta)\,d\alpha_1 d\alpha_2 d\beta}$$

$$K(\alpha_1,\alpha_2,\beta|data) \propto \alpha_1^{d_1+\lambda_1-1}\alpha_2^{d_2+\lambda_2-1}\beta^{d+\lambda_3-1}\prod_{i=1}^{d} t_i^{\beta-1} e^{\sum\limits_{i=1}^{d} t_i^{\beta}}\, e^{-[(\alpha_1+\alpha_2)\xi(\beta)+\theta_1\alpha_1+\theta_2\alpha_2+\theta_3\beta]} \tag{17}$$

The joint posterior density in (17) is very complicated and hence no closed-form inferences appear to be possible. We, therefore, make use of Gibbs sampler, a MCMC method, proposed by Geman and Geman [18]. It makes us easy to generate a sequence of random variables from the full conditional probability distributions using the current values of the given parameters. MCMC is a class of methods in which one can simulate draws that are slightly dependent and approximately from the posterior distribution. By means of this procedure, our aim is to get the ergodic chains of the parameters which are irreducible, aperiodic and positive recurrent. For implementing Gibbs sampling procedure, the full conditional posterior distributions $\alpha_1,\ \alpha_2$and β are

$$\pi_1(\alpha_1|data,\alpha_2,\beta)\,\tilde{}\,Gamma\,[d_1+\lambda_1,\xi(\beta)+\theta_1] \tag{18}$$
$$\pi_2(\alpha_2|data,\alpha_1,\beta)\,\tilde{}\,Gamma\,[d_2+\lambda_2,\xi(\beta)+\theta_2] \tag{19}$$

$$\pi_3(\beta|data,\alpha_1,\alpha_2) \propto \beta^{d+\lambda_3-1}\prod_{i=1}^{d} t_i^{\beta-1} e^{\sum\limits_{i=1}^{d} t_i^{\beta}}\, e^{-\{(\alpha_1+\alpha_2)\xi(\beta)+\theta_3\beta\}} \tag{20}$$

For obtaining Bayes estimates with Non-informative prior, we can work along the same lines by only putting the values of all prior parameter equal to zero i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \theta_1 = \theta_2 = \theta_3 = 0$. For generating samples from (18)-(20), we make use of the foolwing Gibbs algorithm:

Gibbs Algorithm:

1.Generate α_1 from the posterior density $\pi_1(\alpha_1|data,\alpha_2,\beta)$ as given in (18).
2.Generate α_2 from the posterior density $\pi_2(\alpha_2|data,\alpha_1,\beta)$ as given in (19).
3.Generate $\beta$from the posterior density $\pi_3(\beta|data,\alpha_1,\alpha_2)$ as given in (20) using Metropolis-Hastings algorithm [19, 20].
4.Repeat steps 1-3 N times and record the sequence of $\Delta = (\alpha_1,\alpha_2,\beta)$after M burn-in iterations have occurred to eliminate the effects of the starting values i.e.$(\Delta_{M+1},\Delta_{M+2}.......\Delta_N)$
5.The Bayes estimate of Λsay Λ^*under squared error loss function is

$$\Lambda^* = \frac{1}{N-M}\sum_{k=M+1}^{n}\Lambda_k$$

6.The posterior variance of Λ is

$$V(\Lambda^*) = \frac{1}{N-M}\sum_{k=M+1}^{N}(\Lambda_k-\Lambda^*)^2$$

7.Let $\Lambda^*{}_{(M+1)} \le \Lambda^*{}_{(M+2)} \le .... \le \Lambda^*{}_{(N)}$ respectively denote the ordered values of$\Lambda^*{}_{M+1},\Lambda^*{}_{M+2},....,\Lambda^*{}_{N}$. Then, following Chen and Shao [21], the respective $100(1-\gamma)$ % HPD intervals for $\Lambda$is$\left(\Lambda^*{}_{(M+j^*)},\Lambda^*{}_{(M+j^*+[(1-\gamma)(N-M)])}\right)$. Where, j^* is chosen so that
$\Lambda^*{}_{(M+j^*+[(1-\gamma)(N-M)])} - \Lambda^*{}_{(M+j^*)} = \min\limits_{M\le j\le (N-M)-[(1-\gamma)(N-M)}\left(\Lambda^*{}_{(M+j+[(1-\gamma)(N-M)])} - \Lambda^*{}_{(M+j)}\right)$
i.e., we pick that $100(1-\gamma)$ % credible interval which has smallest width among all credible intervals.

Here, it is notable that the sampling from the posterior distribution given in (20) is not easy as they cannot be simplified to the well-known distributions. Therefore, Metropolis-Hastings algorithm [20,21] has been used to generate $\pi_3(\beta|data,\alpha_1,\alpha_2)$ in step 3. The Metropolis-Hastings algorithm is as follows-

Metropolis-Hastings Algorithm:

Step-1: Start with any value satisfying target density $\pi_3\left(\beta^{(0)}\right) > 0$

Step-2: Using current $v^{(0)}$ value, generate a proposal point v_prop from the proposal density$q\left(\beta^{(1)},\beta^{(2)}\right) = P\left(\beta^{(1)} \to \beta^{(2)}\right)$ i.e., the probability of returning a value of $\beta^{(2)}$ given a previous value of$\beta^{(1)}$.

Table 1: Summary for the electrical appliances data

Estimation	α_1 Estimate (SE/PSE) [Confidence/HPD Interval]	α_2 Estimate (SE/PSE) [Confidence/HPD Interval]	β Estimate (SE/PSE) [Confidence/HPD Interval]
ML Method	0.001082 (0.000925) [0, 0.002895]	0.000608 (0.000537) [0, 0.001662]	0.2339084 (0.016754) [0.20106 , 0.26674]
Jeffreys Bayes	0.001321 (0.000442) [0.000538 , 0.002142]	0.000739 (0.000308) [0.000193, 0.001295]	0.2301705 (0.004652) [0.220911, 0.238491]
Gamma Bayes	0.001193 (0.000421) [0.000559 , 0.002097]	0.000672 (0.000297) [0.000199, 0.001219]	0.231781 (0.004176) [0.220989, 0.238219]

We assume proposal density for the distribution in (20) as $N\left(\vartheta,\delta^2\right)$ respectively. The values of ϑ,δ^2have been set according to the corresponding assumed values of β.

Step-3: Calculate the ratio at the proposal point β_prop and current $\beta^{(i-1)}$ as: $\rho = \log\left[\frac{f(\beta\_prop)q\left(\beta\_prop,v^{(i-1)}\right)}{f\left(\beta^{(i-1)}\right)q\left(\beta^{(i-1)},\beta\_prop\right)}\right]$

Step-4: Generate U from uniform on (0, 1) and take Z=log U.

Step-5: If $Z<\rho$, accept the move i.e., β_prop and set $\beta^{(0)}=\beta\_prop$ and return to Step 1. Otherwise reject it and return to Step- 2.

5 Real Data Application

Here, analysis of a real data set is considered for illustrative purposes. We consider the data from an experiment in which small electrical appliances were being tested, and it has been taken from Lawless [22]. The appliances were operated repeatedly by an automatic testing machine; the lifetimes given were the number of cycles of use until the appliances failed. Total 36 appliances were used, and there were 18 different modes according to which the appliance could have failed. Failure due to ninth mode was considered as cause 1 and the remaining as cause 2. This data set has already been analysed by Kundu and Basu (2000) and Bhattacharya et al. (2014) using Weibull latent failure time distributions with equal shape parameter in context of complete and hybrid censored setup. For the analysis purpose, first, we have created artificial hybrid censored data from the complete sample (n=36) and considering R=25 and T=3000 similarly as considered in Bhattacharya et al. (2014). The hybrid data set is as follows:
(11, 1), (35, 1), (49, 1), (170, 1), (329, 1), (381, 1), (708, 1), (958, 1), (1062, 1), (1167, 2), (1594, 1), (1925, 2), (1990, 2), (2223, 2), (2327, 1), (2400, 2), (2451, 1), (2471, 2), (2551, 2), (2565, 1), (2568, 2), (2694, 2), (2702, 1), (2761, 1), (2831, 1)
Based on the above data, we computed the MLEs, and Bayes estimates for the model parameters along with SE/PSEs and confidence/HPD interval.

To obtain MLEs for the model parameters, First of all, we have used the iterative procedure given in equation (13) to get the ML estimate of β until the absolute difference of two consecutive iteration is less than a very small quantity (0.0001 in our case). To start the iterative procedure in equation (13), initial value have guess by plotting the profile log-likelihood function given in equation (12). Plot of profile log-likelihood in figure 1 clearly shows that the initial value for the iterative procedure can be taken near 0.23. After obtaining ML estimate ofβ, one can easily calculate ML estimates of α_1 and α_2 using equation (10) and (11) respectively.

For obtaining Bayes estimates of the parameters, we have considered both the informative and non-informative priors. We use Gibbs sampler to generate Markov chains with 25,000 realizations. Initially, we run algorithm several times with different starting values of the parameters to check the convergence of the sequences of α_1,α_2 and β for the stationary distributions. It was observed that all the markov chains reached to the stationary condition very rapidly. However, still for removing the effect of initial values of the model parameters, 1000 burn-in iterations have been discarded from the study. Also, strong autocorrelation is observed among the generated chain of model parameters. Therefore, for reducing the autocorrelation among the generated values of model parameters, we only record every 10th generated values of each parameter. The resulting sampling run, posterior density, running mean, and autocorrelation for each model parameter are plotted in Figure 2-4. Note that, for obtaining Bayes estimates with gamma priors, we set the values of priors parameters by considering$\alpha_1=E[\alpha_1]=\lambda_1/\theta_1$,$\alpha_2=E[\alpha_2]=\lambda_2/\theta_2$and $\beta=E[\beta]=\lambda_3/\theta_3$and put the values of all priors parameters as zero to obtain Bayes estimates with non-informative prior. The results of the study have been summarized in Table 1. From Table 1, it is clear that, Bayes estimates with informative prior perform better than non-informative prior as well as ML estimates in terms SE and length of the interval. For the numerical computations of various estimates and confidence/HPD intervals, the programs are developed in R-environment and are available with the authors.

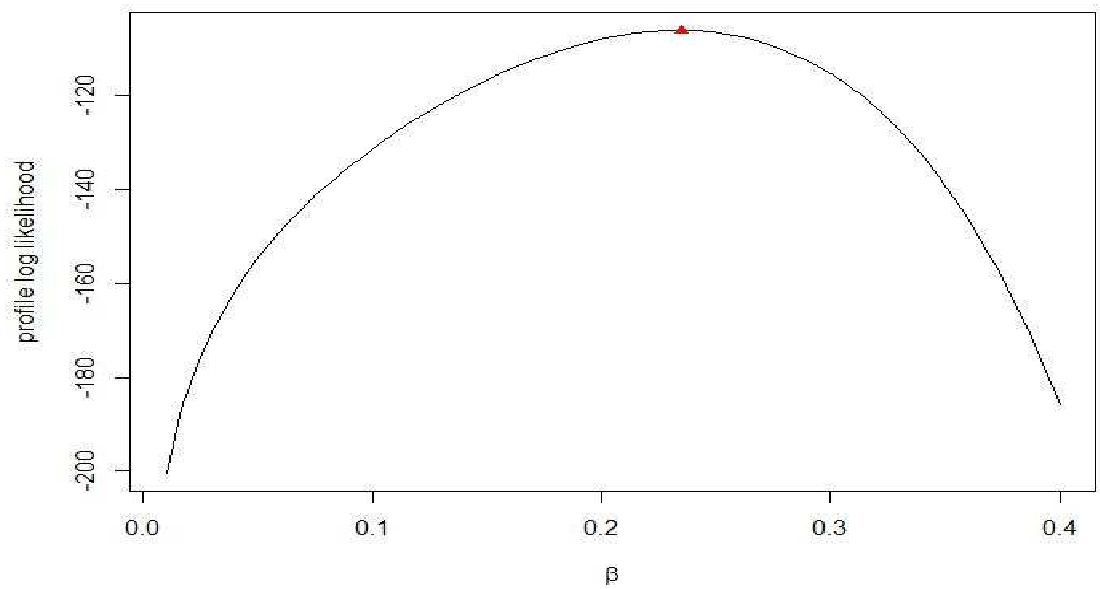

Fig. 1: : profile log-likelihood function for β

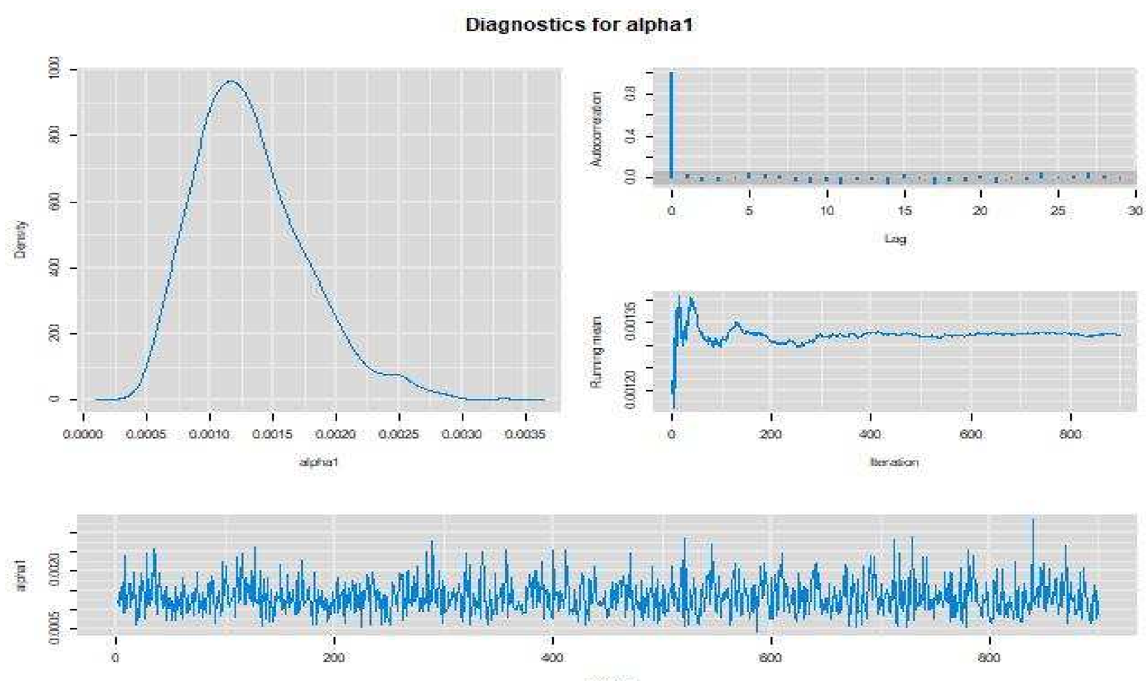

Fig. 2: : : Diagnostics for α_1

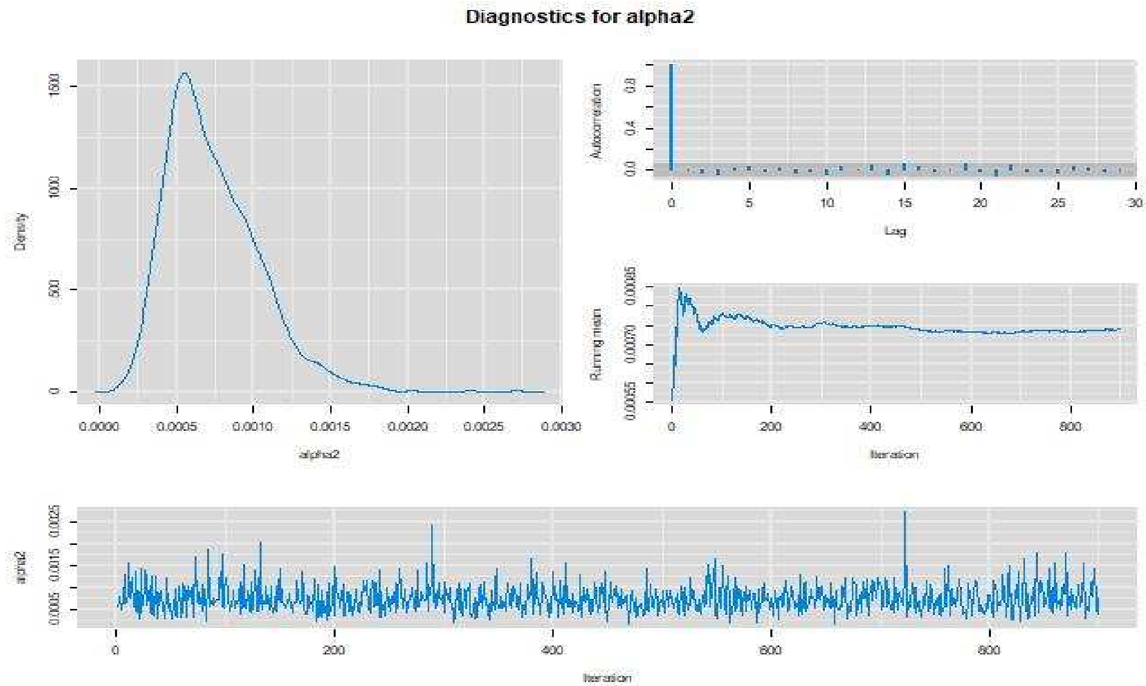

Fig. 3: : : Diagnostics for α_2

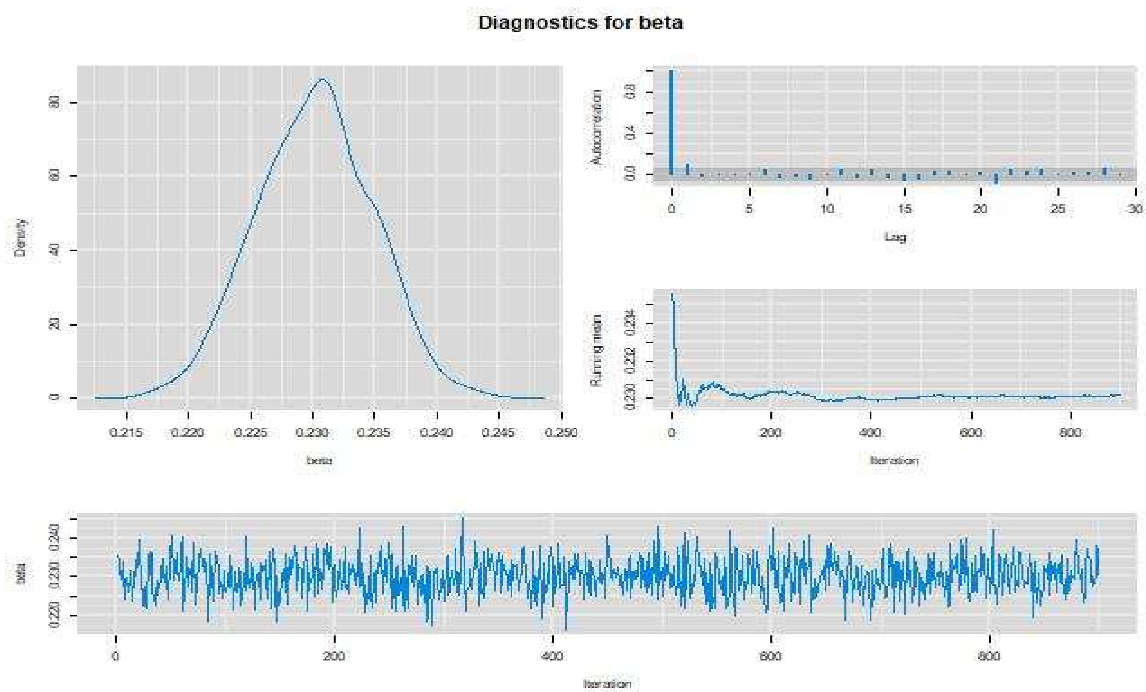

Fig. 4: :: Diagnostics for β

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