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Vectorial Fractional Approximation by Linear Operators

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Abstract: In this article we study quantitatively with rates the convergence of sequences of linear operators applied on Banach space valued functions. The results are pointwise and uniform estimates. To prove our main results we use an elegant boundedness property of our linear operators by their companion positive linear operators. Our inequalities are fractional involving the right and left vector Caputo type fractional derivatives, built in vector moduli of continuity. We treat very general classes of Banach space valued functions. We give applications to vectorial Bernstein operators.

Keywords: Vector fractional derivative, Bochner integral, vector fractional Taylor formula, vector modulus of continuity, linear operators, positive linear operators.

1 Motivation

Let $(X, \|\cdot\|)$ be a Banach space, $N \in \mathbb{N}$. Consider $g \in C([0,1])$ and the classic Bernstein polynomials

$$\left(\widetilde{B}_N g\right)(t) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{1}$$

Let also $f \in C([0,1], X)$ and define the vector valued in X Bernsein linear operators

$$(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{2}$$

That is $(B_N f)(t) \in X$.

Clearly here $\|f\| \in C([0,1])$.

We notice that

$$\|(B_N f)(t)\| \leq \sum_{k=0}^{N} \left\| f\left(\frac{k}{N}\right)\right\| \binom{N}{k} t^k (1-t)^{N-k} = \left(\widetilde{B}_N(\|f\|)\right)(t), \tag{3}$$

$\forall\, t \in [0,1]$.

The property

$$\|(B_N f)(t)\| \leq \left(\widetilde{B}_N(\|f\|)\right)(t), \quad \forall t \in [0,1], \tag{4}$$

is shared by almost all summation/integration similar operators and motivates our work here.

If $f(x) = c \in X$ the constant function, then

$$(B_N c) = c. \tag{5}$$

If $g \in C([0,1])$ and $c \in X$, then $cg \in C([0,1], X)$ and

$$(B_N(cg)) = c\widetilde{B}_N(g). \tag{6}$$

Again (5), (6) are fulfilled by many summation/integration operators.

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In fact here (6) implies (5), when $g \equiv 1$.

The above can be generalized from $[0,1]$ to any interval $[a,b] \subset \mathbb{R}$. All this discussion motivates us to consider the following situation.

Let $L_N : C([a,b],X) \hookrightarrow C([a,b],X)$, $(X, \|\cdot\|)$ a Banach space, L_N is a linear operator, $\forall\, N \in \mathbb{N}$, $x_0 \in [a,b]$. Let also $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, a sequence of positive linear operators, $\forall\, N \in \mathbb{N}$.

We assume that

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{7}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x_0 \in X$, $\forall\, f \in C([a,b],X)$.

When $g \in C([a,b])$, $c \in X$, we assume that

$$(L_N(cg)) = c\widetilde{L}_N(g). \tag{8}$$

The special case of

$$\widetilde{L}_N(1) = 1, \tag{9}$$

implies

$$L_N(c) = c, \quad \forall\, c \in X. \tag{10}$$

We call $\widetilde{L}_N$ the companion operator of L_N.

Based on the above fundamental properties we study the fractional approximation properties of the sequence of linear operators $\{L_N\}_{N\in\mathbb{N}}$, i.e. their fractional convergence to the unit operator. No kind of positivity property of $\{L_N\}_{N\in\mathbb{N}}$ is assumed. Other important motivation comes from [1], [2], [5].

2 Background

All vector integrals here are of Bochner type ([7]).

We need

Definition 1.([6]) *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \quad \forall\, x \in [a,b]. \tag{11}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [9], p. 83), and also set $D_{*a}^{0} f := f$.

By [6], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b],X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [6], $D_{*a}^{\alpha} f \in C([a,b],X)$, hence $\|D_{*a}^{\alpha} f\| \in C([a,b])$.

We mention

Lemma 1.([5]) *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b],X)$ and $f^{(m)} \in L_\infty([a,b],X)$. Then $D_{*a}^{\alpha} f(a) = 0$.*

We mention

Definition 2.([4]) *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b],X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\,dz, \quad \forall\, x \in [a,b]. \tag{12}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [4], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b],X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [4], $D_{b-}^{\alpha} f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha} f\right\| \in C([a,b])$.

We need

Lemma 2.([5]) *Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D^{\alpha}_{b-} f(b) = 0$.*

We mention the left fractional Taylor formula

Theorem 1.([6]) *Let $m \in \mathbb{N}$ and $f \in C^{m-1}([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\alpha > 0 : m = \lceil \alpha \rceil$. Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{13}$$

where $x \in [a,b]$.

Assume that $f^{(m)}$ exists outside a λ-null Borel set $B_x \subseteq [a,x]$ (λ is the Lebesgue measure) such that

$$h_1(F_x(B_x)) = 0, \ \forall\, x \in [a,b] \tag{14}$$

(h_1 is the Hausdorff measure of order 1, see [10]). We also assume that $f^{(m)} \in L_1([a,b],X)$. Then

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D^{\alpha}_{*a} f\right)(z)\, dz, \tag{15}$$

$\forall\, x \in [a,b]$.

We also mention the right fractional Taylor formula

Theorem 2.([4]) *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b],X)$. Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [x,b], \tag{16}$$

where $x \in [a,b]$.

Assume that $f^{(m)}$ exists outside a λ-null Borel set $B_x \subseteq [x,b]$, such that

$$h_1(F_x(B_x)) = 0, \ \forall\, x \in [a,b]. \tag{17}$$

We also assume that $f^{(m)} \in L_1([a,b],X)$. Then

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D^{\alpha}_{b-} f\right)(z)\, dz, \tag{18}$$

$\forall\, x \in [a,b]$.

We define the following classes of functions:

Definition 3.([5]) *We call ($x_0 \in [a,b] \subset \mathbb{R}$)*

$$H^{(1)}_{x_0} := \{ f \in C^{m-1}([a,b],X) : [a,b] \subset \mathbb{R}, (X, \|\cdot\|) \tag{19}$$

is a Banach space, $\alpha > 0 : m = \lceil \alpha \rceil$; $f^{(m)} \in L_\infty([a,b],X)$; $F^{(1)}_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t)$ is defined $\forall\, t \in [x,x_0]$,with $x \in [a,x_0]$ and $f^{(m)}$ exists outside a λ-null Borel set $B^{(1)}_x \subseteq [x,x_0]$, such that $h_1\left(F^{(1)}_x\left(B^{(1)}_x\right)\right) = 0$, $\forall\, x \in [a,x_0]$; $F^2_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t)$ is defined $\forall\, t \in [x_0,x]$,with $x \in [x_0,b]$ and $f^{(m)}$ exists outside a λ-null Borel set $B^{(2)}_x \subseteq [x_0,x]$, such that $h_1\left(F^{(2)}_x\left(B^{(2)}_x\right)\right) = 0$, $\forall\, x \in [x_0,b]\}$,

$$H^{(2)} := \{ f \in C^m([a,b],X) : [a,b] \subset \mathbb{R}, \tag{20}$$

X is a Banach space, $\alpha > 0 : m = \lceil \alpha \rceil\}$.

Notice that

$$H^{(2)} \subset H^{(1)}_{x_0}, \ \forall\, x_0 \in [a,b]. \tag{21}$$

Convention 1 *We assume that*

$$D^{\alpha}_{*x_0} f(x) = 0, \text{ for } x < x_0, \quad \text{and} \quad D^{\alpha}_{x_0-} f(x) = 0, \text{ for } x > x_0, \tag{22}$$

for all $x, x_0 \in [a,b]$.

We need

Definition 4.([5]) *Let* $f \in C([a,b],X)$, $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ *a Banach space. We define the first modulus of continuity of* f *as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]: \\ |x-y|\le\delta}} \|f(x) - f(y)\|, \ \ 0 < \delta \le b-a. \tag{23}$$

If $\delta > b-a$, *then* $\omega_1(f,\delta) = \omega_1(f, b-a)$.
Notice $\omega_1(f,\delta)$ *is increasing in* $\delta > 0$.

Clearly f is uniformly continuous and $\omega_1(f,\delta) < \infty$. For $f \in B([a,b],X)$ (bounded functions) $\omega_1(f,\delta)$ is defined the same way.

Lemma 3.([5]) *We have* $\omega_1(f,\delta) \to 0$ *as* $\delta \downarrow 0$ *iff* $f \in C([a,b],X)$.

We mention

Proposition 2.([5]) *Let* $f \in C^n([a,b],X)$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D^{\nu}_{*a} f(x)$ *is continuous in* $x \in [a,b]$.

Proposition 3.([5]) *Let* $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D^{\nu}_{b-} f(x)$ *is continuous in* $x \in [a,b]$.

We also mention

Proposition 4.([5]) *Let* $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D^{\alpha}_{*x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{24}$$

for all $x, x_0 \in [a,b] : x \ge x_0$.
Then $D^{\alpha}_{*x_0} f(x)$ *is continuous in* x_0.

Proposition 5.([5]) *Let* $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D^{\alpha}_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{25}$$

for all $x, x_0 \in [a,b] : x_0 \ge x$.
Then $D^{\alpha}_{x_0-} f(x)$ *is continuous in* x_0.

Corollary 1.([5]) *Let* $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. *Then* $D^{a}_{*x_0} f(x)$, $D^{a}_{x_0-} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $[a,b]^2$ *into* X, X *is a Banach space.*

We need

Theorem 3.([5]) *Let* $f : [a,b]^2 \to X$ *be jointly continuous,* X *is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot,x), \delta, [x,b]), \tag{26}$$

$\delta > 0$, $x \in [a,b]$.
Then G *is continuous on* $[a,b]$.

Theorem 4.(*[5]) Let $f:[a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1(f(\cdot,x), \delta, [a,x]), \tag{27}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We mention and need

Remark.([5]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then

$$\|D_{*a}^{\nu} f(x)\| \leq \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \ \forall x \in [a,b], \tag{28}$$

and it follows that

$$\omega_1(D_{*a}^{\nu} f, \delta) \leq \frac{2\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{29}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{30}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{31}$$

and

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{x_0-}^{\alpha} f, \delta\right)_{[a,x_0]} \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{32}$$

*Remark.*Let μ be a finite positive measure on Borel σ-algebra of $[a,b]$.
Let $\alpha > 0$, then by Hölder's inequality we obtain

$$\int_{[a,x_0]} (x_0-x)^{\alpha} d\mu(x) \leq \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu([a,x_0])^{\frac{1}{(\alpha+1)}}, \tag{33}$$

and

$$\int_{(x_0,b]} (x-x_0)^{\alpha} d\mu(x) \leq \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu((x_0,b])^{\frac{1}{(\alpha+1)}}. \tag{34}$$

Let now $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1, ..., m-1$. Then by applying again Hölder's inequality we obtain

$$\int_{[a,b]} |x-x_0|^k d\mu(x) \leq \left(\int_{[a,b]} |x-x_0|^{\alpha+1} d\mu(x) \right)^{\frac{k}{(\alpha+1)}} \mu([a,b])^{\frac{\alpha+1-k}{(\alpha+1)}}. \tag{35}$$

We need

Lemma 4.(*[1], p. 208, Lemma 7.1.1) Let $f \in B([a,b],X)$, $(X, \|\cdot\|)$ is a Banach space. Then*

$$\|f(x) - f(x_0)\| \leq \omega_1(f,h) \left\lceil \frac{|x-x_0|}{h} \right\rceil \leq \omega_1(f,h)\left(1 + \frac{|x-x_0|}{h}\right), \tag{36}$$

$\forall\, x, x_0 \in [a,b]$, $h > 0$.

We need

Terminology 1_Let $\widetilde{L}_N : C([a,b]) \to C([a,b])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. By Riesz representation theorem (see [8], p. 304) we have_

$$\widetilde{L}_N(f, x_0) = \int_{[a,b]} f(t)\, d\mu_{Nx_0}(t), \tag{37}$$

$\forall\, x_0 \in [a,b]$, where μ_{Nx_0} is a unique positive finite measure on σ-Borel algebra of $[a,b]$. Call

$$\widetilde{L}_N(1, x_0) = \mu_{Nx_0}([a,b]) = M_{Nx_0}. \tag{38}$$

We also make

*Remark.*Let $\widetilde{L}_N : C([a,b]) \to C([a,b])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. Using (37) and Hölder's inequality we obtain ($x \in [a,b]$, $k = 1, \ldots, m-1, \, , m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$) for $k = 1, \ldots, m-1$ that

$$\left\| \widetilde{L}_N \left(|\cdot - x|^k, x \right) \right\|_\infty \leq \left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1}, x \right) \right\|_\infty^{\frac{1}{(\alpha+1)}} \left\| \widetilde{L}_N 1 \right\|_\infty^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}. \tag{39}$$

Here χ stands for the characteristic function.

Also we observe that

$$C([a,b]) \ni |\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot) \leq |\cdot - x|^{\alpha+1}, \quad \forall\, x \in [a,b], \tag{40}$$

and

$$C([a,b]) \ni |\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot) \leq |\cdot - x|^{\alpha+1}, \quad \forall\, x \in [a,b].$$

By positivity of L_N we obtain

$$\left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x \right) \right\|_\infty \leq \left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1}, x \right) \right\|_\infty, \tag{41}$$

and

$$\left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x \right) \right\|_\infty \leq \left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1}, x \right) \right\|_\infty. \tag{42}$$

So if the right side of each of (41), (42) tends to zero, so do the left hand sides of these.

We also make

*Remark.*Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Take $a \leq x \leq x_0$, then

$$(x_0 - x)^{\alpha+1} \leq (x_0 - x)^{\alpha+1} 1 + 0.$$

Similarly, for $x_0 \leq x \leq b$, we get

$$(x - x_0)^{\alpha+1} \leq 0 + (x - x_0)^{\alpha+1} 1.$$

So we have

$$|\cdot - x|^{\alpha+1} \leq |\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot) + |\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), \quad \forall\, x \in [a,b]. \tag{43}$$

Thus, by positivity of $\widetilde{L}_N$, we obtain

$$\left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1}, x \right) \right\|_\infty \leq \left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x \right) \right\|_\infty + \left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x \right) \right\|_\infty. \tag{44}$$

So if both $\left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1} \chi_{[a,x]}(\cdot), x \right) \right\|_\infty, \left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1} \chi_{[x,b]}(\cdot), x \right) \right\|_\infty \to 0$, as $N \to \infty$, then $\left\| \widetilde{L}_N \left(|\cdot - x|^{\alpha+1}, x \right) \right\|_\infty \to 0$.

3 Main Results

It follows our first main result

Theorem 5.*Let $N \in \mathbb{N}$ and $L_N : C([a,b],X) \to C([a,b],X)$, where $(X, \|\cdot\|)$ is a Banach space and L_N is a linear operator. Let the positive linear operators $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, such that*

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{45}$$

$\forall\, N \in \mathbb{N}$, where $f \in C([a,b],X)$, and $x_0 \in [a,b]$.

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\, g \in C([a,b]), \forall\, c \in X. \tag{46}$$

Here we consider $f \in H_{x_0}^{(1)}$; $r_1, r_2 > 0$, $0 < \alpha \notin \mathbb{N}$. Furthermore the unique positive finite measure μ_{Nx_0} is as in (37).

Then

$$\begin{gathered}\left\|(L_N(f))(x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| \leq \\ \frac{1}{\Gamma(\alpha+1)}\left\{\left[\mu_{Nx_0}([a,x_0])^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1)r_1}\right]\right. \\ \omega_1\left(D_{x_0-}^{\alpha}f, r_1\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]} \\ \left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)} + \\ \left[\mu_{Nx_0}((x_0,b])^{\frac{1}{(\alpha+1)}} + \frac{1}{r_2(\alpha+1)}\right] \\ \omega_1\left(D_{*x_0}^{\alpha}f, r_2\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{(x_0,b]} \\ \left.\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right\}.\end{gathered} \tag{47}$$

*Proof.*For a fixed $x_0 \in [a,b]$ we have

$$\Delta(x_0) := \left\|(L_N(f))(x_0) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| = \tag{48}$$

$$\left\|\left(L_N\left(f(\cdot) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(\cdot - x_0)^k\right)\right)(x_0)\right\| \leq \tag{49}$$

$$\left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(\cdot - x_0)^k\right\|\right)\right)(x_0) \overset{(37)}{=}$$

$$\int_{[a,b]}\left\|f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right\|d\mu_{Nx_0}(x) = \tag{50}$$

$$\int_{[a,x_0]}\left\|f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right\|d\mu_{Nx_0}(x) +$$

$$\int_{(x_0,b]}\left\|f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right\|d\mu_{Nx_0}(x) \overset{\text{by } ((18),\,(15))}{=}$$

$$\frac{1}{\Gamma(\alpha)}\left[\int_{[a,x_0]}\left\|\int_x^{x_0}(z-x)^{\alpha-1}\left(D_{x_0-}^{\alpha}f\right)(z)\,dz\right\|d\mu_{Nx_0(x)}+\right.$$
$$\left.\int_{(x_0,b]}\left\|\int_{x_0}^{x}(x-z)^{\alpha-1}\left(D_{*x_0}^{\alpha}f\right)(z)\,dz\right\|d\mu_{Nx_0(x)}\right]\leq \tag{51}$$

(above the integrands are continuous functions in x, also $D_{x_0-}^{\alpha}f, D_{*x_0}^{\alpha}f\in L_1([a,b],X)$)

$$\frac{1}{\Gamma(\alpha)}\left[\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left\|\left(D_{x_0-}^{\alpha}f\right)(z)-\left(D_{x_0-}^{\alpha}f\right)(x_0)\right\|dz\right)d\mu_{Nx_0(x)}+\right.$$
$$\left.\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left\|\left(D_{*x_0}^{\alpha}f\right)(z)-\left(D_{*x_0}^{\alpha}f\right)(x_0)\right\|dz\right)d\mu_{Nx_0(x)}\right]\leq \tag{52}$$

(let $h_1,h_2>0$, by (36))

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(1+\frac{x_0-z}{h_1}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}\right.$$
$$\left.+\left[\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(1+\frac{z-x_0}{h_2}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\}. \tag{53}$$

I.e. it holds

$$\Delta(x_0)\leq\frac{1}{\Gamma(\alpha)}\cdot$$
$$\left\{\left[\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(1+\frac{x_0-z}{h_1}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}\right.$$
$$\left.+\left[\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(1+\frac{z-x_0}{h_2}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\}= \tag{54}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[a,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\left(\int_x^{x_0}(x_0-z)^{2-1}(z-x)^{\alpha-1}dz\right)\right)d\mu_{Nx_0}(x)\right]\right.$$
$$\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}+$$
$$\left[\int_{(x_0,b]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}(z-x_0)^{2-1}dz\right)\right)d\mu_{Nx_0}(x)\right]$$
$$\left.\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\}= \tag{55}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[a,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\frac{(x_0-x)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}+\right.$$
$$\left.\left[\int_{(x_0,b]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\frac{(x-x_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu_{Nx}(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\}.$$

Therefore it holds

$$\Delta(x_0)\leq\frac{1}{\Gamma(\alpha)}\cdot$$
$$\left\{\left[\frac{1}{\alpha}\int_{[a,x_0]}(x_0-x)^{\alpha}d\mu_{Nx_0}(x)+\frac{1}{h_1\alpha(\alpha+1)}\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\right]\right. \tag{56}$$
$$\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}+$$
$$\left[\frac{1}{\alpha}\int_{(x_0,b]}(x-x_0)^{\alpha}d\mu_{Nx_0}(x)+\frac{1}{h_2\alpha(\alpha+1)}\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\right]$$

$$\omega_1 \left(D^{\alpha}_{*x_0} f, h_2 \right)_{[x_0,b]} \Big\} .$$

Momentarily we assume the positive choices of

$$h_1 = r_1 \left(\int_{[a,x_0]} (x_0 - x)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\frac{1}{(\alpha+1)}} > 0, \tag{57}$$

and

$$h_2 = r_2 \left(\int_{(x_0,b]} (x - x_0)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\frac{1}{(\alpha+1)}} > 0. \tag{58}$$

Consequently we obtain

$$\begin{aligned} \Delta(x_0) \le \frac{1}{\Gamma(\alpha+1)} \cdot \\ \left\{ \left[\mu_{Nx_0}([a,x_0])^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1) r_1} \right] \omega_1 \left(D^{\alpha}_{x_0-} f, h_1 \right)_{[a,x_0]} \left(\frac{h_1}{r_1} \right)^{\alpha} + \right. \\ \left. \left[\left(\mu_{Nx_0}((x_0,b]) \right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r_2(\alpha+1)} \right] \omega_1 \left(D^{\alpha}_{*x_0} f, h_2 \right)_{[x_0,b]} \left(\frac{h_2}{r_2} \right)^{\alpha} \right\} = \end{aligned} \tag{59}$$

$$\begin{aligned} \frac{1}{\Gamma(\alpha+1)} \left\{ \left[\mu_{Nx_0}([a,x_0])^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1) r_1} \right] \right. \\ \omega_1 \left(D^{\alpha}_{x_0-} f, r_1 \left(\int_{[a,x_0]} (x_0 - x)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \\ \left(\int_{[a,x_0]} (x_0 - x)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} + \\ \left[\left(\mu_{Nx_0}((x_0,b]) \right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r_2(\alpha+1)} \right] \\ \omega_1 \left(D^{\alpha}_{*x_0} f, r_2 \left(\int_{(x_0,b]} (x - x_0)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \\ \left. \left(\int_{(x_0,b]} (x - x_0)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right\} . \end{aligned} \tag{60}$$

So far we have proved

$$\begin{aligned} \Delta(x_0) \le \frac{1}{\Gamma(\alpha+1)} \left\{ \left[\mu_{Nx_0}([a,x_0])^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1) r_1} \right] \right. \\ \omega_1 \left(D^{\alpha}_{x_0-} f, r_1 \left(\int_{[a,x_0]} (x_0 - x)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \\ \left(\int_{[a,x_0]} (x_0 - x)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} + \\ \left[\left(\mu_{Nx_0}((x_0,b]) \right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r_2(\alpha+1)} \right] \\ \omega_1 \left(D^{\alpha}_{*x_0} f, r_2 \left(\int_{(x_0,b]} (x - x_0)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \\ \left. \left(\int_{(x_0,b]} (x - x_0)^{\alpha+1} d\mu_{Nx_0}(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right\}, \text{ for } h_1, h_2 > 0. \end{aligned} \tag{61}$$

Below we discuss special cases. If $\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)=0$, then $(x-x_0)=0$, a.e. on $(x_0,b]$, that is $x=x_0$ a.e. on $(x_0,b]$, more precisely $\mu_{Nx_0}\{x\in(x_0,b]:x\neq x_0\}=0$, hence $\mu_{Nx_0}(x_0,b]=0$.

Therefore μ_{Nx_0} concentrates on $[a,x_0]$.

Hence inequality (61) is written as

$$\Delta(x_0)\leq\frac{1}{\Gamma(\alpha+1)}\left\{\left[\left(\mu_{Nx_0}([a,x_0])\right)^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r_1}\right]\right.$$

$$\omega_1\left(D^{\alpha}_{x_0-}f,r_1\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\tag{62}$$

$$\left.\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right\}.$$

Since $(b,b]=\emptyset$ and $\mu(\emptyset)=0$, in the case of $x_0=b$, we obtain again (62) written for $x_0=b$. Thus, the inequality (62) is a valid inequality when $\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\neq0$.

If additionally we assume that $\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)=0$, then $(x_0-x)=0$, a.e. on $[a,x_0]$, that is $x=x_0$ a.e. on $[a,x_0]$, which means $\mu_{Nx_0}\{x\in[a,x_0]:x\neq x_0\}=0$. Hence $\mu_{Nx_0}=\delta_{x_0}M$, where δ_{x_0} denotes the unit Dirac measure and $M=\mu_{Nx_0}([a,b])>0$.

In that case the right hand side of (62) equals zero. Furthermore

$$\left(\widetilde{L}_N\left(\left\|f(\cdot)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(\cdot-x_0)^k\right\|\right)\right)(x_0)$$

(by (50))

$$=\|f(x_0)-f(x_0)\|M=0,$$

implying $\Delta(x_0)=0$, by (48), (49). That is (62) is valid trivially.

Finally let us go the other way around. Let us assume that
$\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)=0$, then reasoning similarly as before we get that μ_{Nx_0} over $[a,x_0]$ concentrates at x_0. That is $\mu_{Nx_0}=\delta_{x_0}\mu_{Nx_0}([a,x_0])$, on $[a,x_0]$.

In that case (61) is written and it holds as

$$\Delta(x_0)\leq\frac{1}{\Gamma(\alpha+1)}\left[\left(\mu((x_0,b])\right)^{\frac{1}{(\alpha+1)}}+\frac{1}{r_2(\alpha+1)}\right]$$

$$\omega_1\left(D^{\alpha}_{*x_0}f,r_2\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\tag{63}$$

$$\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}.$$

If $x_0=a$ then (63) can be redone and rewritten, just replace $(x_0,b]$ by $[a,b]$ all over.

So inequality (63) is valid when

$$\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\neq0.$$

If additionally we assume that $\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)=0$, then as before $\mu_{Nx_0}(x_0,b]=0$. Then the right hand side of (63) is zero. Then
$\left(\widetilde{L}_N\left(\left\|f(\cdot)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(\cdot-x_0)^k\right\|\right)\right)(x_0)=0$, by (50). Hence by (48), (49) we get $\Delta(x_0)=0$. Therefore (63) is valid trivially. The proof of (47) now has been completed in all possible cases.

Corollary 2. *All as in Theorem 5, with $r_1 = r_2 = r > 0$. Then*

$$\left\| \left(L_N(f)\right)(x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) \right\| \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \left[\left(\left(\widetilde{L}_N(1)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r(\alpha+1)} \right]$$

$$\left[\omega_1 \left(D^{\alpha}_{x_0-} f, r \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} + \right.$$

$$\left. \omega_1 \left(D^{\alpha}_{*x_0} f, r \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \right]$$

$$\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \quad \forall\, N \in \mathbb{N}. \tag{64}$$

Proof. We observe that

$$\text{Right hand side (47)} \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \left[\left(\left(\widetilde{L}_N(1)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r(\alpha+1)} \right]$$

$$\left[\omega_1 \left(D^{\alpha}_{x_0-} f, r \left(\left(\widetilde{L}_N\left(\chi_{[a,x_0]}(\cdot)\,|x_0 - \cdot|^{\alpha+1}\right)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \right.$$

$$\left(\left(\widetilde{L}_N\left(\chi_{[a,x_0]}(\cdot)\,|x_0 - \cdot|^{\alpha+1}\right)\right)(x_0) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} + \tag{65}$$

$$\omega_1 \left(D^{\alpha}_{*x_0} f, r \left(\left(\widetilde{L}_N\left(\chi_{[x_0,b]}(\cdot)\,|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]}$$

$$\left. \left(\left(\widetilde{L}_N\left(\chi_{[x_0,b]}(\cdot)\,|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right] \overset{\text{(by (40))}}{\leq}$$

$$\frac{1}{\Gamma(\alpha+1)} \left[\left(\left(\widetilde{L}_N(1)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r(\alpha+1)} \right]$$

$$\left[\omega_1 \left(D^{\alpha}_{x_0-} f, r \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} + \right.$$

$$\left. \omega_1 \left(D^{\alpha}_{*x_0} f, r \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \right]$$

$$\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \tag{66}$$

proving (64).

Corollary 3. *All as in Theorem 5, with $r_1 = r_2 = r > 0$. Then*

1)

$$\left\| \left(L_N(f)\right)(x_0) - f(x_0) \right\| \leq$$

$$\| f(x_0) \| \left| \left(\widetilde{L}_N(1)\right)(x_0) - 1 \right| +$$

$$\sum_{k=0}^{m-1} \frac{\left\| f^{(k)}(x_0) \right\|}{k!} \left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0) +$$

$$\frac{1}{\Gamma(\alpha+1)}\left[\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}+\frac{1}{r(\alpha+1)}\right]$$

$$\left[\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}+\right.$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f, r\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right]$$

$$\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \ \ \forall N\in\mathbb{N}. \tag{67}$$

2) additionally assume that $f^{(k)}(x_0)=0$, $k=0,1,\dots,m-1$, *we have*

$$||(L_N(f))(x_0)||\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left[\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}+\frac{1}{r(\alpha+1)}\right]$$

$$\left[\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}+\right.$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f, r\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right]$$

$$\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \ \ \forall N\in\mathbb{N}. \tag{68}$$

Proof. We have that

$$\|(L_N(f))(x_0)-f(x_0)\|=$$

$$\left\|(L_N(f))(x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot-x_0)^k\right)\right)(x_0)+\right.$$

$$\left.\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot-x_0)^k\right)\right)(x_0)-f(x_0)\right\|\leq \tag{69}$$

$$\left\|(L_N(f))(x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot-x_0)^k\right)\right)(x_0)\right\|+$$

$$\left\|\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot-x_0)^k\right)\right)(x_0)\right\|+$$

$$\|f(x_0)\|\left|\left(\widetilde{L}_N(1)\right)(x_0)-1\right|\overset{(64)}{\leq}$$

$$\|f(x_0)\|\left|\left(\widetilde{L}_N(1)\right)(x_0)-1\right|+$$

$$\sum_{k=1}^{m-1}\frac{\left\|f^{(k)}(x_0)\right\|}{k!}\left(\widetilde{L}_N\left(|\cdot-x_0|^k\right)\right)(x_0)+$$

$$\frac{1}{\Gamma(\alpha+1)}\left[\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}+\frac{1}{r(\alpha+1)}\right] \tag{70}$$

$$\left[\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}+\right.$$

$$\omega_1\left(D^{\alpha}_{*x_0}f, r\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\Bigg]$$

$$\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \quad \forall\, N \in \mathbb{N}.$$

We make

Definition 5.(*[5]*) *We call* ($x_0 \in [a,b] \subset \mathbb{R}$)

$$\widetilde{H}^{(1)}_{x_0} := \{f \in C([a,b],X) : [a,b] \subset \mathbb{R}, (X, \|\cdot\|) \tag{71}$$

is a Banach space, $0 < \alpha < 1$*;* $f' \in L_\infty([a,b],X)$*;* f' *exists outside a* λ*-null Borel set* $B^{(1)}_x \subseteq [x,x_0]$*, such that* $h_1\left(f\left(B^{(1)}_x\right)\right) = 0$*,* $\forall\ x \in [a,x_0]$*;* f' *exists outside a* λ*-null Borel set* $B^{(2)}_x \subseteq [x_0,x]$*, such that* $h_1\left(f\left(B^{(2)}_x\right)\right) = 0$*,* $\forall$ $x \in [x_0,b]\}$*.*

Notice that $C^1([a,b],X) \subset \widetilde{H}^{(1)}_{x_0}$*,* $\forall\, x_0 \in [a,b]$*.*

The last Definition 5 simplifies a lot Definition 3 when $m = 1$.

Because h_1 is an outer measure on the power set $\mathscr{P}(X)$ we can further simplify Definition 5, based on $f(\emptyset) = \emptyset$, $h_1(\emptyset) = 0$, and $A \subset B$ implies $h_1(A) \leq h_1(B)$, as follows:

Remark.([5]) Let $x_0 \in [a,b] \subset \mathbb{R}$. We have that

$$\widetilde{H}^{(1)}_{x_0} := \{f \in C([a,b],X) : (X, \|\cdot\|) \tag{72}$$

is a Banach space, $0 < \alpha < 1$; $f' \in L_\infty([a,b],X)$; f' exists outside a λ-null Borel set $B_a \subseteq [a,x_0]$, such that $h_1(f(B_a)) = 0$; f' exists outside a λ-null Borel set $B_b \subseteq [x_0,b]$, such that $h_1(f(B_b)) = 0\}$.

We give

Corollary 4.*All as in Theorem 5, with* $r_1 = r_2 = r > 0$*,* $f \in \widetilde{H}^{(1)}_{x_0}$*,* $0 < \alpha < 1$*. Then*

$$\begin{gathered}\|(L_N(f))(x_0) - f(x_0)\| \leq \\ \|f(x_0)\|\left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \\ \frac{1}{\Gamma(\alpha+1)}\left[\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}} + \frac{1}{r(\alpha+1)}\right] \\ \Bigg[\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]} + \\ \omega_1\left(D^{\alpha}_{*x_0}f, r\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\Bigg] \\ \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \quad \forall\, N \in \mathbb{N}.\end{gathered} \tag{73}$$

*Proof.*Use of (67) and $m = \lceil\alpha\rceil = 1$.

We make

*Remark.*By (35) we obtain that

$$\left(\widetilde{L_N}\left(|\cdot - x_0|^k\right)\right)(x_0) \leq \left(\left(\widetilde{L_N}\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{k}{\alpha+1}}\left(\left(\widetilde{L_N}(1)\right)(x_0)\right)^{\frac{\alpha+1-k}{(\alpha+1)}}, \tag{74}$$

for $k = 1,\ldots,m-1$; where $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$.

In case of $\left(\widetilde{L_N}(1)\right)(x_0) \to 1$, as $N \to \infty$, then $\left(\widetilde{L_N}(1)\right)(x_0)$ is bounded. Assume also that $\left(\widetilde{L_N}\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0) \to 0$, as $N \to \infty$. Then, by (67) and (74), we easily derive that $(L_N(f))(x_0) \to f(x_0)$, as $N \to \infty$, $\forall\ f \in H^{(1)}_{x_0}$.

The same conclusion derives from (73).

It follows our second main result

Theorem 6. *Let $N \in \mathbb{N}$ and $L_N : C([a,b],X) \to C([a,b],X)$, where $(X, \|\cdot\|)$ is a Banach space and L_N is a linear operator. Let the positive linear operators $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, such that*

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{75}$$

$\forall\, N \in \mathbb{N}$, $\forall\, f \in C([a,b],X)$, and $\forall\, x_0 \in [a,b]$.

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\, g \in C([a,b]), \forall\, c \in X. \tag{76}$$

Let $0 < \alpha \notin \mathbb{N}$, and $m = \lceil \alpha \rceil$, and let $r > 0$.

Here we consider $f \in C^m([a,b],X)$. Then

$$\begin{gathered}
\|\|L_N(f) - f\|\|_{\infty,[a,b]} \leq \\
\|\|f\|\|_{\infty,[a,b]} \left\|\widetilde{L}_N(1) - 1\right\|_{\infty,[a,b]} + \\
\sum_{k=1}^{m-1} \frac{\left\|\left\|f^{(k)}\right\|\right\|_{\infty,[a,b]}}{k!} \left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]} + \\
\frac{1}{\Gamma(\alpha+1)} \left[\left\|\widetilde{L}_N(1)\right\|_{\infty,[a,b]}^{\frac{1}{(\alpha+1)}} + \frac{1}{r(\alpha+1)}\right] \\
\left[\sup_{x_0 \in [a,b]} \omega_1\left(D_{x_0-}^{\alpha} f, r \left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]} + \right. \\
\left. \sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, r \left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right] \\
\left(\left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \quad \forall\, N \in \mathbb{N}.
\end{gathered} \tag{77}$$

Proof. We use (67) and the fact that $C^m([a,b],X) \subset H_{x_0}^{(1)}$, $\forall\, x_0 \in [a,b]$.

Corollary 5. *(to Theorem 6) All as in Theorem 6. Here $0 < \alpha < 1$, and $f \in C^1([a,b],X)$. Then*

$$\begin{gathered}
\|\|L_N(f) - f\|\|_{\infty,[a,b]} \leq \\
\|\|f\|\|_{\infty,[a,b]} \left\|\widetilde{L}_N(1) - 1\right\|_{\infty,[a,b]} + \\
\frac{1}{\Gamma(\alpha+1)} \left[\left\|\widetilde{L}_N(1)\right\|_{\infty,[a,b]}^{\frac{1}{(\alpha+1)}} + \frac{1}{r(\alpha+1)}\right] \\
\left[\sup_{x_0 \in [a,b]} \omega_1\left(D_{x_0-}^{\alpha} f, r \left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]} + \right. \\
\left. \sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, r \left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right] \\
\left(\left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \quad \forall\, N \in \mathbb{N}.
\end{gathered} \tag{78}$$

Proof. By (77). Here $m = 1$ and $C^1([a,b],X) \subset \widetilde{H}_{x_0}^{(1)}$, $\forall\, x_0 \in [a,b]$.

We make

*Remark.*By [2], we get that $\left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0)$, $k = 1,...,m-1$;
$\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)$ are all continuous functions in $x_0 \in [a,b]$, thus their uniform norms are finite.

By (31), (32), the $\sup_{x_0\in[a,b]} \omega_1\left(D^{\alpha}_{x_0-}f,\cdot\right)_{[a,x_0]}$, $\sup_{x_0\in[a,b]} \omega_1\left(D^{\alpha}_{*x_0}f,\cdot\right)_{[x_0,b]}$ are finite in both (77), (78).

By (74) we derive

$$\left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0)\right\|_{\infty,x_0\in[a,b]} \leq$$

$$\left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{\alpha+1}\right)\right)(x_0)\right\|^{\frac{k}{\alpha+1}}_{\infty,x_0\in[a,b]} \left\|\widetilde{L}_N(1)\right\|^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}_{\infty,[a,b]}, \tag{79}$$

for $k = 1,...,m-1$; where $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$.

Based on Theorem 6 we have proved the following important Korovkin type convergence theorem.

Theorem 7.*All as in Theorem 6. Assume $\widetilde{L_N}(1) \xrightarrow{u} 1$, uniformly, as $N \to \infty$, and $\left(\widetilde{L_N}\left(|x - x_0|^{\alpha+1}\right)\right)(x_0) \xrightarrow{u} 0$, uniformly in $x_0 \in [a,b]$, as $N \to \infty$. Then $L_N(f) \xrightarrow{u} f$, uniformly, as $N \to \infty$, $\forall f \in C^m([a,b],X)$.*

Note: 1) From above notice that $\left\|\widetilde{L}_N(1)\right\|_{\infty,[a,b]}$ is bounded, so that everything works fine out of (77).

2) Theorem 7 also derives from Corollary 5, case of $0 < \alpha < 1$, $\forall f \in C^1([a,b],X)$.

4 Application

Here $[a,b] = [0,1]$.

Consider $g \in C([0,1])$ and the classic Bernstein polynomials

$$\left(\widetilde{B_N g}\right)(t) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad \forall t \in [0,1], N \in \mathbb{N}. \tag{80}$$

Let $x_0 \in [0,1]$ be fixed, then

$$\left(\widetilde{B_N g}\right)(x_0) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right)\binom{N}{k} x_0^k (1-x_0)^{N-k}. \tag{81}$$

We have that $\left(\widetilde{B_N}1\right) = 1$, and $\widetilde{B_N}$ are positive linear operators. The last means $\left(\widetilde{B_N}1\right)(x_0) = 1$.

Let $(X, \|\cdot\|)$ be a Banach space, and $f \in \widetilde{H}^{(1)}_{x_0}$; $r > 0$, $0 < \alpha < 1$.

We consider the vector valued in X Bernstein linear operators

$$(B_N f)(x_0) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} x_0^k (1-x_0)^{N-k}, \quad N \in \mathbb{N}. \tag{82}$$

That is $(B_N f)(x_0) \in X$.

By Corollary 4 we get

Corollary 6.*It holds*

$$\|(B_N f)(x_0) - f(x_0)\| \leq \frac{1}{\Gamma(\alpha+1)}\left[1 + \frac{1}{(\alpha+1)r}\right]$$

$$\left[\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\left(\widetilde{B_N}\left(|x-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[0,x_0]} + \right.$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f, r\left(\left(\widetilde{B_N}\left(|x-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,1]}\right]$$

$$\left(\left(\widetilde{B_N}\left(|x-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \quad \forall N \in \mathbb{N}. \tag{83}$$

Next let $\alpha = \frac{1}{2}$, $r = \frac{1}{\alpha+1}$, that is $r = \frac{2}{3}$. Notice $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.

Corollary 7. *It holds*

$$\|(B_N f)(x_0) - f(x_0)\| \leq$$
$$\frac{4}{\sqrt{\pi}}\left[\omega_1\left(D^{\frac{1}{2}}_{x_0-}f, \frac{2}{3}\left(\left(\widetilde{B_N}\left(|x-x_0|^{\frac{3}{2}}\right)\right)(x_0)\right)^{\frac{2}{3}}\right)_{[0,x_0]} + \right.$$
$$\left.\omega_1\left(D^{\frac{1}{2}}_{*x_0}f, \frac{2}{3}\left(\left(\widetilde{B_N}\left(|x-x_0|^{\frac{3}{2}}\right)\right)(x_0)\right)^{\frac{2}{3}}\right)_{[x_0,1]}\right]$$
$$\left(\left(\widetilde{B_N}\left(|x-x_0|^{\frac{3}{2}}\right)\right)(x_0)\right)^{\frac{1}{3}}, \quad \forall\, N \in \mathbb{N}. \tag{84}$$

We have that (see [3])

$$\left(\widetilde{B_N}\left(|x-x_0|^{\frac{3}{2}}\right)\right)(x_0) \leq \frac{1}{(4N)^{\frac{3}{4}}}, \quad \forall\, x_0 \in [0,1]. \tag{85}$$

We have proved

Corollary 8. *Here* $[a,b] = [0,1]$, $x_0 \in [0,1]$. *Let* $f \in \widetilde{H}^{(1)}_{x_0}$, $\alpha = \frac{1}{2}$, $N \in \mathbb{N}$. *Then*

$$\|(B_N f)(x_0) - f(x_0)\| \leq$$
$$\frac{2^{\frac{3}{2}}}{\sqrt{\pi}\sqrt[4]{N}}\left[\omega_1\left(D^{\frac{1}{2}}_{x_0-}f, \frac{1}{3\sqrt{N}}\right)_{[0,x_0]} + \omega_1\left(D^{\frac{1}{2}}_{*x_0}f, \frac{1}{3\sqrt{N}}\right)_{[x_0,1]}\right]. \tag{86}$$

Notice that $\frac{2^{\frac{3}{2}}}{\sqrt{\pi}} \approx 1.59$.

So as $N \to \infty$ *we conclude that* $(B_N f)(x_0) \overset{\|\cdot\|}{\to} f(x_0)$, *quantitatively, where* $x_0 \in [0,1]$.

We finish with

Corollary 9. *Let* $f \in C^1([0,1], X)$, $(X, \|\cdot\|)$ *is a Banach space. Then*

$$\|\|(B_N f)(x_0) - f(x_0)\|\|_{\infty,[0,1]} \leq$$
$$\frac{2^{\frac{3}{2}}}{\sqrt{\pi}\sqrt[4]{N}}\left[\sup_{x_0\in[0,1]}\omega_1\left(D^{\frac{1}{2}}_{x_0-}f, \frac{1}{3\sqrt{N}}\right)_{[0,x_0]} + \sup_{x_0\in[0,1]}\omega_1\left(D^{\frac{1}{2}}_{*x_0}f, \frac{1}{3\sqrt{N}}\right)_{[x_0,1]}\right], \tag{87}$$

$\forall\, N \in \mathbb{N}$.

So as $N \to \infty$, *we conclude that* $B_N f \to f$, *uniformly with rates.*

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