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On the Nature of Solutions of Neutral Differential Equations with Periodic Coefficients

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Abstract: By this work, we construct certain specific assumptions guaranteeing the asymptotic stability (AS) of trivial solution to a retarded linear neutral differential equation with periodic coefficients, and we estimate the decay rate of the solutions of the considered equation. To reach the desirable results, we benefit from a Lyapunov functional. We give an example to show applicability of the constructed assumptions and use MATLAB-Simulink to show the behaviors of the paths of the solutions of the considered equation.

Keywords: Neutral type differential equation, periodic coefficients, asymptotic stability, decay rate, Lyapunov functional

1 Introduction

When one checks the relative literature, it can be confirmed that the existence of many interesting results on the (AS) of the trivial solutions of differential equations with constant retardation (see [9,11,16] and [17]). A technique for finding estimates related to the solutions of the linear differential equation with constant retardation

$$\frac{d}{dt}y(t) = Ay(t) + By(t-\tau),$$

was offered in ([1,12,15,20]), where A and B are constant matrices and $\tau > 0$ is a constant retardation. Some similar results were established in [2] for the case where the entries of A and B are periodic functions of t with the common period. On the other hand, a lot of different similar investigations for certain nonlinear differential equations were proceeded in [19].

This paper is a continuation of the former works on the nature of solutions to functional differential equations with constant retardation (see [1]-[8], [13,18,21]).

In 2015, Skvortsova [22] considered the neutral linear differential equation with a variable retardation

$$\frac{d}{dt}[y(t) + D(t-\tau(t))] = Ay(t) + By(t-\tau(t)), \tag{1}$$

where $y \in \Re^n$, A, B and D are constant matrices and $\tau(t) \in C^1([0,\infty))$. The author described sufficient conditions for the (AS) of the trivial solution of Eq. (1) by using a Lyapunov functional like

$$\begin{aligned} V(t,y) =& \langle H(y(t) + D(y-\tau(t))), (y(t) + Dy(t-\tau(t)))\rangle \\ &+ \int_{t-\tau(t)}^{t} \langle K(t-s)y(s), y(s)\rangle ds, \end{aligned} \tag{2}$$

where $H = H^* > 0$, $K(s) = K^*(s) > 0$, $s \in [0, \tau_2]$.

In this paper, principally motivated by the ideas in Skvortsova [22] and that in [**?**], [**?**]), we consider the linear neutral differential equation with periodic coefficients and periodic variable retardation of the from

$$\frac{d}{dt}[y(t) + Dy(t-\tau(t))] = A(t)y(t) + B(t)y(t-\tau(t)), \tag{3}$$

where $t \geq 0, y \in \Re^n$, $t-\tau(t) \geq 0$, D is an $n \times n$-constant matrix, $A(t)$ and $B(t)$ are $n \times n$- matrices with continuous T-periodic entries, $(T > 0)$, that is,

$$A(t+T) \equiv A(t), \quad B(t+T) \equiv B(t)$$

and $\tau(t) \in C^1([0,\infty])$, $\tau(t)$ is T-periodic variable retardation, and it fulfills that

$$\begin{aligned} &\tau(t+T) = \tau(t), \\ &0 < \tau_1 \leq \tau(t) \leq \tau_2 < \infty \end{aligned} \tag{4}$$

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and

$$\tau'(t) \le \alpha < 1, \quad \alpha \in (0,1), \tag{5}$$

where τ_1, τ_2 and α are some positive constants.

The aim of this paper is to construct new sufficient conditions to ensure the (AS) stability of the trivial solution Eq. (3), and we estimate the decay rate of the solutions of Eq. (3). It is obvious that Eq. (1) is special cases of Eq. (3). In addition, instead of the constant matrices, we replace variable matrices with periodic entries. To come at the desirable results of this paper, we introduce an auxiliary Lyapunov functional and use it in the proofs. By the results of this paper, we improve the results of Skvortsova [22] from the case of the constant matrices to the case of variable periodic matrices and retardation. Besides, we do a contribution to the literature (see the references of [1]-[26]). These facts show the improvement and newness of the present paper.

To discuss the nature of solutions of Eq. (3), we describe an auxiliary functional by

$$\begin{aligned} V(0,\vartheta) =&\langle H(0)(\vartheta(0)+D\vartheta(-\tau(0))), \\ &(\vartheta(0)+D\vartheta(-\tau(0)))\rangle \\ &+\int_{-\tau(0)}^{0}\langle K(-s)\vartheta(s),\vartheta(s)\rangle ds, \\ &\vartheta(s)\in C[-\tau_2,0], \end{aligned} \tag{6}$$

where $H \in C^1([0,T])$ and $K \in C^1([0,\tau_2])$ are matrix valued functions such that

$$H(t) = H^*(t), \;\; H(t) = H(t+T) > 0, \;\; t \ge 0, \tag{7}$$

$$K(s) = K^*(s) > 0, \;\; \frac{d}{ds}K(s) < 0, \;\; s \in [0,\tau_2], \tag{8}$$

with H^* and K^* are the conjugate transposes of H and K, respectively, and, in addition, it is presumed that

$$C(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{12}^*(t) & C_{22}(t) \end{pmatrix} \tag{9}$$

is a positive definite matrix for all $t \in [0,T]$, where

$$\begin{aligned} C_{11}(t) &= -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - K(0), \\ C_{12}(t) &= -H(t)B(t) + H(t)A(t)D + K(0)D, \\ C_{22}(t) &= -D^*K(0)D + (1-\alpha)K(\tau_2). \end{aligned}$$

It is notable that C_{12}^* is the conjugate transpose of C_{12}. We consider the initial value problem (IVP)

$$\begin{aligned} &\frac{d}{dt}[y(t)+Dy(t-\tau(t))] = A(t)y(t)+B(t)y(t-\tau(t)), \; t \ge 0, \\ &y(t) = \vartheta(t), t \in [-\tau_2,0], \\ &y(0^+) = \vartheta(0), \end{aligned} \tag{10}$$

where $\vartheta(t) \in C^1([-\tau_2,0])$ is a given vector-valued function.

In this paper, it is assumed that $H(t)$ and $K(s)$ satisfy assumptions (7) and (8), respectively, and the symbol $\|D\|$ represents the spectral norm of the matrix D. We use the below notations for dot product and vector norm, respectively:

$$\langle x,y\rangle = \sum_{j=1}^{n} x_j\bar{y}_j, \;\; \|x\| = \sqrt{\langle x,x\rangle}.$$

We now state the definitions of the stability and asymptotic stability.

Definition 1. The trivial solution of (IVP) (10) is said Lyapunov stable if for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that $\max_{t\in[-\tau_2,0]}\|\vartheta(t)\| < \delta_0$ implies $\|y(t)\| < \varepsilon$ for all $t > 0$.

Definition 2. The trivial solution of (IVP) (10) is said (AS) if the trivial solution of (IVP) (10) is Lyapunov stable, and in addition there exists $\delta_0 > 0$ such that $\max_{t\in[-\tau_2,0]}\|\vartheta(t)\| < \delta_0$ implies $\|y(t)\| \to 0$ as $t \to \infty$.

Let

$$y_t : \theta \to y(t+\theta), \;\; \theta \in [-\tau_2,0].$$

Then, it follows from (6) that

$$\begin{aligned} V(.) &= \langle H(0)(y_t(0)+Dy_t(-\tau(0))),(y_t(0)+Dy_t(-\tau(0)))\rangle \\ &+\int_{-\tau(0)}^{0}\langle K(-\theta)y_t(\theta),y_t(\theta)\rangle d\theta \\ &= \langle H(t)(y(t)+Dy(t-\tau(t))),(y(t)+Dy(t-\tau(t)))\rangle \\ &+\int_{t-\tau(t)}^{t}\langle K(t-s)y(s),y(s)\rangle ds. \end{aligned} \tag{11}$$

The following lemma is needed for proof of the main result of this paper.

Lemma 1. We assume that $C(t)$ given by (9) is positive definite. Then the spectrum of the matrix D is contained in the disc $\{\lambda \in \mathbb{C} : |\lambda| < \sqrt{1-\alpha}\}$.

Proof. The proof of Lemma 1 is similar that of Skvortsova [22, Lemma 1]. Therefore, we leave it.

In view of (9) and Lemma 1, it follows that there exists a solution $\tilde{H}$ to the discrete Lyapunov matrix equation

$$\tilde{H} - \frac{1}{1-\alpha}D^*\tilde{H}D = I, \tag{12}$$

where I is the identity matrix. In addition, we have the counterpart of the Krein inequality

$$\|D^j\| \le \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}\rho^j, \;\; j \in N, \tag{13}$$

where

$$\rho = \sqrt{(1-\alpha)\left(1-\frac{1}{\|\tilde{H}\|}\right)} \tag{14}$$

(see Godunov [10]).

We benefit from this estimate for the proof of the principal result of this paper.

Let $c(t)$ be the minimal eigenvalue of $C(t)$ such that

$$c(t) > 0, C(t) \geq c(t)\begin{bmatrix} H(t) & 0 \\ 0 & 0 \end{bmatrix} \tag{15}$$

and by $k > 0$ the maximal number such that

$$\frac{d}{ds}K(s) + kK(s) \leq 0, s \in [0, \tau_2], \tag{16}$$

$$\varepsilon(t) = min\{c(t), k\}. \tag{17}$$

We denote the minimal eigenvalue of $H(t)$ by $h_{min}(t) > 0$. Let

$$\Phi = \max_{t\in[-\tau_2,0]} \|\vartheta(t)\|, \tag{18}$$

$$\mu = \max_{t\in[0,T]} \sqrt{\frac{V(0,\vartheta)}{h_{min}(t)}}, \tag{19}$$

$$\beta(t) = \varepsilon(t)/2, \beta^+ = \max_{t\in[0,T]} \beta(t), \beta^- = \min_{t\in[0,T]} \beta(t). \tag{20}$$

2 Nature of solutions

The main result of this paper is formulated by the following theorem.

Theorem 1. Let the assumption given by (9) holds. Then the following assertions are true:

1) If $\rho < e^{-\beta^+\tau_2}$, where ρ and β^+ are defined in (14) and (20), respectively, then the solution $y(t)$ of (IVP) (10) satisfies the estimate

$$\|y(t)\| \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu(1-\rho e^{\beta^+\tau_2})^{-1}e^{-\int_0^t \beta(s)ds} + \rho^{\max\{t/\tau_2,1\}}\Phi], \tag{21}$$

where $\tilde{H}, \Phi, \mu$ and $\beta(t)$ are defined by (12), (18), (19) and (20), respectively.

2) If $\rho = e^{-\beta^+\tau_2}$, then the solution $y(t)$ of (IVP) (10) satisfies the estimate

$$\|y(t)\| \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu(t/\tau_1+1)e^{-\int_0^t \beta(s)ds} + \rho^{\max\{t/\tau_2,1\}}\Phi]. \tag{22}$$

3) If $e^{-\beta^+\tau_2} < \rho < e^{-(\beta^+\tau_2-\beta^-\tau_1)}$, where β^- is defined in (20), then the solution $y(t)$ of (IVP) (10) satisfies the estimate

$$\|y(t)\| \leq \sqrt{\|\tilde{H}\|\|\tilde{H}^{-1}\|}[\mu(1-(\rho e^{\beta^+\tau_2})^{-1})^{-1} \times (\rho e^{\beta^+\tau_2-\beta^-\tau_1})^{t/\tau_1} + \rho^{\max\{t/\tau_2,1\}}\Phi]. \tag{23}$$

We now start with auxiliary assertions.

The next lemma is also needed in the proof of main result of this paper.

Lemma 2. Let assumption (9) holds. Then the solution $y(t)$ of (IVP) (10) satisfies the inequality

$$\|y(t)+Dy(t-\tau(t))\| \leq \sqrt{\frac{V(0,\vartheta)}{h_{\min}(t)}}e^{-\int_0^t \beta(s)ds}, t > 0, \tag{24}$$

where $V(0,\vartheta)$ and $\beta(t)$ are defined in (6) and (20), respectively, $h_{\min(t)} > 0$ is the minimal eigenvalue of the matrix $H(t)$.

Proof We follow the strategy proceeded in Demidenko and Matveeva [1]. Let $y(t)$ be a solution of (IVP) (10). Let $y(.) = y(t-\tau(t))$. Differentiating the functional $V(t,y_t)$ along the solutions of the equation in (10), one can obtain

$$\begin{aligned}
\frac{d}{dt}V(t,y_t) &= \langle \frac{d}{dt}H(t)(y(t)+Dy(.)), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)\frac{d}{dt}(y(t)+Dy(.)), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)(y(t)+Dy(.)), \frac{d}{dt}(y(t)+Dy(.))\rangle \\
&+ \langle K(0)y(t),y(t)\rangle - (1-\tau'(t))\langle K(\tau(t))y(.),y(.)\rangle \\
&+ \int_{t-\tau(t)}^t \langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds \\
&= \langle \frac{d}{dt}H(t)(y(t)+Dy(.)), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)(A(t)y(t)+B(t)y(.)), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)(y(t)+Dy(.)), (A(t)y(t)+B(t)y(.))\rangle \\
&+ \langle K(0)y(t),y(t)\rangle - (1-\tau'(t))\langle K(\tau(t))y(.),y(.)\rangle \\
&+ \int_{t-\tau(t)}^t \langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds \\
&= \langle \frac{d}{dt}H(t)(y(t)+Dy(.)), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)A(t)y(t), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)B(t)y(.), (y(t)+Dy(.))\rangle \\
&+ \langle H(t)(y(t)+Dy(.)), A(t)y(t)\rangle \\
&+ \langle H(t)(y(t)+Dy(.)), B(t)y(.)\rangle \\
&+ \langle K(0)y(t),y(t)\rangle - (1-\tau'(t))\langle K(\tau(t))y(.),y(.)\rangle \\
&+ \int_{t-\tau(t)}^t \langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds.
\end{aligned} \tag{25}$$

Let us consider the expression

$$(1-\tau'(t))\langle K(\tau(t))y(.),y(.)\rangle.$$

By (5) and the condition $K(s) = K^*(s) > 0$, $s \in [0, \tau_2]$, it is clear that

$$\begin{aligned}
&(1-\tau'(t))\langle K(\tau(t))y(.),y(.)\rangle \\
&\geq (1-\alpha)\langle K(\tau(t))y(.),y(.)\rangle.
\end{aligned}$$

Using the assumptions $\frac{d}{ds}K(s) < 0$ and $\tau(t) \leq \tau_2$, we have $K(\tau(t)) \geq K(\tau_2)$. Hence, it is obvious that

$$\begin{aligned}
&(1-\tau'(t))\langle K(\tau(t))y(.),y(.)\rangle \\
&\geq (1-\alpha)\langle K(\tau_2)y(.),y(.)\rangle.
\end{aligned}$$

By (25), we may have

$$\begin{aligned}\frac{d}{dt}V(t,y_t) \leq & \langle \frac{d}{dt}H(t)(y(t)+Dy(.)),(y(t)+Dy(.))\rangle \\ & + \langle H(t)A(t)(y(t)+Dy(.)),(y(t)+Dy(.))\rangle \\ & - \langle H(t)A(t)Dy(.),(y(t)+Dy(.))\rangle \\ & + \langle H(t)B(t)y(.),(y(t)+Dy(.))\rangle \\ & + \langle A^*(t)H(t)(y(t)+Dy(.)),(y(t)+Dy(.))\rangle \\ & - \langle A^*(t)H(t)(y(t)+Dy(.)),Dy(.)\rangle \\ & + \langle B^*(t)H(t)(y(t)+Dy(.)),y(.)\rangle \\ & + \langle K(0)(y(t)+Dy(.)),(y(t)+Dy(.))\rangle \\ & - \langle K(0)Dy(.),(y(t)+Dy(.))\rangle \\ & - \langle D^*K(0)(y(t)+Dy(.)),y(.)\rangle \\ & + \langle D^*K(0)Dy(.),y(.)\rangle \\ & - (1-\alpha)\langle K(\tau_2)y(.),y(.)\rangle \\ & + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds.\end{aligned}$$

Taking into account the matrix given by (9) and its positiveness, it is easy to obtain

$$\begin{aligned}\frac{d}{dt}V(t,y_t) + \left\langle C(t)\begin{pmatrix} y(t)+Dy(.) \\ y(.)\end{pmatrix}, \begin{pmatrix} y(t)+Dy(.) \\ y(.)\end{pmatrix}\right\rangle \\ - \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds \leq 0.\end{aligned}$$

Using (15) and (16), we find

$$\begin{aligned}\frac{d}{dt}V(t,y_t) + c(t)\langle H(t)(y(t)+Dy(.)),(y(t)+Dy(.))\rangle \\ + k\int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s),y(s)\rangle ds \leq 0.\end{aligned}$$

Taking into account (11) and (17), we can obtain

$$\frac{d}{dt}V(t,y_t) + \varepsilon(t)V(t,y_t) \leq 0$$

so that

$$V(t,y_t) \leq V(0,y_0)e^{-\int_0^t \varepsilon(s)ds}.$$

In addition, it can be arrived at $V(0,y_0) = V(0,\vartheta)$,

$$\langle H(t)(y(t)+Dy(.)),(y(t)+Dy(.))\rangle \leq V(t,y_t)$$

and

$$\begin{aligned}\|y(t)+Dy(.)\| \leq \sqrt{\frac{V(t,y_t)}{h_{\min}(t)}} \leq \sqrt{\frac{V(0,\vartheta)}{h_{\min}(t)}}e^{-\int_0^t \varepsilon(s)/2ds} \\ = \sqrt{\frac{V(0,\vartheta)}{h_{\min}(t)}}e^{-\int_0^t \beta(s)ds},\end{aligned}$$

by (20) and (6), respectively, where $h_{\min}(t) > 0$ is the minimal eigenvalue of $H(t)$. This is the desired result.

We now estimate $||y(t)||$. Let $t > 0$. Consider the functions

$$\begin{aligned}\gamma_0(t) &= t, \\ \gamma_1(t) &= t-\tau(t), \\ &\dots \\ \gamma_l(t) &= \gamma_{l-1}(t) - \tau(\gamma_{l-1}(t)), l \geq 1\end{aligned}$$

or, in the equivalent form,

$$\begin{aligned}&\gamma_0(t) = t, \\ &\gamma_1(t) = t-\tau(\gamma_0(t)), \\ &\dots \\ &\gamma_l(t) = t - \sum_{j=0}^{l-1}\tau(\gamma_j(t)), l \geq 1.\end{aligned} \tag{26}$$

Let $m \in N$ be the minimal number such that

$$\gamma_m(t) \in [-\tau_2, 0). \tag{27}$$

Lemma 3. Let $C(t)$ in (9) be positive definite. Then the solution $y(t)$ of (IVP)(10) satisfies the estimate

$$\begin{aligned}||y(t)|| \leq \sqrt{||\tilde{H}||||\tilde{H}^{-1}||}(\mu\sum_{j=0}^{m-1}(\rho e^{\beta^+\tau_2})^j e^{-\int_0^t \beta(s)ds} \\ + \rho^{max\{t/\tau_2,1\}}\Phi),\end{aligned} \tag{28}$$

where $\tilde{H}, \rho, \Phi, \mu,$ and β^+ and $\beta(t)$ are defined in (12), (14), (18), (19) and (20), respectively.

Proof. Using $\{\gamma_l(t)\}_{l\geq 1}$, we represent $y(t)$ as

$$\begin{aligned}y(t) = & [y(\gamma_0(t)) + Dy(\gamma_1(t))] - D[y(\gamma_1(t)) + Dy(\gamma_2(t))] \\ & + \ldots + (-1)^{m-1}D^{m-1}[y(\gamma_{m-1}(t)) + Dy(\gamma_m(t))] \\ & + (-1)^m D^m y(\gamma_m(t))\end{aligned}$$

which implies that

$$\begin{aligned}||y(t)|| \leq & ||y(\gamma_0(t)) + Dy(\gamma_1(t))|| + ||D||||y(\gamma_1(t)) + Dy(\gamma_2(t))|| \\ & + \ldots + ||D^{m-1}||||y(\gamma_{m-1}(t)) + Dy(\gamma_m(t))|| \\ & + ||D^m y(\gamma_m(t))|| \\ = & ||y(\gamma_0(t)) + Dy(\gamma_0(t) - \tau(\gamma_0(t)))|| \\ & + ||D||||y(\gamma_1(t)) + Dy(\gamma_1(t) - \tau(\gamma_1(t)))|| + \ldots \\ & + ||D^{m-1}||||y(\gamma_{m-1}(t)) + Dy(\gamma_{m-1}(t) - \tau(\gamma_{m-1}(t)))|| \\ & + ||D^m y(\gamma_m(t))||.\end{aligned}$$

It is now notable that

$$\begin{aligned}||y(t)|| \leq & \mu e^{-\int_0^{\gamma_0(t)}\beta(s)ds} + ||D||\mu e^{-\int_0^{\gamma_1(t)}\beta(s)ds} + \ldots \\ & + ||D^{m-1}||\mu e^{-\int_0^{\gamma_{m-1}(t)}\beta(s)ds} + ||D^m y(\gamma_m(t))|| \\ = & \sum_{j=0}^{m-1}||D^j||\mu e^{-\int_0^{\gamma_j(t)}\beta(s)ds} + ||D^m y(\gamma_m(t))||\end{aligned}$$

by (19) and (24).

In view of (13), it may be followed that

$$\begin{aligned}||y(t)|| &\leq \sqrt{||\tilde{H}||||\tilde{H}^{-1}||}\\ &\times [\mu \sum_{j=0}^{m-1} \rho^j e^{-\int_0^{\gamma_j(t)} \beta(s)ds} + \rho^m \Phi]\\ &= \sqrt{||\tilde{H}||||\tilde{H}^{-1}||}\\ &\times [\mu \sum_{j=0}^{m-1} \rho^j e^{\int_{\gamma_j(t)}^t \beta(s)ds} e^{-\int_0^t \beta(s)ds} + \rho^m \Phi].\end{aligned}$$

We note that (26) and (4) imply that $\gamma_j(t) \geq t - j\tau_2$. In particular, by (27), $0 > \gamma_m(t) \geq t - m\tau_2$, which implies $m > t/\tau_2$. Hence, we can reach that

$$\begin{aligned}||y(t)|| &\leq \sqrt{||\tilde{H}||||\tilde{H}^{-1}||}\\ &\times [\mu \sum_{j=0}^{m-1} \rho^j e^{\beta^+(t-\gamma_j(t))} e^{-\int_0^t \beta(s)ds} + \rho^{max\{t/\tau_2,1\}} \Phi]\\ &\leq \sqrt{||\tilde{H}||||\tilde{H}^{-1}||}\\ &\times [\mu \sum_{j=0}^{m-1} \rho^j e^{\beta^+ j\tau_2} e^{-\int_0^t \beta(s)ds} + \rho^{max\{t/\tau_2,1\}} \Phi].\end{aligned}$$

Using inequality (28), it is easy to prove estimates given by (21)-(23). Indeed, let $\rho < e^{-\beta^+\tau_2}$. By the estimate

$$\sum_{j=0}^{m-1} (\rho e^{\beta^+\tau_2})^j \leq \sum_{j=0}^{\infty} (\rho e^{\beta^+\tau_2})^j = (1-\rho e^{\beta^+\tau_2})^{-1}$$

and inequality (28), we can arrive at inequality (21).

Let $\rho = e^{-\beta^+\tau_2}$. By (27), we have $\gamma_{m-1}(t) \geq 0$. Moreover, by (26) and (4), we get $\gamma_{m-1}(t) \leq t-(m-1)\tau_1$, which implies

$$m \leq \frac{t}{\tau_1} + 1. \tag{29}$$

Hence,

$$\sum_{j=0}^{m-1} (\rho e^{\beta^+\tau_2})^j = m \leq \frac{t}{\tau_1} + 1.$$

The former equality and inequality (28) imply (22).

Finally, let $e^{-\beta^+\tau_2} < \rho < e^{-(\beta^+\tau_2 - \beta^-\tau_1)}$. By (29), we have

$$\begin{aligned}\sum_{j=0}^{m-1} (\rho e^{\beta^+\tau_2})^j &= \sum_{j=0}^{m-1} (\rho e^{\beta^+\tau_2})^{m-1-j} \leq \sum_{j=0}^{m-1} (\rho e^{\beta^+\tau_2})^{t/\tau_1 - j}\\ &\leq \sum_{j=0}^{\infty} (\rho e^{\beta^+\tau_2})^{-j} (\rho e^{\beta^+\tau_2})^{t/\tau_1}\\ &= (1-(\rho e^{\beta^+\tau_2})^{-1})^{-1} (\rho e^{\beta^+\tau_2})^{t/\tau_1}.\end{aligned}$$

By (20) and inequality (28), we have

$$\begin{aligned}y(t) &\leq \sqrt{\|\tilde{H}\|\tilde{H}^{-1}\|}\|[\mu(1-(\rho e^{\beta^+\tau_2})^{-1})^{-1}\\ &\times (\rho e^{\beta^+\tau_2 - \beta^-\tau_1})^{t/\tau_1} + \rho^{\max\{t/\tau_2,1\}} \Phi].\end{aligned}$$

Hence, we obtain inequality (23). This completes the proof of Theorem 1.

Example. For $n = 2$ as a special case of Eq. (3), we consider the following a time-varying delay system of linear neutral differential equations with periodic coefficients

$$\begin{aligned}&\frac{d}{dt}\left(\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 0.1 & 0.14 \\ -0.03 & 0.12 \end{bmatrix} \times \begin{bmatrix} y_1(t - \frac{1+\sin^2(t)}{10}) \\ y_2(t - \frac{1+\sin^2(t)}{10}) \end{bmatrix}\right)\\ &= \begin{bmatrix} -8+0.2\cos t & 1-0.4\cos t \\ 2+0.3\cos t & -16-0.1\cos t \end{bmatrix} \times \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}\\ &+ \begin{bmatrix} 0.4\sin t & 0 \\ -0.5\sin t & 0.2\cos t \end{bmatrix} \times \begin{bmatrix} y_1(t - \frac{1+\sin^2(t)}{10}) \\ y_2(t - \frac{1+\sin^2(t)}{10}) \end{bmatrix}\end{aligned} \tag{30}$$

for $t > 0$. When we compare Eq. (30) with Eq. (3), it can be seen that

$$D = \begin{pmatrix} 0.1 & 0.14 \\ -0.03 & 0.12 \end{pmatrix}, \quad A(t) = \begin{pmatrix} -8+0.2\cos t & 1-0.4\cos t \\ 2+0.3\cos t & -16-0.1\cos t \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0.4\sin t & 0 \\ -0.5\sin t & 0.2\cos t \end{pmatrix}$$

and

$$\tau_1 = \frac{1}{10} \leq \tau(t) = \frac{1+\sin^2 t}{10} \leq \frac{1}{5} = \tau_2.$$

In addition, it can be followed that

$$H(t) = \begin{pmatrix} 4-0.2\sin t & 1-1.2\sin t \\ 1-1.2\sin t & 6+3.2\sin t \end{pmatrix}$$

and

$$K(s) = e^{-ks}K_0, \quad k = 0.07, \quad K_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Let $h_{\min}(t) > 0$ be the minimal eigenvalue of the matrix $H(t)$. Hence, it is notable that

$$1.2 \leq h_{\min}(t) \leq 3.86, \quad 5.8 \leq \|H(t)\| \leq 9.2.$$

Therefore, for the former particular choices, one can easily check that the matrix $C(t)$ is positive definite for all $t \in [0, 2\pi]$ and the minimal eigenvalue $c_{\min}(t)$ of the matrix $C(t)$ satisfies $c_{\min}(t) \geq 0.0945$ by MATLAB-Simulink. Finally, we have

$$\varepsilon(t) = \min\{c(t), k\} = k, \ \beta(t) = \beta^+ = \beta^- = \frac{k}{2} = 0.035$$

and

$$\rho < e^{-\beta^+\tau_2}$$

so that

$$\|y(t)\| \leq r \max_{-\tau_2 \leq s \leq 0} \|y(s)\| e^{-0.035t}, \; t \geq 0$$

for a proper positive constant r.

As a result, it is seen that all the assumptions of Theorem 1 can be held.

Let

$$\tau(t) = \frac{1+\sin^2(t)}{10}, \, t > 0.$$

Benefited from by MATLAB-Simulink, the desired result for the behaviors of the orbits of solutions of the considered differential system is shown by the following graph.

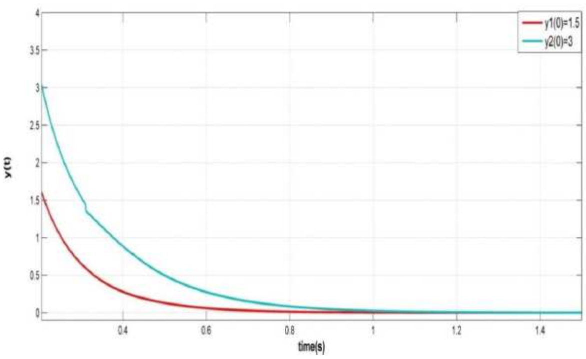

Fig. 1: Trajectories of solutions $y(t)$ of system (30) when $\tau(t) = \frac{1+\sin^2(t)}{10}$, $t > 0$.

Remark 1. For the choice of $\tau(t) = \frac{1+\sin^2(t)}{10}$, $t > 0$, it is obvious that the solution $y(t)$ of Eq. (30) is asymptotic stable.

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