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On a Functional Equation Arising from a Network Model

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Abstract: Recently, modeling many systems in communication and networks produce difference equations characterizing the dynamics of such systems. A certain class of functional equations arises from such difference equations. The functional equation of our interest arises from a queueing model for a gateway linking two Ethernet-type local area networks. It stems from a second order difference equation with boundary conditions reflecting the dynamics of the gateway. The functional equation is not yet solved to find the exact system distribution. In this article, on one hand we investigate the possible singularities of the unknowns of the two-place functional equation. On the other hand we introduce some application of computing the possible singularities, and we use the generating functions of the system distribution to compute some expectations of interest. It is hoped that computing the possible singularities of the unknowns of such equation will be a step forward in the road towards a general solution theory for this interesting class of equations.

Keywords: functional equations, difference equations, queueing theory, singularity.

1 Introduction

Functional equations (FEs) are defined as the equations where the unknowns are functions rather than variables [1, 2, 3, 4, 5]. They are a relatively old subject of mathematics, but their theory has flourished principally through the work of the prolific mathematician J. Aczél [5] who identified many of their classes, illustrating efficient methods for their solutions as well as criteria for the existence and uniqueness of those solutions [6]. FEs arise abundantly in models of many fields [7], such as population ethics [8], astronomy [9], neural networks [10], and wireless networks [11]. Specifically, each of these models can be formulated so as to eventually lead to a FE that can yield precise quantitative relationships.

FEs can be in one variable or two variables, depending on whether the underlying model is one-dimensional or two dimensional. Specifically, there is no universal solution technique for these FEs, but rather almost each equation is solved in another way.

Figure 1 summarizes the process between many communication and networks systems on one hand and a certain class of two-place functional equations on the other hand: Computer scientists who are interested in some communication elements like switches, and they want to study some performance measures like the delay or the waiting time. They start by describing their system mathematically to obtain the mathematical model which in most cases is consisting of a difference equation characterizing the dynamics of the underlying systems together with some boundary conditions. They used a probability generating function (PGF) to map such difference equations to challenging functional equations.

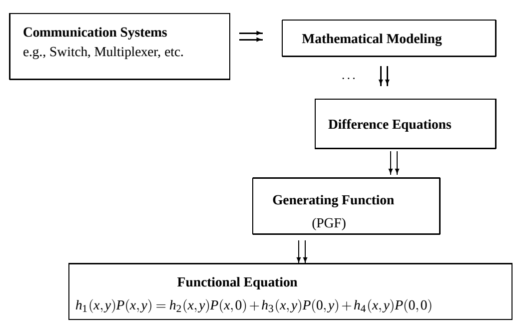

Fig. 1: Communication Systems and functional equations–Relation Summary

In particular, we find out that the following general class of two-place functional equations

$$h_1(x,y)P(x,y) = h_2(x,y)P(x,0) + h_3(x,y)P(0,y) \\ +h_4(x,y)P(0,0), \tag{1}$$

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where $h_i(x,y), i = 1,2,3,4$, are given polynomials in two complex variables x,y, arises from different communication and networks systems. Special cases of (1) arise from: a 2×2 switch see [12], a multiplexer see [13], the gateway see [14], a tandem queueing model with coupled processors see [15], two coupled processors see [16], arise in the analysis of stochastic models see [17], from two parallel queues created by arrivals with two demands [18], from a queueing model [27], from two processors [26], from two processors with coupled inputs [35], and from databases [20].

The current article is mainly concerned with locating the possible singularities of the unknowns of a two-place FE arising from two-queue model of a network queueing system originally published in [14]. The sequel of the paper is laid as follows: In the next section we recall the functional equation from the original article [14], and we introduce the main idea used to find the intersection points between two polynomials to be used in computing the possible singularities. In section 3 we find the potential singularities of the first unknown $P(x,0)$, in section 4 we find the potential singularities of the second unknown $P(0,y)$, in section 5 we discuss the singularities obtained and we give an example in which the marginal distribution cannot be calculated. In section 6 we give pointers to one possible application of computing the possible singularities, in section 7 we compute some expectations, and in section 8 we conclude our work.

2 The functional equation

In [14], the difference equation characterizing the dynamics of the gateway in steady state together with the boundary conditions are given by

$$\left(1-\overline{\xi_1}\,\overline{\xi_2}\right) p_{0,0} - \overline{\xi_1}\overline{r_2}p_{0,0} - \overline{r_1}\,\overline{r_2}p_{1,1} - \overline{r_1}\overline{\xi_2}p_{0,1} = 0 \tag{2}$$

$$\begin{aligned}(1-\overline{\xi_1}r_2\overline{s_2}-\xi_1\overline{r_2})p_{1,0}-\xi_1\overline{\xi_2}p_{0,0}-\overline{\xi_1}\overline{r_2}p_{2,0}\\ -\overline{r_1}\,\overline{r_2}p_{2,1}-\overline{r_1}r_2\overline{s_2}p_{1,1}=0\end{aligned} \tag{3}$$

$$\begin{aligned}(1-r_1\overline{s_1}r_2\overline{s_2}-\xi_1\overline{r_2}-\overline{r_1}\xi_2)p_{1,1}-\xi_1\xi_2p_{0,0}\\ -\overline{\xi_1}\xi_2p_{1,0}-r_1\overline{s_1}\overline{r_2}p_{2,1}-\overline{r_1}\overline{r_2}p_{2,2}\\ -\overline{r_1}r_2\overline{s_2}p_{1,2}-\xi_1\overline{\xi_2}p_{0,1}=0\end{aligned} \tag{4}$$

$$\begin{aligned}(1-r_1\overline{s_1}\xi_2-\overline{r_1}\xi_1)p_{0,1}-\overline{\xi_1}\xi_2p_{0,0}-\overline{r_1}\overline{\xi_2}p_{0,2}\\ -\overline{r_1}\,\overline{r_2}p_{1,2}-r_1\overline{s_1}\,\overline{r_2}p_{1,1}=0\end{aligned} \tag{5}$$

$$\begin{aligned}(1-\overline{\xi_1}r_2\overline{s_2}-\xi_1\overline{r_2})p_{m,0}-\xi_1r_2\overline{s_2}p_{m-1,0}\\ -\overline{\xi_1}\overline{r_2}p_{m+1,0}-\overline{r_1}\,\overline{r_2}p_{m+1,1}-\overline{r_1}r_2\overline{s_2}p_{m,1}=0,\\ m=2,3,\cdots\end{aligned} \tag{6}$$

$$\begin{aligned}(1-r_1\overline{s_1}r_2\overline{s_2}-\xi_1\overline{r_2}-\overline{r_1}\xi_2)p_{m,1}-\xi_1\xi_2p_{m-1,0}\\ -\overline{\xi_1}\xi_2p_{m,0}-r_1\overline{s_1\overline{r_2}}p_{m+1,1}-\overline{r_1\overline{r_2}}p_{m+1,2}\\ -\overline{r_1}r_2\overline{s_2}p_{m,2}-\xi_1r_2\overline{s_2}p_{m-1,1}=0,\\ m=2,3,\cdots\end{aligned} \tag{7}$$

$$\begin{aligned}(1-r_1\overline{s_1}r_2\overline{s_2}-\xi_1\overline{r_2}-\overline{r_1}\xi_2)p_{m,n}-\xi_1\xi_2p_{m-1,n-1}\\ -r_1\overline{s_1}\xi_2p_{m,n-1}-r_1\overline{s_1\overline{r_2}}p_{m+1,n}-\overline{r_1\overline{r_2}}p_{m+1,n+1}\\ -\overline{r_1}r_2\overline{s_2}p_{m,n+1}-\xi_1r_2\overline{s_2}p_{m-1,n}=0,\\ m,n=2,3,\cdots\end{aligned} \tag{8}$$

$$\begin{aligned}(1-r_1\overline{s_1}r_2\overline{s_2}-\xi_1\overline{r_2}-\overline{r_1}\xi_2)p_{1,n}-\xi_1\xi_2p_{0,n-1}\\ -r_1\overline{s_1}\xi_2p_{1,n-1}-r_1\overline{s_1\overline{r_2}}p_{2,n}-\overline{r_1\overline{r_2}}p_{2,n+1}\\ -\overline{r_1}r_2\overline{s_2}p_{1,n+1}-\xi_1\overline{\xi_2}p_{0,n}=0,\\ n=2,3,\cdots\end{aligned} \tag{9}$$

$$\begin{aligned}(1-r_1\overline{s_1}\xi_2-\overline{r_1}\xi_1)p_{0,n}-r_1\overline{s_1}\xi_2p_{0,n-1}\\ -\overline{r_1}\overline{\xi_2}p_{0,n+1}-\overline{r_1\overline{r_2}}p_{1,n+1}-r_1\overline{s_1\overline{r_2}}p_{1,n}=0,\\ n=2,3,\cdots\end{aligned} \tag{10}$$

where $0 < r_i, s_i, \xi_i = r_is_i < 1, i = 1,2$ are some operational parameters, $\overline{w} = 1-w$, for $w \in [0,1]$, see [14] for more details. Applying the generating function approach see e.g., [21] to (2)–(10), the author in [14] ends up after some nontrivial manipulation with the following challenging two-place functional equation

$$\begin{aligned}(M(x,y)-xy)P(x,y) = (1-y)(M(x,0)\\ +\overline{r_1}\xi_2xy)P(x,0)+(1-x)(M(0,y)\\ +\overline{r_2}\xi_1xy)P(0,y)-(1-x)(1-y)\\ M(0,0)P(0,0),\end{aligned} \tag{11}$$

where

$$P(x,y) = \sum_{m,n=0}^{\infty} p_{m,n}x^my^n, \quad |x| \le 1, |y| \le 1$$

is the probability generating function (PGF) of the sequence $p_{m,n}$, which is defined in [14],

$$P(x,0) = \sum_{m=0}^{\infty} p_{m,0}x^m, \quad |x| \le 1 \tag{12}$$

is the generating function of the sequence $p_{m,0}$,

$$P(0,y) = \sum_{n=0}^{\infty} p_{0,n}y^n, \quad |y| \le 1$$

is the generating function of the sequence $p_{0,n}$,

$$P(0,0) = p_{0,0},$$

is the probability that the gateway is empty, and

$$M(x,y) = (\overline{r_1} + r_1\overline{s_1}y + \xi_1 xy)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 xy).$$

For simplicity reasons (11) can be rewritten as follows

$$h_1(x,y)P(x,y) = h_2(x,y)P(x,0) + h_3(x,y)P(0,y) + h_4(x,y)P(0,0), \qquad (13)$$

where

$$h_1(x,y) = (\overline{r_1} + r_1\overline{s_1}y + \xi_1 xy)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 xy) - xy,$$

$$h_2(x,y) = (1-y)\overline{r_1}(\overline{r_2} + r_2\overline{s_2}x + \xi_2 xy),$$
$$h_3(x,y) = (1-x)\overline{r_2}(\overline{r_1} + r_1\overline{s_1}y + \xi_1 xy),$$

and

$$h_4(x,y) = (x-1)(1-y)\overline{r_1 r_2}.$$

Equation (13) is a functional equation in which one main unknown function $P(x,y)$ is defined by three other unknowns namely, $P(x,0), P(0,y)$, $P(0,0)$, and four known polynomials $h_i(x,y), i = 1,\ldots,4$. The author in [14] leaves the equation unsolved, and hence the current article is seen as a step in the way to solve it. Equation (13) cannot be solved directly for $P(x,y)$ because it contains other unknown functions namely $P(x,0)$, $P(0,y)$, and $P(0,0)$. In the next subsection we describe the main idea used in computing the singularities of the unknowns.

2.1 The main idea

The main idea of locating the possible singularities of the unknowns of the functional equation (13) will be merely to find the intersection points between two functions. So that it is reasonable to introduce first the idea of computing the intersection points between two general polynomials. Generally speaking, when we have two polynomials in two variables, say,

$$f(x,y) = a_0(y) + a_1(y)x + \cdots + a_{n_1}(y)x^{n_1}, \qquad (14)$$

and

$$g(x,y) = b_0(y) + b_1(y)x + \cdots + b_{n_2}(y)x^{n_2}, \qquad (15)$$

the resultant $Res_x(f,g;y)$ of the polynomials f and g (see e.g. [22] in Appendix B) with respect to x is the determinant of the matrix

$$\left.\begin{pmatrix} a_{n_1} & a_{n_1-1} & \cdots & a_0 & 0 & \cdots & \cdots \\ 0 & a_{n_1} & a_{n_1-1} & \cdots & a_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & a_{n_1} & a_{n_1-1} & \cdots & a_0 \\ b_{n_2} & b_{n_2-1} & \cdots & b_0 & 0 & \cdots & \cdots \\ 0 & b_{n_2} & b_{n_2-1} & \cdots & b_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & b_{n_2} & b_{n_2-1} & \cdots & b_0 \end{pmatrix}\begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n_2 - rows \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n_1 - rows \end{matrix}\right. \qquad (16)$$

which is a polynomial in y. The resultant with respect to x is 0 at y_0 if the polynomials f and g have a common nontrivial root (x_0, y_0) or the leading coefficients are zero. During the next two sections we will use the idea (16) in computing the potential singularities of the unknowns of the functional equation (13). It should be noted that the most important thing is to check wether the obtained potential singularities represent removable singularities or not. For this reason a somehow detailed analysis of the potential singularities is introduced in section 5.

3 Potential singularity of $P(x,0)$

Since the main PGF $P(x,y)$ in (13) is an analytic function in the unit disk, then this means that if

$$h_1(x,y) = 0,$$

then also

$$h_2(x,y)P(x,0) + h_3(x,y)P(0,y) + h_4(x,y)P(0,0) = 0,$$

so

$$P(x,0) + \frac{h_3(x,y)}{h_2(x,y)}P(0,y) + \frac{h_4(x,y)}{h_2(x,y)}P(0,0) = 0,$$

which is equivalent to

$$P(x,0) = -\frac{h_3(x,y)P(0,y) + h_4(x,y)P(0,0)}{h_2(x,y)}. \qquad (17)$$

We observe from the above equation that the unknown function $P(x,0)$ could have a potential singularity at some point x if for some y, $h_1(x,y) = 0$ and $h_2(x,y) = 0$. This leads us to study the intersection points of the curves $h_1(x,y) = 0$ and $h_2(x,y) = 0$. If such a point x exists, then y is a root of the resultant $\mathrm{Res}_x(h_1,h_2;y)$ in x of the polynomials $h_1(x,y)$ and $h_2(x,y)$. It should be noted that computing the potential singularities of the function $P(x,0)$ might be used to estimate the sequence $p_{m,0}$ defined by $P(x,0)$ in (12) using the theorem below.

3.1 Resultant in x

The two functions $h_1(x,y)$ and $h_2(x,y)$ can be written as

$$h_1(x,y) = (\tilde{r}_1 + r_1\tilde{s}_1 y + \xi_1 xy)(\tilde{r}_2 + r_2\tilde{s}_2 x + \xi_2 xy) - xy = a_0(y) + a_1(y)x + a_2(y)x^2, \qquad (18)$$

and

$$h_2(x,y) = (1-y)\tilde{r}_1(\tilde{r}_2 + r_2\tilde{s}_2 x + \xi_2 xy) = b_0(y) + b_1(y)x, \qquad (19)$$

where

$$a_2(y) = \xi_2\xi_1 y^2 + r_2\tilde{s}_2\xi_1 y,$$
$$a_1(y) = \xi_2 r_1\tilde{s}_1 y^2 + \tilde{r}_1\xi_2 y + r_2\tilde{s}_2 r_1\tilde{s}_1 y + \tilde{r}_2\xi_1 y - y + \tilde{r}_1 r_2\tilde{s}_2,$$
$$a_0(y) = \tilde{r}_2 r_1\tilde{s}_1 y + \tilde{r}_1\tilde{r}_2,$$
$$b_1(y) = \tilde{r}_1 r_2\tilde{s}_2 + \tilde{r}_1\xi_2 y - \tilde{r}_1 r_2\tilde{s}_2 y - \tilde{r}_1\xi_2 y^2,$$

and

$$b_0(y) = \tilde{r}_1\tilde{r}_2 - \tilde{r}_1\tilde{r}_2 y.$$

Using (16), (18), (19) we can write the resultant in x of $h_1(x,y)$ and $h_2(x,y)$ as the determinant of the matrix,

$$M_1 = \begin{pmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & 0 \\ 0 & b_1 & b_0 \end{pmatrix}. \tag{20}$$

From (16) and (20), the resultant of $h_1(x,y)$ and $h_2(x,y)$ in x as a function in y is given by

$$\begin{aligned} Res_x(h_1,h_2;y) &= \det(M_1) \\ &= b_1^2 a_0 + b_0^2 a_2 - a_1 b_1 b_0, \end{aligned} \tag{21}$$

after some algebraic manipulations we get

$$\begin{aligned} Res_x(h_1,h_2;y) &= \tilde{r}_1^2 \tilde{r}_2 r_2 y(y-1) \\ &\{s_2 y^2 + (1-2s_2)y + s_2 - 1\}. \end{aligned} \tag{22}$$

We conclude that the resultant in x of the functions $h_1(x,y)$ and $h_2(x,y)$ as a function of y is zero at the zeros of (22) which are

1. $y_1 = 0$
2. $y_2 = 1$
3. $y_3 = 1$
4. $y_4 = 1 - \frac{1}{s_2}$.

3.2 Resultant in y

In order to find the resultant in y of the two functions $h_1(x,y)$ and $h_2(x,y)$ we start by rewriting them as functions in y with some coefficients in x. The two functions $h_1(x,y)$ and $h_2(x,y)$ can be written as

$$\begin{aligned} h_1(x,y) &= (\tilde{r}_1 + r_1\tilde{s}_1 y + \xi_1 xy)(\tilde{r}_2 + r_2\tilde{s}_2 x + \xi_2 xy) - xy \\ &= c_0(x) + c_1(x)y + c_2(x)y^2, \end{aligned} \tag{23}$$

and

$$\begin{aligned} h_2(x,y) &= (1-y)\tilde{r}_1(\tilde{r}_2 + r_2\tilde{s}_2 x + \xi_2 xy) \\ &= d_0(x) + d_1(x)y + d_2(x)y^2, \end{aligned} \tag{24}$$

where

$$c_2(x) = \xi_2\xi_1 x^2 + \xi_2 r_1 \tilde{s}_1 x,$$

$$c_1(x) = r_2\tilde{s}_2\xi_1 x^2 + \tilde{r}_1\xi_2 x + r_2\tilde{s}_2 r_1\tilde{s}_1 x + \tilde{r}_2\xi_1 x - x + \tilde{r}_2 r_1\tilde{s}_1,$$

$$c_0(x) = \tilde{r}_1 r_2\tilde{s}_2 x + \tilde{r}_1\tilde{r}_2,$$

$$d_2(x) = -\tilde{r}_1\xi_2 x,$$

$$d_1(x) = \tilde{r}_1\xi_2 x - \tilde{r}_1\tilde{r}_2 - \tilde{r}_1 r_2\tilde{s}_2 x,$$

and

$$d_0(x) = \tilde{r}_1\tilde{r}_2 + \tilde{r}_1 r_2\tilde{s}_2 x.$$

The resultant in y of $h_1(x,y)$ and $h_2(x,y)$ is using (16), (23), (24) the determinant of the matrix

$$M_2 = \begin{pmatrix} c_2 & c_1 & c_0 & 0 \\ 0 & c_2 & c_1 & c_0 \\ d_2 & d_1 & d_0 & 0 \\ 0 & d_2 & d_1 & d_0 \end{pmatrix}. \tag{25}$$

From (25) after some algebraic manipulations we can write that

$$\begin{aligned} Res_y(h_1,h_2;x) &= \det(M_2), \\ &= \xi_2\tilde{r}_1 x^2(\tilde{r}_1 r_2\tilde{s}_2 x + \tilde{r}_1\tilde{r}_2) \\ &\quad \{\xi_1 r_2 x^2 + (\tilde{\xi}_1 r_2 + \xi_1\tilde{r}_2 - 1)x + \tilde{r}_2\tilde{\xi}_1\}. \end{aligned} \tag{26}$$

We conclude that the resultant in y of the functions $h_1(x,y)$ and $h_2(x,y)$ as a function of x is zero at the zeros of (26) which are:

1. $x_1 = 0$,
2. $x_2 = 0$,
3. $x_3 = -\frac{\tilde{r}_2}{r_2\tilde{s}_2}$,
4. $x_4 = 1$,
5. $x_5 = \frac{1-\xi_1-r_2+\xi_1 r_2}{r_2\tilde{\xi}_1}$,

the obtained values are possible singularities of the function $P(x,0)$.

4 Potential singularity of $P(0,y)$

Since in the main functional equation in (13) of this paper we have two unknown functions namely, $P(x,0)$ and $P(0,y)$ then it is reasonable also to investigate the potential singularities of the other unknown $P(0,y)$. Since the main PGF $P(x,y)$ in (13) is an analytic function in the unit disk, then this means that if

$$h_1(x,y) = 0,$$

then also

$$h_2(x,y)P(x,0) + h_3(x,y)P(0,y) + h_4(x,y)P(0,0) = 0,$$

so that

$$P(0,y) + \frac{h_2(x,y)}{h_3(x,y)}P(x,0) + \frac{h_4(x,y)}{h_3(x,y)}P(0,0) = 0,$$

which is equivalent to

$$P(0,y) = -\frac{h_2(x,y)P(x,0) + h_4(x,y)P(0,0)}{h_3(x,y)}.$$

We observe from the above equation that the unknown function $P(0,y)$ could have a singularity at some point y if for some x, $h_1(x,y) = 0$ and $h_3(x,y) = 0$. This leads us to study the intersection points between the two functions $h_1(x,y)$ and $h_3(x,y)$.

4.1 Resultant in y

The two functions $h_1(x,y)$ and $h_3(x,y)$ can be written as

$$\begin{aligned} h_1(x,y) &= (\tilde{r}_1 + r_1\tilde{s}_1 y + \xi_1 xy)(\tilde{r}_2 + r_2\tilde{s}_2 x + \xi_2 xy) - xy \\ &= a_0(x) + a_1(x)y + a_2(x)y^2, \end{aligned}$$

and

$$\begin{aligned} h_3(x,y) &= (1-x)\tilde{r}_2(\tilde{r}_1 + r_1\tilde{s}_1 y + \xi_1 xy), \\ &= b_0(x) + b_1(x)y. \end{aligned}$$

Using (16), the resultant of $h_1(x,y)$ and $h_3(x,y)$ in y as a function in x is the determinant of the matrix,

$$M_3 = \begin{pmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & 0 \\ 0 & b_1 & b_0 \end{pmatrix} \tag{27}$$

where

$$a_2(x) = \xi_1\xi_2 x^2 + r_1\tilde{s}_1\xi_2 x,$$

$$a_1(x) = r_2\tilde{s}_2\xi_1 x^2 + \tilde{r}_1\xi_2 x + r_2\tilde{s}_2 r_1\tilde{s}_1 x + \tilde{r}_2\xi_1 x - x + \tilde{r}_2 r_1\tilde{s}_1,$$

$$a_0(x) = \tilde{r}_1 r_2\tilde{s}_2 x + \tilde{r}_1\tilde{r}_2,$$

$$b_1(x) = r_1\tilde{r}_2\tilde{s}_1 + \tilde{r}_2\xi_1 x - r_1\tilde{r}_2\tilde{s}_1 x - \tilde{r}_2\xi_1 x^2,$$

and

$$b_0(x) = \tilde{r}_1\tilde{r}_2 - \tilde{r}_1\tilde{r}_2 x.$$

From (27) we can write that

$$\begin{aligned} &Res_y(C_1,C_3;x) = \det(M_3) \\ &= x(x-1)\tilde{r}_2^2\tilde{r}_1 r_1\{s_1x^2 + (1-2s_1)x + s_1 - 1\}, \end{aligned}$$

so we conclude that the resultant in y of the functions $h_1(x,y)$ and $h_3(x,y)$ as a function of x is zero at

1. $x_1 = 0$
2. $x_2 = 1$
3. $x_3 = 1$
4. $x_4 = 1 - \frac{1}{s_1}$.

4.2 Resultant in x

The two functions $h_1(x,y)$ and $h_3(x,y)$ can also be written as follows

$$\begin{aligned} h_1(x,y) &= (\tilde{r}_1 + r_1\tilde{s}_1 y + \xi_1 xy)(\tilde{r}_2 + r_2\tilde{s}_2 x + \xi_2 xy) - xy \\ &= c_0(y) + c_1(y)x + c_2(y)x^2, \end{aligned}$$

and

$$\begin{aligned} h_3(x,y) &= (1-x)\tilde{r}_2(\tilde{r}_1 + r_1\tilde{s}_1 y + \xi_1 xy), \\ &= d_0(y) + d_1(y)x + d_2(y)x^2. \end{aligned}$$

Using (16), the resultant of $h_1(x,y)$ and $h_3(x,y)$ in x as a function in y is the determinant of the matrix,

$$M_4 = \begin{pmatrix} c_2 & c_1 & c_0 & 0 \\ 0 & c_2 & c_1 & c_0 \\ d_2 & d_1 & d_0 & 0 \\ 0 & d_2 & d_1 & d_0 \end{pmatrix} \tag{28}$$

where

$$c_2(y) = \xi_1\xi_2 y^2 + r_2\tilde{s}_2\xi_1 y,$$

$$c_1(y) = r_1\tilde{s}_1\xi_2 y^2 + \tilde{r}_1\xi_2 y + r_2\tilde{s}_2 r_1\tilde{s}_1 y + \tilde{r}_2\xi_1 y - y + \tilde{r}_1 r_2\tilde{s}_2,$$

$$c_0(y) = \tilde{r}_2 r_1\tilde{s}_1 y + \tilde{r}_1\tilde{r}_2,$$

$$d_2(y) = -(\tilde{r}_2\xi_1 y),$$

$$d_1(y) = \tilde{r}_2\xi_1 y - \tilde{r}_1\tilde{r}_2 - r_1\tilde{r}_2\tilde{s}_1 y$$

and

$$d_0(y) = r_1\tilde{r}_2\tilde{s}_1 y + \tilde{r}_1\tilde{r}_2.$$

From (28) we can write that

$$\begin{aligned} &Res_x(C_1,C_3;y) = \det(M_4) \\ &= \xi_1\tilde{r}_2 y^2(\tilde{r}_2 r_1\tilde{s}_1 y + \tilde{r}_2\tilde{r}_1) \\ &\qquad \{\xi_2 r_1 y^2 + (\tilde{\tilde{\xi}}_2 r_1 + \xi_2\tilde{r}_1 - 1)y + \tilde{r}_1\tilde{\tilde{\xi}}_2\}, \end{aligned}$$

so we can conclude that the resultant in x of the functions $h_1(x,y)$ and $h_3(x,y)$ as a function of y is zero at the zeros of the above equation which are:

1. $y_1 = 0$,
2. $y_2 = 0$,
3. $y_3 = -\frac{\tilde{r}_1}{r_1\tilde{s}_1}$
4. $y_4 = 1$
5. $y_5 = \frac{1 - r_1 - \xi_2 + r_1\xi_2}{r_1\xi_2}$

the obtained values are possible singularities of the function $P(0,y)$.

5 Discussion of the singularities

Here we discuss the (potential) singularities we have obtained so far for the unknowns. In the table in figure 2 below we show the set of ordered pairs that represents the intersection points of $h_1(x,y) = 0$ and $h_2(x,y) = 0$. In other words the set of ordered pairs at which the unknown function $P(x,0)$ has potential singularities. Since it is well-known that the unknown function $P(x,0)$ e.g. is by definition an analytic function in the unit disk, then such unknown cannot have a singularity at (x_j,y_j) that lie inside the unit disk. If it happens that such unknown has singularities that lie inside the unit disk, then it must be a removable singularity, i.e. the numerator of the equation (13) must be zero at such (x_j,y_j). Then we have to test whether the obtained potential singularities that lie inside the unit disk represent removable singularities or not. For instance, it is easy to see that the pair $(0,0)$ in the table

(x_i, y_i)	$y_1 = 0$	$y_2 = 1$	$y_3 = 1 - \frac{1}{s_2}$	$y_4 = -\frac{\tilde{r}_1}{r_1\tilde{s}_1}$	$y_5 = \frac{1-\xi_2-r_1+r_1\xi_2}{r_1\xi_2}$
$x_1 = 0$	♣		♣		
$x_2 = 1$		⊛	⊛		
$x_3 = 1 - \frac{1}{s_1}$	♣	⊛	⊛		
$x_4 = -\frac{\tilde{r}_2}{r_2\tilde{s}_2}$	♣				
$x_5 = \frac{1-\xi_1-r_2+r_2\xi_1}{r_2\xi_1}$					

The sign ♣ corresponds to the ordered pairs that lie or may lie inside the unit disk while the sign ⊛ corresponds to the pairs that lie or may lie on the disk

Fig. 2: The set of ordered pairs (x_i, y_i) at which the function $P(x,0)$ has potential singularities

represents a removable singularity since it easy to check that the numerator of (17) is zero or

$$h_3(x,y)P(0,y) + h_4(x,y)P(0,0) = 0 \tag{29}$$

at this pair. But in the mean time it is interesting to note that the pair $(0,0)$ is not a common zero of the functions $h_1(x,y)$ and $h_2(x,y)$. On the other hand, the pair $(1,1)$ represents a zero of (29) and in the mean time it is a common zero of the functions $h_1(x,y)$ and $h_2(x,y)$. It should also be noted that the pair $(x,1)$ for any x represents a zero of the function $h_2(x,y)$. Similarly the pair $(1,y)$ for any value of y represents a zero for the function $h_3(x,y)$. Also, it is interesting to find out that either the pair $(x,1)$ or $(1,y)$ for any values of x and y respectively represents zeros of the function $h_4(x,y)$.

Now, since in the table some of the ordered pairs are given as functions of the system parameters $r_i, s_i, \xi_i; i = 1,2$, then it is reasonable to discuss such ordered pairs with respect to the parameters values. It is a well-known fact in queueing theory (see e.g. [28]) that for a queueing system with arrival rate λ and service rate μ, the stability condition is that the arrival rate does not exceed the service rate, that is to say $\lambda < \mu$. In the current case of this paper the system producing the functional equation (13) is a network gateway modeled as two back-to-back interfering queues see [14]. The arrival rate of the first queue is ξ_1 and the service rate of such queue is $1 - r_2$, and similarly the arrival rate of queue 2 is ξ_2 and the service rate is $1 - r_1$. For the current functional equation the system is stable only if the parameters $r_i, s_i, \xi_i; i = 1,2$ satisfy the following requirement

$$\xi_1 < 1 - r_2,\ \xi_2 < 1 - r_1. \tag{30}$$

One can conclude that the parameters values should not be selected in a random fashion. In other words, in selecting values of the parameters in the table we must select them in accordance to (30).

Having a look at the table we can conclude that the pairs that need to be checked are those which corresponding cells are highlighted by the sign ♣. This is because such pairs lie inside the unit disk even for some special values of the system parameters. The ordered pairs for which the corresponding cells are highlighted by the sign ⊛ represent the pairs that lie or may lie on the disk itself. Other ordered pairs outside the disk for any values of the parameters. If the function $P(x,0)$ has some singularity either on the circle of convergence or outside it then we don't have problems. An example of the potential singularities that lie outside the unit disk for any system parameters satisfying (30) is x_5 in the table. This is because

$$x_5 = \frac{1 - \xi_1 - r_2 + r_2\xi_1}{r_2\xi_1} = 1 + \frac{1 - r_2 - \xi_1}{r_2\xi_1} > 1,$$

and according to the stability condition (30) we have $1 - r_2 > \xi_1$ then x_5 is always outside the unit disk. In a complete symmetric way one can find easily that $y_5 > 1$. In the table it is interesting to find out that the rest of the ordered pairs even if they lie inside the unit disk and represent singularities that are not removable they will not make any problems. This is because they are not common zeros of the functions $h_1(x,y)$ and $h_2(x,y)$. This means that we will not care about them. For instance, the pair $(0,1)$ is not a common zero of the functions $h_1(x,y)$ and $h_2(x,y)$.

It should be noted that the pair $(1,1)$ represents a removable singularity also it is easy to show that on the set $\{(x,y) : h_1(x,y) = 0\}$ the limit

$$\lim_{(x,y)\to(1,1)} \frac{1-x}{1-y}$$

exists. This can be shown easily using the Taylor's expansion of $h_1(x,y)$ around the point $(1,1)$. It should be noted that since we have a complete symmetric equation, then a similar analysis can be obtained for the other unknown function $P(0,y)$. The main reason behind computing the possible singularities is to estimate the sequences defined by such unknowns as introduced by theorem 1 below. We also hope that computing the potential singularities of the unknowns will help us in the future to solve such a challenging class of equations.

*Example 1.*The following equation arose in [23] from two parallel queues with the customer joining the shorter

$$\begin{aligned}(x(2\rho x+1)-2(1+\rho)xy+y^2)P(x,y)\\ =y(y-x)P(0,y)+(x(2\rho x+1)\\ -(1+\rho)xy-\rho xy^2)P(x,0),\end{aligned} \tag{31}$$

where $\rho < 1$. In order to use the idea (16) we first rewrite the above equation in the form

$$h_1(x,y)P(x,y)=h_2(x,y)P(x,0)+h_3(x,y)P(0,y),$$

where

$$h_1(x,y)=x(2\rho x+1)-2(1+\rho)xy+y^2,$$

$$h_2(x,y)=x(2\rho x+1)-(1+\rho)xy-\rho xy^2,$$

and

$$h_3(x,y)=y(y-x).$$

5.1 Singularity of P(0,y)

Following the same argument as in the previous two sections to find that

5.1.1 Resultant in x of h_1 and h_3

First we write the two functions h_1 and h_3 as

$$\begin{aligned}h_1(x,y)&=x(2\rho x+1)-2(1+\rho)xy+y^2,\\ &=y^2+(1-2y-2\rho y)x+2\rho x^2\\ &=a_0+a_1x+a_2x^2,\end{aligned} \tag{32}$$

and

$$\begin{aligned}h_3(x,y)&=y(y-x),\\ &=y^2-yx\\ &=b_0+b_1x.\end{aligned} \tag{33}$$

The resultant $Res_x(h_1,h_3,y)$ of the two functions h_1 and h_3 is the determinant of the matrix

$$A=\begin{pmatrix}a_2 & a_1 & a_0\\ b_1 & b_0 & 0\\ 0 & b_1 & b_0\end{pmatrix}, \tag{34}$$

which can be rewritten as:

$$\begin{aligned}Res_x(h_1,h_3,y)&=\det(A)\\ &=a_2b_0^2-b_1(a_1b_0-a_0b_1)=y^3(1-y).\end{aligned} \tag{35}$$

The zeros of the above equation are $y=0$ with multiplicity 3, and $y=1$.

5.1.2 Resultant in y of h_1 and h_3

Rewrite the two functions h_1 and h_3 as

$$\begin{aligned}h_1(x,y)&=x(2\rho x+1)-2(1+\rho)xy+y^2,\\ &=2\rho x^2+x-2x(1+\rho)y+y^2\\ &=c_0+c_1y+c_2y^2,\end{aligned} \tag{36}$$

and

$$\begin{aligned}h_3(x,y)&=y(y-x),\\ &=-xy+y^2\\ &=d_1y+d_2y^2.\end{aligned} \tag{37}$$

The resultant $Res_y(h_1,h_3,x)$ of the two functions h_1 and h_3 is the determinant of the matrix

$$B=\begin{pmatrix}c_2 & c_1 & c_0 & 0\\ 0 & c_2 & c_1 & c_0\\ d_2 & d_1 & d_0 & 0\\ 0 & d_2 & d_1 & d_0\end{pmatrix}. \tag{38}$$

From (38) we get

$$\begin{aligned}Res_y(h_1,h_3,x)&=\det(B)\\ &=c_0d_1^2c_2-c_1^2d_1d_2+c_0c_1d_2^2,\end{aligned} \tag{39}$$

the zeros of the above equation are $x=0$ of multiplicity two, and

$$x=\frac{-4\rho-5\pm\sqrt{(4\rho+5)^2+8\rho(2+2\rho)}}{4\rho}$$

Combining the results of the two previous subsections to conclude that the unknown function $P(0,y)$ has a potential singularity at

1. $y=0$ of multiplicity three
2. $y=1$
3. $y=\frac{-4\rho-5\pm\sqrt{(4\rho+5)^2+8\rho(2+2\rho)}}{4\rho}$.

6 Application of the singularity

Generally speaking, asymptotic analysis endeavors are to find a solution that closely approximates the exact solution [24]. One possible application of computing the possible singularities of the unknowns of functional equations is to obtain the asymptotic behavior of the sequences defined by such unknowns. The relation between the singularities and the asymptotic expansion comes from the fact that an asymptotic expansion of a function near some singularity is mapped to matching asymptotic expansion of its coefficients [24]. There are many techniques for deriving tail asymptotics for random walks in the first quarter plane see [22, 24], each technique depends on some properties of the function defining sequences. The functional equation of our interest is in fact an equation in which the unknowns are generating functions, namely $P(x,y)$, $P(x,0)$, $P(0,y)$ with interesting coefficients $p_{m,n}, p_{m,0}, p_{0,n}$ respectively, that have no simple closed form. In this section we give pointers to a technique for obtaining the asymptotic of the marginal distribution $p_{m,0}$ for large m, given $P(x,0) = \sum_{m=0}^{\infty} p_{m,0}x^m$. If the function $P(x,0)$ has singularities, then Darboux's theorem can use these singularities to estimate the sequence $p_{m,0}$. The statement of the theorem is due to [28] chapter 1 as follows:

Theorem 1.*(Darboux's method). Suppose* $P(x) = \sum_{m=0}^{\infty} p_{m,0}x^m$ *with positive real coefficients* $p_{m,0}$ *is analytic near* 0 *and has only algebraic singularities* x_k *on its circle of convergence* $|x| = R$*, in other words, in a neighborhood of* x_k *we have*

$$P(x) \approx (1-\frac{x}{x_k})^{-w_k} G_k(x),$$

where $w_k \in \{1,2,3,\cdots\}$ *and* $G_k(x)$ *denotes a nonzero analytic function near* x_k*. Let* $w = max_k \Re(w_k)$ *denote the maximum of the real parts of the* w_k*. Then we have*

$$p_{m,0} = \sum_j \frac{G_j(x_j)}{\Gamma(w_j)} m^{w_j-1} x_j^{-m} + o(m^{w-1}R^{-m})$$

with the sum taken over all j *with* $\Re(w_j) = w$ *and* $\Gamma(w)$ *the Gamma function of* w *(with* $\Gamma(n) = (n-1)!$ *for* n *a positive integer).*

7 Expectations

In this section we find the expected number of customers (packets) in both queues of the gateway using the corresponding generating functions. It is easy to see using (13) that the generating function of the number of packets in the first queue is given by

$$\begin{aligned} P(x,1) &= \sum_{m=0}^{\infty} P(N_1 = m)x^m \\ &= \frac{(1-x)\overline{r_2}(\overline{r_1}+r_1\overline{s_1}+\xi_1 x)P(0,1)}{(\overline{r_1}+r_1\overline{s_1}+\xi_1 x)(\overline{r_2}+r_2\overline{s_2}x+\xi_2 x)-x}, \end{aligned} \quad (40)$$

where N_1 is the number of packets in the first queue. Using the normalization condition $P(1,1) = 1$ in (40) to find $P(0,1)$ by applying de l'Hôpital's rule to get

$$P(0,1) = \frac{\overline{r_2}-\xi_1}{\overline{r_2}}. \quad (41)$$

Using (41) in (40) we will compute the expected number of packets in queue 1. It is well known see e.g. [13] that the expected number of packets in queue 1 is given by

$$\begin{aligned} E[N_1] &= \frac{\partial}{\partial x} P(x,1) \mid_{x=1} \\ &= \frac{0}{0}. \end{aligned}$$

Therefore by applying de l'Hôpital's rule we get

$$\mathbb{E}[N_1] = \frac{2\xi_1(1-\xi_1)+\xi_2}{2(\tilde{r}_2-\xi_1)}. \quad (42)$$

Similarly, the generating function of the number of packets

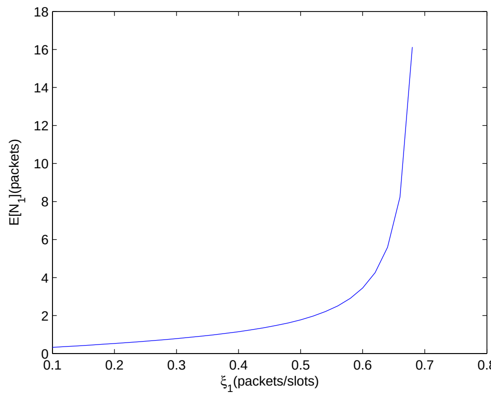

Fig. 3: The expected number of packets in queue I vs. the arrival rate ξ_1 for fixed $r_2 = 0.3$, $s_2 = 0.7$, and $\xi_2 = 0.21$

in the second queue is

$$\begin{aligned} P(1,y) &= \sum_{n=0}^{\infty} \Pr(N_2 = n)y^n \\ &= \frac{(1-y)\tilde{r}_1(\tilde{r}_2+r_2\tilde{s}_2+\xi_2 y)f(1,0)}{(\tilde{r}_1+r_1\tilde{s}_1 y+\xi_1 y)(\tilde{r}_2+r_2\tilde{s}_2+\xi_2 y)-y}, \end{aligned}$$

where N_2 is the number of packets in the second queue. Following the same way to end up with the expected number of packets in the other queue namely,

$$\mathbb{E}[N_2] = \frac{2\xi_2(1-\xi_2)+\xi_1}{2(\tilde{r}_1-\xi_2)}.$$

In figure 3 we used (42) to plot the expected number of packets in queue I ($\mathbb{E}[N_1]$) versus the queue I arrival rate for fixed values of queue II parameters namely, for $r_2 = 0.3$, $s_2 = 0.7$ and $\xi_2 = 0.21$. It is clear from the figure that the higher the arrival rate the higher the expected number of packets in queue I which clearly makes perfect sense. We also notice from the figure 3 that the expected number of packets increases almost linearly with the arrival rate. This is the typical behavior of queues in general (see Woodward [25] chapter 4.). On the other hand, in Figure 4 below, we plot the expected number of packets in queue I ($\mathbb{E}[N_1]$) versus the queue I arrival rate whenever there are no packets in LAN II. In other words when queue I service rate is 1. From figure 4 its a perfect linear relation meaning that the higher the arrival rate the higher the expected number of packets. This clearly makes perfect sense.

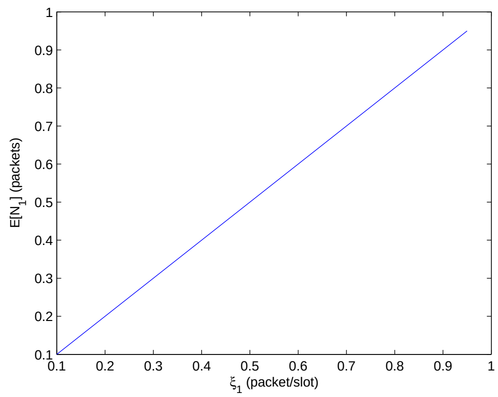

Fig. 4: The expected number of packets in queue I vs. the arrival rate ξ_1 when $r_2 = 0$, $s_2 = 0$, and $\xi_2 = 0$

8 Conclusions

We have computed the potential singularities of the unknown functions of a challenging functional equation which is not yet solved. The potential singularities of the unknowns of this equation are obtained by finding the intersection points between the kernel function and the corresponding coefficient of each unknown. As one application of computing the possible singularities we state, and give some pointers to an estimation of the sequences defined by the two unknowns. Another contribution is computing some expectations of interest. Possible extension of this work could be to find an analytical solution of the functional equation by utilizing the knowledge of the singularities of the unknowns of the equation.

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