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Coupled Coincidence and Common Fixed Point Theorems for Mappings in Partially Ordered Metric Spaces

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Abstract: In this paper, we prove coupled coincidence points and common fixed points theorem for two mappings satisfying various contractive conditions in partially ordered metric spaces. Our results will generalize and extend some recent results in fixed point theory.

Keywords: Common fixed point, coupled fixed point, coupled coincidence point, mixed g-monotone property, partially ordered set

1 Introduction and Preliminaries

Throughout this article, unless otherwise specified, we always suppose that $\mathbb{N}$ is the set of positive integers and X is a non empty set. Some authors generalized the Banach contraction principle theorem [4]in different ways (see for example [1,2,9,10,11,13,15,16]). Recently, Bhashkar and Lakshmikantham [5] coupled coincidence points, coupled fixed points, coupled common fixed points and common fixed points of nonlinear mappings with two variables expressed. After the publication of this work, several coupled fixed point and coincidence point results have appeared in the recent literature. Works noted in ([3, 6,7,8,12,14,17,18,19])are some relevant examples. The aim of this article is to make further studies on such problems, and to generalize and complement some known results. Next, let us recall some related definitions:

Definition 11*Let X be a nonempty set, then $(X,d,\preceq)$ is called an partially ordered metric space if:*

$(i)(X,d)$ *is a metric space,*
$(ii)(X,\preceq)$ *is a partially ordered set.*

Definition 12*Let $(X,\preceq)$ be a partially ordered set, then $x,y\in X$ are called comparable if $x\preceq y$ or $y\preceq x$.*

Definition 13*[5] Let $(X,\preceq)$ be a partially ordered set, and $F:X\times X\longrightarrow X$. The mapping F has the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y\in X$,*

$$x_1,x_2\in X, x_1\preceq x_2 \Longrightarrow F(x_1,y)\preceq F(x_2,y),$$

and

$$y_1,y_2\in X, y_1\preceq y_2 \Longrightarrow F(x,y_1)\succeq F(x,y_2).$$

Definition 14*[5] An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F:X\times X\longrightarrow X$ if*

$$F(x,y)=x,\ F(y,x)=y.$$

The main results of Bhaskar and Lakshmikantham in [5] are the following coupled fixed point theorems.

Theorem 15*[5] Let $(X,\preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F:X\times X\longrightarrow X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k\in[0,1)$ with*

$$d(F(x,y),F(u,v))\leq \frac{k}{2}[d(x,u)+d(y,v)]$$

for all $u\preceq x$ and $y\preceq v$. If there exist two elements $x_0,y_0\in X$ with $x_0\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq y_0$, then F has a coupled fixed point.

Recently, R. Bhardwaj [6] proved some generalizations of the main results in [5].

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Theorem 16[6] *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Suppose that there exist $\lambda \in [0,1), T: X\times X \longrightarrow X$ such that*

$$d(T(x,y),T(u,v)) \leq \lambda\ max\{\frac{d(x,T(x,y)).d(u,T(u,v))}{d(x,u)}, \frac{d(u,T(x,y)).d(x,T(u,v))}{d(x,u)}, d(x,u)\},$$

for all $x,y,u,v \in X$ with $x \preceq u, y \succeq v$ and $x \neq u$. Suppose also that T is continuous, has the mixed monotone property on X. If there exist $(x_0,y_0) \in X\times X$ such that $x_0 \preceq T(x_0,y_0)$ and $T(y_0,x_0) \preceq y_0$, then there exist $x,y \in X$ such that $x = T(x,y)$ and $y = T(y,x)$.

In [12], Lakshmikantham and Ćirić introduced the concept of mixed g-monotone property which present these definitions and results in the following.

Definition 17[12] *Let $(X,\leq)$ be a partially ordered set, and $F: X\times X \longrightarrow X$ and $g: X \longrightarrow X$. We say F has the mixed g-monotone property if F is non-decreasing g-monotone in its first argument and is non-increasing g-monotone in its second argument, that is, for any $x,y \in X$*

$$x_1,x_2 \in X, gx_1 \leq gx_2 \Longrightarrow F(x_1,y) \leq F(x_2,y),$$

and

$$y_1,y_2 \in X, gy_1 \leq gy_2 \Longrightarrow F(x,y_1) \geq F(x,y_2).$$

Note that if g is the identity mapping, then Definition 18 reduces to Definition 14.

Definition 18[12] *An element $(x,y) \in X\times X$ is called a coupled coincidence point of a mapping $F: X\times X \longrightarrow X$ and a mapping $g: X \longrightarrow X$ if*

$$F(x,y) = gx,\ F(y,x) = gy.$$

Similarly, note that if g is the identity mapping, then Definition 19 reduces to Definition 17.

Definition 19[12] *An element $x \in X$ is called a common fixed point of a mapping $F: X\times X \longrightarrow X$ and $g: X \longrightarrow X$ if*

$$F(x,x) = gx = x.$$

Definition 110[12] *Let X be a nonempty set and $F: X\times X \longrightarrow X$ and $g: X \longrightarrow X$. One says F and g are commutative if for all $x,y \in X$,*

$$F(gx,gy) = g(F(x,y)).$$

2 Main results

Throughout the article, let $\Psi[0,\infty)$ be the family of all functions $\psi: [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

$(a)\psi$ is continuous,
$(b)\psi$ nondecreasing,
$(c)\psi(t) = 0$ if and only if $t = 0$.

We denote by $\Phi[0,\infty)$ the set of all functions $\phi: [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

$(a)\phi$ is lower semi-continuous,
$(b)\phi(t) = 0$ if and only if $t = 0$,

and $\Theta[0,\infty)$ the set of all continuous functions $\theta: [0,\infty) \longrightarrow [0,\infty)$ with $\theta(t) = 0$ if and only if $t = 0$.
Our first result is the following.

Theorem 21*Suppose that $(X,d,\leq)$ be a partially ordered complete metric space. Let $T: X\times X \longrightarrow X$, $g: X \longrightarrow X$, $\psi \in \Psi[0,\infty), \varphi \in \Phi[0,\infty)$ and $\theta \in \Theta[0,\infty)$ satisfy following condition*

$$\psi(d(T(x,y),T(u,v))) \leq \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) + L\theta(N(x,y,u,v)), \qquad (2.1)$$

for all $x,y,u,v \in X$ with $gx \leq gu$ and $gy \geq gv$ and $gx \neq gu$, $L \geq 0$ and

$$M(x,y,u,v) = max\{\frac{d(gx,T(x,y)).d(gu,T(u,v))}{d(gx,gu)}, \frac{d(gu,T(x,y)).d(gx,T(u,v))}{d(gx,gu)}, d(gx,gu)\},$$

and

$$N(x,y,u,v) = min\{d(gx,T(x,y)),d(gu,T(u,v)), d(gu,T(x,y)),d(gx,T(u,v))\}.$$

Also, assume T and g are continuous mappings such that T has the mixed g-monotone property, g commutes with T and $T(X\times X) \subseteq g(X)$. If there exists $(x_0,y_0) \in X\times X$ such that $gx_0 \leq T(x_0,y_0)$ and $gy_0 \geq T(y_0,x_0)$, then T and g have coupled coincidence point in X.

Proof. By the given assumptions, there exists $(x_0,y_0) \in X\times X$ such that $gx_0 \leq T(x_0,y_0)$ and $gy_0 \geq T(y_0,x_0)$. Since $T(X\times X) \subseteq g(X)$, we can define $(x_1,y_1) \in X\times X$ such that $gx_1 = T(x_0,y_0)$ and $gy_1 = T(y_0,x_0)$, then $gx_0 \leq T(x_0,y_0) = gx_1$ and $gy_0 \geq T(y_0,x_0) = gy_1$. Also there exists $(x_2,y_2) \in X\times X$ such that $gx_2 = T(x_1,y_1)$ and $gy_2 = T(y_1,x_1)$. Since T has the mixed g-monotone property, we have

$$gx_1 = T(x_0,y_0) \leq T(x_0,y_1) \leq T(x_1,y_1) = gx_2,$$

and

$$gy_2 = T(y_1,x_1) \leq T(y_0,x_1) \leq T(y_0,x_0) = gy_1.$$

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all n=0,1,2

$$gx_{n+1} = T(x_n,y_n)\ \ and\ \ gy_{n+1} = T(y_n,x_n), \qquad (2.2)$$

for which

$$gx_0 \leq gx_1 \leq gx_2 \leq ... \leq gx_n \leq gx_{n+1} \leq ..., \qquad (2.3)$$
$$gy_0 \geq gy_1 \geq gy_2 \geq ... \geq gy_n \geq gy_{n+1} \geq ....$$

If there exists $k_0 \in \mathbb{N}$ such that $gx_{k_0+1} = gx_{k_0}$ and $gy_{k_0+1} = gy_{k_0}$, then $gx_{k_0} = T(x_{k_0}, y_{k_0})$ and $gy_{k_0} = T(y_{k_0}, x_{k_0})$. This means that (x_{k_0}, y_{k_0}) is a coupled coincidence point of T, g and the proof is finished. Thus, that $gx_{n+1} \neq gx_n$ and $gy_{n+1} \neq gy_n$ for all $n \in \mathbb{N}$. From (2.2) and (2.3) and the inequality (2.1) with $(x,y) = (x_n, y_n)$ and $(u,v) = (x_{n+1}, y_{n+1})$, we have

$$\begin{aligned}\psi(d(gx_{n+1},gx_{n+2})) &\leq \psi(d(gx_{n+1},gx_{n+2}))\\ &= \psi(d(T(x_n,y_n),T(x_{n+1},y_{n+1})))\\ &\leq \psi(M(x_n,y_n,x_{n+1},y_{n+1}))\\ &-\phi(M(x_n,y_n,x_{n+1},y_{n+1}))\\ &+L\theta(N(x_n,y_n,x_{n+1},y_{n+1})),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}&M(x_n,y_n,x_{n+1},y_{n+1}) =\\ &= max\{\frac{d(gx_n,T(x_n,y_n)).d(gx_{n+1},T(x_{n+1},y_{n+1}))}{d(gx_n,gx_{n+1})},\\ &\frac{d(gx_{n+1},T(x_n,y_n)).d(gx_n,T(x_{n+1},y_{n+1}))}{d(gx_n,gx_{n+1})},d(gx_n,gx_{n+1})\}\\ &= max\{\frac{d(gx_n,gx_{n+1}).d(gx_{n+1},gx_{n+2})}{d(gx_n,gx_{n+1})},\\ &\frac{d(gx_{n+1},gx_{n+1}).d(gx_n,gx_{n+2})}{d(gx_n,gx_{n+1})},d(gx_n,gx_{n+1})\}\\ &= max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\},\end{aligned}$$

and

$$\begin{aligned}&N(x_n,y_n,x_{n+1},y_{n+1}) =\\ &= min\{d(gx_n,T(x_n,y_n)),d(gx_{n+1},T(x_{n+1},y_{n+1})),\\ &d(gx_{n+1},T(x_n,y_n)),d(gx_n,T(x_{n+1},y_{n+1}))\}\\ &= min\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),d(gx_{n+1},gx_{n+1})\\ &,d(gx_n,gx_{n+2})\} = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\psi(d(gx_{n+1},gx_{n+2})) \leq\\ &\leq \psi(max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\})\\ &-\phi(max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\}),\end{aligned} \tag{2.5}$$

We prove that for all $n \in \mathbb{N}$,

$$d(gx_{n+1},gx_{n+2}) \leq d(gx_n,gx_{n+1}). \tag{2.6}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $d(gx_{n_0+1},gx_{n_0+2}) > d(gx_{n_0},gx_{n_0+1})$, from (2.5), we have

$$\begin{aligned}&\psi(d(gx_{n_0+1},gx_{n_0+2})) \leq\\ &\leq \psi(max\{d(gx_{n_0},gx_{n_0+1}),d(gx_{n_0+1},gx_{n_0+2})\})\\ &-\phi(max\{d(gx_{n_0},gx_{n_0+1}),d(gx_{n_0+1},gx_{n_0+2})\})\\ &= \psi(d(gx_{n_0+1},gx_{n_0+2}))-\phi(d(gx_{n_0+1},gx_{n_0+2}))\\ &< \psi(d(gx_{n_0+1},gx_{n_0+2})),\end{aligned}$$

which is a contradiction. Hence,

$$d(gx_{n+1},gx_{n+2}) \leq d(gx_n,gx_{n+1})$$

for all $n \in \mathbb{N}$. Similarly, we can show that $d(gy_{n+1},gy_{n+2}) \leq d(gy_n,gy_{n+1})$ for all $n \in \mathbb{N}$. It follows that the sequences $\{d(gx_n,gx_{n+1})\}$ and $\{d(gy_n,gy_{n+1})\}$ are monotone decreasing sequences of non-negative real numbers and consequently there exists $\delta_1, \delta_2 \geq 0$ such that

$$\lim_{n\longrightarrow\infty} d(gx_n,gx_{n+1}) = \delta_1 \text{ and } \lim_{n\longrightarrow\infty} d(gy_n,gy_{n+1}) = \delta_2. \tag{2.7}$$

We shall show that $\delta_1 = \delta_2 = 0$. Suppose, to the contrary, that $\delta_1 > 0$. Taking the (upper) limit as $n \longrightarrow \infty$ in (2.5) and using the properties of the functions ψ and ϕ we get

$$\begin{aligned}\psi(\delta_1) &\leq \psi(\delta_1)\\ &-\liminf_{n\longrightarrow\infty}\phi(max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\}\\ &\leq \psi(\delta_1)-\phi(\delta_1) < \psi(\delta_1),\end{aligned}$$

which is a contradiction. Therefore, $\delta_1 = 0$, that is,

$$\lim_{n\longrightarrow\infty} d(gx_n,gx_{n+1}) = 0. \tag{2.8}$$

Similarly, we can show that

$$\lim_{n\longrightarrow\infty} d(gy_n,gy_{n+1}) = 0. \tag{2.9}$$

Now, we claim that

$$\lim_{n,m\longrightarrow\infty} d(gx_n,gx_m) = 0. \tag{2.10}$$

Assume the contrary. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$ with $m(k) > n(k) \geq k$ such that

$$d(gx_{n(k)},gx_{m(k)}) \geq \varepsilon. \tag{2.11}$$

Additionally, corresponding to $n(k)$, we may choose $m(k)$ such that it is the smallest integer satisfying (2.11) and $m(k) > n(k) \geq k$. Thus,

$$d(gx_{n(k)},gx_{m(k)-1}) < \varepsilon. \tag{2.12}$$

We have

$$\begin{aligned}\varepsilon &\leq d(gx_{m(k)},gx_{n(k)})\\ &\leq d(gx_{m(k)},gx_{m(k)-1})+d(gx_{m(k)-1},gx_{n(k)})\\ &< d(gx_{m(k)},gx_{m(k)-1})+\varepsilon.\end{aligned}$$

Taking the upper limit as $k \longrightarrow \infty$ and using (2.8) we obtain

$$\lim_{k\longrightarrow\infty} d(gx_{n(k)},gx_{m(k)}) = \varepsilon. \tag{2.13}$$

Also

$$\begin{aligned}d(gx_{n(k)},gx_{m(k)}) &\leq d(gx_{m(k)},gx_{m(k)-1})\\ &+d(gx_{m(k)-1},gx_{n(k)-1})+d(gx_{n(k)-1},gx_{n(k)}),\end{aligned}$$

$$\begin{aligned} d(gx_{m(k)-1}, gx_{n(k)-1}) &\leq d(gx_{m(k)-1}, gx_{m(k)}) \\ &+ d(gx_{m(k)}, gx_{n(k)}) + d(gx_{n(k)}, gx_{n(k)-1}), \end{aligned}$$

$$d(gx_{n(k)-1}, gx_{m(k)}) \leq d(gx_{n(k)-1}, gx_{n(k)}) + d(gx_{n(k)}, gx_{m(k)}),$$

$$d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}).$$

So from (2.8),(2.13) and taking $k \longrightarrow \infty$ in the above inequalities, we have

$$\lim_{k\longrightarrow\infty} d(gx_{n(k)-1}, gx_{m(k)-1}) = \lim_{k\longrightarrow\infty} d(gx_{n(k)-1}, gx_{m(k)}) = \varepsilon. \tag{2.14}$$

Also

$$\begin{aligned} d(gx_{m(k)-1}, gx_{n(k)-1}) &\leq d(gx_{m(k)-1}, gx_{n(k)}) \\ &+ d(gx_{n(k)}, gx_{n(k)-1}) \\ &< \varepsilon + d(gx_{n(k)}, gx_{n(k)-1}). \end{aligned}$$

So from (2.8),(2.14) and taking $k \longrightarrow \infty$ in the above inequalities, we have

$$\lim_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)-1}) = \varepsilon. \tag{2.15}$$

By the definition of $M(x,y,u,v)$ and from (2.8),(2.14) and (2.15) we get

$$\begin{aligned} &\lim_{k\longrightarrow\infty} M(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1}) = \varepsilon, \\ &\lim_{k\longrightarrow\infty} N(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1}) = 0. \end{aligned} \tag{2.16}$$

Indeed,

$$\begin{aligned} &M(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1}) = \\ &= max\{\frac{d(gx_{n(k)-1}, T(x_{n(k)-1}, y_{n(k)-1}))}{d(gx_{n(k)-1}, gx_{m(k)-1})} \\ &\times d(gx_{m(k)-1}, T(x_{m(k)-1}, y_{m(k)-1})), \\ &\frac{d(gx_{m(k)-1}, T(x_{n(k)-1}, y_{n(k)-1}))}{d(gx_{n(k)-1}, gx_{m(k)-1})} \\ &\times d(gx_{n(k)-1}, T(x_{m(k)-1}, y_{m(k)-1})), \\ &d(gx_{n(k)-1}, gx_{m(k)-1})\} \\ &= max\{\frac{d(gx_{n(k)-1}, gx_{n(k)}).d(gx_{m(k)-1}, gx_{m(k)})}{d(gx_{n(k)-1}, gx_{m(k)-1})} \\ &\frac{d(gx_{m(k)-1}, gx_{n(k)}).d(gx_{n(k)-1}, gx_{m(k)})}{d(gx_{n(k)-1}, gx_{m(k)-1})} \\ &d(gx_{n(k)-1}, gx_{m(k)-1})\} \end{aligned}$$

and

$$\begin{aligned} &N(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1}) = \\ &= min\{d(gx_{n(k)-1}, T(x_{n(k)-1}, y_{n(k)-1})), \\ &d(gx_{m(k)-1}, T(x_{m(k)-1}, y_{m(k)-1})) \\ &d(gx_{m(k)-1}, T(x_{n(k)-1}, y_{n(k)-1})), \\ &d(gx_{n(k)-1}, T(x_{m(k)-1}, y_{m(k)-1}))\} \\ &= min\{d(gx_{n(k)-1}, gx_{n(k)}), d(gx_{m(k)-1}, gx_{m(k)}) \\ &d(gx_{m(k)-1}, gx_{n(k)}), d(gx_{n(k)-1}, gx_{m(k)})\}. \end{aligned}$$

Taking the limit as $k \longrightarrow \infty$ and using (2.8),(2.14) and (2.15) we get (2.16). Since $m(k) > n(k)$, $gx_{m(k)-1} \geq gx_{n(k)-1}$ and $gy_{m(k)-1} \leq gy_{n(k)-1}$ from (2.1), we have

$$\begin{aligned} &\psi(d(gx_{m(k)}, gx_{n(k)}) = \\ &= \psi(d(T(x_{n(k)-1}, y_{n(k)-1}), T(x_{m(k)-1}, y_{m(k)-1}))) \\ &\leq \psi(M(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1})) \\ &-\phi(M(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1})) \\ &+L\theta(N(x_{n(k)-1}, y_{n(k)-1}, x_{m(k)-1}, y_{m(k)-1})), \end{aligned}$$

taking the upper limit as $k \longrightarrow \infty$, and using (2.16) and the properties of the function ϕ we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. Therefore (2.10) holds, and we have

$$\lim_{n,m\longrightarrow\infty} d(gx_n, gx_m) = 0.$$

Similarly, we show that

$$\lim_{n,m\longrightarrow\infty} d(gy_n, gy_m) = 0.$$

Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\longrightarrow\infty} gx_{n+1} = x \ \ and \ \ \lim_{n\longrightarrow\infty} gy_{n+1} = y. \tag{2.17}$$

From the commutativity of T and g, we have

$$\begin{aligned} g(gx_{n+1}) &= g(T(x_n, y_n)) = T(gx_n, gy_n), \\ g(gy_{n+1}) &= g(T(y_n, x_n)) = T(gy_n, gx_n). \end{aligned} \tag{2.18}$$

Now we shall show that

$$gx = T(x,y) \ \ and \ \ gy = T(y,x).$$

Letting $n \longrightarrow \infty$ in (2.18) and from the continuity of T and g, we get

$$\begin{aligned} gx &= \lim_{n\longrightarrow\infty} g(gx_{n+1}) = \lim_{n\longrightarrow\infty} T(gx_n, gy_n) \\ &= T(\lim_{n\longrightarrow\infty} gx_n, \lim_{n\longrightarrow\infty} gy_n) = T(x,y), \\ gy &= \lim_{n\longrightarrow\infty} g(gy_{n+1}) = \lim_{n\longrightarrow\infty} T(gy_n, gx_n) \\ &= T(\lim_{n\longrightarrow\infty} gy_n, \lim_{n\longrightarrow\infty} gx_n) = T(y,x). \end{aligned}$$

This implies that (x,y) is a coupled coincidence point of T and g . This completes the proof.

Corollary 22*Let $(X,d,\leq)$ be a partially ordered complete metric space. Let $T : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property on X, $\psi \in \Psi[0,\infty)$,$\varphi \in \Phi[0,\infty)$ and $\theta \in \Theta[0,\infty)$ satisfy following condition*

$$\begin{aligned} \psi(d(T(x,y), T(u,v))) \leq \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) \\ +L\theta(N(x,y,u,v)), \end{aligned}$$

for all $x,y,u,v \in X$ with $x \leq u$ and $y \geq v$ and $x \neq u$, $L \geq 0$ and

$$M(x,y,u,v) = max\{\frac{d(x,T(x,y)).d(u,T(u,v))}{d(x,u)}, \frac{d(u,T(x,y)).d(x,T(u,v))}{d(x,u)}, d(x,u)\},$$

and

$$N(x,y,u,v) = min\{d(x,T(x,y)), d(u,T(u,v)), d(u,T(x,y)), d(x,T(u,v))\}.$$

If there exists $(x_0,y_0) \in X \times X$ such that $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$, then then there exist $x,y \in X$ such that $x = T(x,y)$ and $y = T(y,x)$.

Proof. Applying Theorems 21 and taking as $g = I_X$ we obtain the corollary 22.

Corollary 23*Let $(X,d,\leq)$ be a partially ordered complete metric space. Let $T : X \times X \longrightarrow X$, $g : X \longrightarrow X$ and there exist a $\lambda \in [0,1)$ satisfy following condition*

$$d(T(x,y),T(u,v))) \leq \lambda\ max\{\frac{d(gx,T(x,y)).d(gu,T(u,v))}{d(gx,gu)}, \frac{d(gu,T(x,y)).d(gx,T(u,v))}{d(gx,gu)}, d(gx,gu)\},$$

for all $x,y,u,v \in X$ with $gx \leq gu$ and $gy \geq gv$ and $gx \neq gu$. Also, assume T and g are continuous mappings such that T has the mixed g-monotone property, g commutes with T and $T(X \times X)0 \subseteq g(X)$. If there exists $(x_0,y_0) \in X \times X$ such that $gx_0 \leq T(x_0,y_0)$ and $gy_0 \geq T(y_0,x_0)$, then then there exist $x,y \in X$ such that $gx = T(x,y)$ and $gy = T(y,x)$.

Proof. In Theorems 21, taking as $L = 0, \phi(t) = (1-\lambda)t$ and $\psi(t) = I_X$ for all $t \in [0,+\infty)$, we get corollary 23.

Corollary 24*Let $(X,d,\leq)$ be a partially ordered complete metric space. Suppose that there exist $\lambda \in [0,1)$, $T : X \times X \longrightarrow X$ such that*

$$d(T(x,y),T(u,v))) \leq \lambda\ max\{\frac{d(x,T(x,y)).d(u,T(u,v))}{d(x,u)}, \frac{d(u,T(x,y)).d(x,T(u,v))}{d(x,u)}, d(x,u)\},$$

for all $x,y,u,v \in X$ with $x \leq u$ and $y \geq v$ and $x \neq u$. Suppose also that T is continuous, has the mixed monotone property on X.If there exists $(x_0,y_0) \in X \times X$ such that $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$, then then there exist $x,y \in X$ such that $x = T(x,y)$ and $y = T(y,x)$.

Proof. Taking $g = I_X$ in Corollary 23, we obtain the corollary 24.

Corollary 25*Let $(X,d,\leq)$ be a partially ordered complete metric space. Suppose that $T : X \times X \longrightarrow X$, $g : X \longrightarrow X$ satisfies the following condition:*

$$d(T(x,y),T(u,v))) \leq a\frac{d(gx,T(x,y)).d(gu,T(u,v))}{d(gx,gu)} + b\frac{d(gu,T(x,y)).d(gx,T(u,v))}{d(gx,gu)} + cd(gx,gu),$$

for all $x,y,u,v \in X$ with $gx \leq gu$ and $gy \geq gv$ and $gx \neq gu$ and for some $a,b,c \geq 0$ with $a+b+c < 1$. Also, assume T and g are continuous mappings such that T has the mixed g-monotone property, g commutes with T and $T(X \times X)0 \subseteq g(X)$. If there exists $(x_0,y_0) \in X \times X$ such that $gx_0 \leq T(x_0,y_0)$ and $gy_0 \geq T(y_0,x_0)$, then then there exist $x,y \in X$ such that $gx = T(x,y)$ and $gy = T(y,x)$.

Proof. For $a,b,c \geq 0, a+b+c < 1$ and for all $x,y,u,v \in X$ with $gx \leq gu$ and $gy \geq gv$ and $gx \neq gu$, we have

$$\begin{aligned} d(T(x,y),T(u,v))) &\leq a\frac{d(gx,T(x,y)).d(gu,T(u,v))}{d(gx,gu)} + b\frac{d(gu,T(x,y)).d(gx,T(u,v))}{d(gx,gu)} + cd(gx,gu) \\ &\leq (a+b+c)max\{\frac{d(x,T(x,y)).d(u,T(u,v))}{d(x,u)}, \frac{d(u,T(x,y)).d(x,T(u,v))}{d(x,u)}, d(x,u)\} \\ &= \lambda\ max\{\frac{d(x,T(x,y)).d(u,T(u,v))}{d(x,u)}, \frac{d(u,T(x,y)).d(x,T(u,v))}{d(x,u)}, d(x,u)\}, \end{aligned}$$

where $\lambda = a+b+c \in [0,1)$. Therefore, we get Corollary 25 of Corollary 24.

Now we give sufficient conditions for uniqueness of the coupled coincidence point. Note that if $(X,\leq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x,y),(z,t) \in X \times X$,

$$(x,y) \leq (z,t) \Longleftrightarrow x \leq z,\ y \geq t.$$

From Theorem 21, it follows that the set of coupled coincidence points of T and g is non-empty.

Theorem 26*By adding to the hypotheses of Theorem 21, the condition:*
for every (x,y) and (z,t) in $X \times X$, there exists a $(u,v) \in X \times X$ such that $(T(u,v),T(v,u))$ is comparable to $(T(x,y),T(y,x))$ and to $(T(z,t),T(t,z))$, then T and g have a unique coupled common fixed point; that is, there exist a unique $(x,y) \in X \times X$ such that

$$x = gx = T(x,y),\ y = gy = T(y,x).$$

Proof. We know, from Theorem 2.1, that there exists at least a coupled coincidence point. Suppose that (x,y) and (z,t) are coupled coincidence points of T and g, that is, $T(x,y) = gx$, $T(y,x) = gy$, $T(z,t) = gz$ and $T(t,z) = gt$. We shall show that $gx = gz$ and $gy = gt$. By the assumptions, there exists $(u,v) \in X \times X$ such that $(T(u,v),T(v,u))$ is comparable to $(T(x,y),T(y,x))$ and to $(T(z,t),T(t,z))$. Without any restriction of the generality, we can assume that

$$(T(x,y),T(y,x)) \leq (T(u,v),T(v,u))$$

and

$$(T(z,t),T(t,z)) \leq (T(u,v),T(v,u)).$$

Put $u_0 = u,\ v_0 = v$ and choose $(u_1,\ v_1) \in X \times X$ such that

$gu_1 = T(u_0, v_0),\ gv_1 = T(v_0, u_0).$

For $n \geq 1$, continuing this process we can construct sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$gu_{n+1} = T(u_n, v_n),\ gv_{n+1} = T(v_n, u_n)$ *for all n*.

Further, set $x_0 = x,\ y_0 = y$ and $z_0 = z,\ t_0 = t$ and on the same way define sequences $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}, \{gt_n\}$. Then, it is easy to see that

$$\begin{aligned} &gx_n \longrightarrow T(x,y),\ gy_n \longrightarrow T(y,x),\\ &gz_n \longrightarrow T(z,t),\ gt_n \longrightarrow T(t,z), \end{aligned} \tag{2.19}$$

for all $n \geq 1$.
Since $(T(x,y), T(y,x)) = (gx, gy) = (gx_1, gy_1)$ is comparable to $(T(u,v), T(v,u)) = (gu_1, gv_1)$, then it is easy to show $(gx, gy) \leq (gu_n, gv_n)$, that is, $gx \leq gu_n$ and $gy \geq gv_n$ for all $n \in \mathbb{N}$. Thus from 2.1, we have

$$\begin{aligned} \psi(d(gx, gu_{n+1})) &= \psi(d(T(x,y), T(u_n, v_n)))\\ &\leq \psi(M(x,y,u_n,v_n)) - \phi(M(x,y,u_n,v_n))\\ &+ L\theta(N(x,y,u_n,v_n)), \end{aligned}$$

where

$$\begin{aligned} M(x,y,u_n,v_n) = max\{&\frac{d(gx, T(x,y)).d(gu_n, T(u_n,v_n))}{d(gx, gu_n)},\\ &\frac{d(gu_n, T(x,y)).d(gx, T(u_n,v_n))}{d(gx, gu_n)}, d(gx, gu_n)\}\\ &= max\{d(gx, gu_n), d(gx, gu_{n+1})\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,u_n,v_n) = min\{&d(gx, T(x,y)), d(gu_n, T(u_n,v_n)),\\ &d(gu_n, T(x,y)), d(gx, T(u_n,v_n)) = 0. \end{aligned}$$

Hence

$$\begin{aligned} \psi(d(gx, gu_{n+1})) \leq\ &\psi(max\{d(gx, gu_n), d(gx, gu_{n+1})\})\\ &- \phi(max\{d(gx, gu_n), d(gx, gu_{n+1})\}). \end{aligned}$$

It is easy to show that

$$\begin{aligned} \psi(d(gx, gu_{n+1})) &\leq \psi(d(gx, gu_n)) - \phi(d(gx, gu_n))\\ &< \psi(d(gx, gu_n)). \end{aligned} \tag{2.20}$$

This implies that $\{d(gx, gu_n)\}$ is a non-increasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(gx, gu_n) = r.$$

Passing the upper limit in 2.20 as $n \longrightarrow \infty$, we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $\phi(r) = 0$ and then, $r = 0$. We deduce that

$$\lim_{n \longrightarrow \infty} d(gx, gu_n) = 0. \tag{2.21}$$

Similarly one can prove that

$$\lim_{n \longrightarrow \infty} d(gy, gv_n) = 0. \tag{2.22}$$

Similarly, one can prove that

$$\lim_{n \longrightarrow \infty} d(gz, gu_n) = \lim_{n \longrightarrow \infty} d(gt, gv_n) = 0. \tag{2.23}$$

By the triangle inequality, 2.22 and 2.23, we get

$$d(gx, gz) \leq d(gx, gu_{n+1}) + d(gz, gu_{n+1}) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

$$d(gy, gt) \leq d(gy, gv_{n+1}) + d(gt, gv_{n+1}) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore, we have $gx = gz$ and $gy = gt$. Since $gx = T(x,y)$ and $gy = T(y,x)$, by commutativity of T and g , we have

$$\begin{aligned} &g(gx) = g(T(x,y)) = T(gx, gy),\\ &g(gy) = g(T(y,x)) = T(gy, gx). \end{aligned} \tag{2.24}$$

Denote $gx = a$ and $gy = b$. Then from 2.24

$$g(a) = T(a,b),\ g(b) = T(b,a). \tag{2.25}$$

Thus, (a,b) is a coupled coincidence point, it follows that $ga = gz$ and $gb = gy$, that is,

$$g(a) = a,\ g(b) = b. \tag{2.26}$$

From 2.25 and 2.26

$$a = g(a) = T(a,b),\ b = g(b) = T(b,a). \tag{2.27}$$

Therefore, (a,b) is a coupled common fixed point of T and g. To prove the uniqueness of the point (a,b), assume that (c,d) is another coupled common fixed point of T and g. Then we have

$$c = gc = T(c,d),\ d = gd = T(d,c).$$

Since (c,d) is a coupled coincidence point of T and g ,we have $gc = gx = a$ and $gd = gy = b$. Thus $c = gc = ga = a$ and $d = gd = gb = b$. Hence, the coupled common fixed point is unique. this completes the proof.
If $g = I$, the identity mapping in Theorem 26, then we deduce the following corollary.

Corollary 27*In addition to the hypotheses of Corollary 22, suppose that for every (x,y) and (z,t) in $X \times X$, there exists a $(u,v) \in X \times X$ such that $(T(u,v), T(v,u))$ is comparable to $(T(x,y), T(y,x))$ and to $(T(z,t), T(t,z))$. Then T has a unique coupled fixed point, that is, there exists a unique $(x,y) \in X \times X$ such that*

$$x = T(x,y),\ y = T(y,x).$$

Now, we state and prove the last theorem of this paper.

Theorem 28*In addition to hypotheses of Theorem 21, if gx_0 and gy_0 are comparable, then T and g have a unique common fixed point, that is, there exists $x \in X$ such that $x = gx = T(x,x)$.*

Proof. By Theorem 21, we can construct two sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $gx_n \longrightarrow x$ and $gy_n \longrightarrow y$, where (x,y) is a unique coupled common fixed point of T and g. We only have to show that $x = y$. Since gx_0 and gy_0 are comparable, we may assume that $gx_0 \le gy_0$, then it is an easy matter to show that

$$gx_n \le gy_n \quad for\ all\ n \ge 0. \tag{2.28}$$

From 2.1 and 2.28, we have

$$\begin{aligned}\psi(d(gx_{n+1}, gy_{n+1}) &= \psi(d(T(x_n,y_n),T(y_n,x_n))) \\ &\le \psi(M(x_n,y_n,y_n,x_n)) \\ &-\phi(M(x_n,y_n,y_n,x_n)),\end{aligned}$$

where

$$\begin{aligned}M(x_n,y_n,y_n,x_n) &= max\{\tfrac{d(gx_n,T(x_n,y_n)).d(gy_n,T(y_n,x_n))}{d(x_n,y_n)}, \\ &\tfrac{d(gy_n,T(x_n,y_n)).d(gx_n,T(y_n,x_n))}{d(x_n,y_n)}, \\ &d(x_n,y_n)\} \\ &= max\{\tfrac{d(gx_n,gx_{n+1}).d(gy_n,gy_{n+1})}{d(gx_n,gy_n))}, \\ &\tfrac{d(gy_n,gx_{n+1}).d(gx_n,gy_{n+1})}{d(gx_n,gy_n)}, \\ &d(gx_n,gy_n)\}\end{aligned}$$

and

$$\begin{aligned}N(x_n,y_n,y_n,x_n) &= min\{d(gx_n,T(x_n,y_n)),d(gy_n,T(y_n,x_n)), \\ &d(gy_n,T(x_n,y_n)),d(gx_n,T(y_n,x_n))\} \\ &= min\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}), \\ &d(gy_n,gx_{n+1}),d(gx_n,gy_{n+1})\}.\end{aligned}$$

By taking the upper limit as $n \longrightarrow \infty$, we get

$$\lim_{n\longrightarrow\infty} M(x_n,y_n,y_n,x_n) = d(x,y), \quad \lim_{n\longrightarrow\infty} N(x_n,y_n,y_n,x_n) = 0.$$

Hence

$$\psi(d(x,y)) \le \psi(d(x,y)) - \phi(d(x,y)),$$

which implies that $\phi(d(x,y)) = 0$. Therefore $x = y$, that is, T and g have a common fixed point. Similar arguments can be used if $gx_0 \ge gy_0$.

If we assume $g = I$ in Theorem 28, then we deduce the following corollary.

Corollary 29*In addition to the hypotheses of Corollary 27, if x_0 and y_0 are comparable, then T has a unique fixed point, that is, there exists $x \in X$ such that $x = T(x,x)$.*

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