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# Green's Function Iterative Method for Solving a Class of Boundary Value Problems Arising in Heat Transfer

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**Abstract:** In this study, a new algorithm based on Green's function fixed point iterations is developed and implemented to solve a class of nonlinear boundary value problems that arise in heat transfer. The method will be employed to determine the efficiency of convective straight fins with temperature dependent thermal conductivity. The main idea of this method is to find an appropriate Green's function that will be incorporated into a linear integral operator. By applying the fixed point theorem, an iterative formula for successive approximations will be obtained. Effectiveness and accuracy were noted when approximate results obtained by the proposed method reveal significant agreement with existed exact solutions and/or approximate solutions obtained by other renowned methods.

**Keywords:** Green's function, fixed point iteration, boundary value problem, heat transfer

## 1 Introduction

Nonlinear differential equations are usually studied as mathematical models of physical phenomena in science and engineering. These equations can be seen in areas such as quantum mechanics, fluid dynamics, electromagnetism, population dynamics, image processing, chemical kinetics, and control theory. However, it is almost impossible to find exact solution to nonlinear differential equations that model real world applications. Fins are used in a large number of applications to increase the rate of heat transfer that extends from a surface to its convective. As such, the design of fin surfaces is of great importance in the study of heat transfer [1]. Determining fin efficiency of convective straight fins with temperature dependent thermal conductivity requires solving a nonlinear boundary value problem, which under usual setting posses no exact solutions. Researchers have applied several well-known numerical methods to obtain approximate or analytic solution in the form of infinite power series to this kind of equations. For example, S. Mosta [2] applied a linearization approach, Mustafa Inc [3] employed the homotopy analysis method, C. Arslanturk [4] applied the Adomian decomposition method, A. Joneidi et al. [5] applied a differential transformation method, A. Aziz and S. Enamul Huq [6] developed a perturbation solution, while the variation iteration method (VIM) was employed by S. Coşkun M. Atay [7]. Recently, in a letter published by S. Khuri and A. Sayfy [8], it was shown that, in many cases, using a fixed point iterative method such as Picard's method for successive approximations of initial value problems of first or higher order, the correction functional of the VIM can be obtained. In this paper, a Green's function fixed point iteration method is developed to solve the underlying nonlinear boundary value problem (BVP) described above. The method begins by identifying the linear and nonlinear terms of the nonlinear BVP and then employ the properties of Green's function to construct the appropriate function that mimics the solution to the homogenous linear term of the equations subject to homogenous boundary conditions. An iterative formula is obtained by applying Picard's fixed point method to a linear integral operator in which the integrand is the product of the constructed Green's function and the BVP. Examples will be provided to (1) show the applicability of the method (2) test the accuracy of the obtained solution by the means of comparison, with exact solutions, if exists, and approximate solutions obtained by other well-established methods.

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## 2 Mathematical formulation of the problem

Consider a straight fin with a temperature-dependent thermal conductivity, arbitrary cross-sectional area $A_c$, perimeter $P$ and length $b$ is attached to a tip-insulated-base surface of temperature $T_b$, that extends into a fluid of temperature $T_a$ (see Fig. 1).

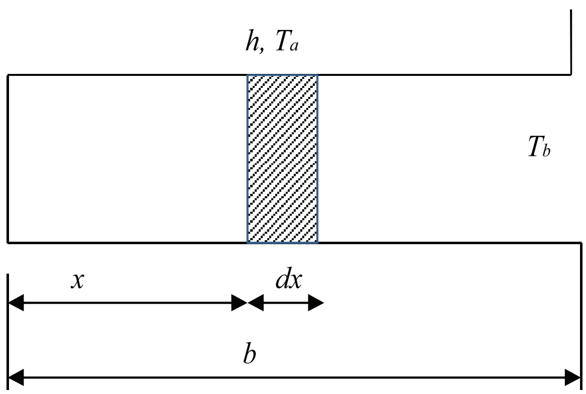


**Fig. 1:** Geometry of a straight fin.

The one dimensional energy balance is given by

$$A_c \frac{d}{dx}\left(K(T)\frac{dT}{dx}\right) - Ph(T_b - T_a) = 0, \tag{1}$$

where $T$ is the temperature, $K(T)$ is the temperature-dependent thermal conductivity of the fin material, and $h$ is the heat transfer coefficient. $K(T)$, expressed as a linear function of the temperature, is given by

$$K(T) = K_a\left(1 + \lambda(T - T_a)\right), \tag{2}$$

where $K_a$ is the thermal conductivity at the ambient fluid temperature of the fin and $\lambda$ is the parameter describing the variation in thermal conductivity.

Let

$$\theta = \frac{T - T_a}{T_b - T_a},\ \ \zeta = \frac{x}{b},\ \ \beta = \lambda(T_b - T_a),\ \ \psi = \left(\frac{hPb^2}{k_a A_c}\right)^{1/2}. \tag{3}$$

Substituting these dimensionless parameters into equation (1) gives the boundary value problem

$$\frac{d^2\theta}{d\zeta^2} + \beta\theta\frac{d^2\theta}{d\zeta^2} + \beta\left(\frac{d\theta}{d\zeta}\right)^2 - \psi^2\theta = 0, \tag{4}$$

subject to

$$\begin{aligned} \theta(1) &= 1, \\ \theta'(0) &= 0. \end{aligned} \tag{5}$$

Using Newton's law of cooling, the heat transfer rate from the fin is determined by integrating the convection heat loss from the surface, that is

$$Q = \int_0^b P(T - T_a)dx. \tag{6}$$

Let $Q_{\text{ideal}}$ represent the heat transfer rate of the entire surface, which depends on the base temperature $T_b$. Then the fin efficiency, denoted $\eta$, is defined as the ratio of $Q$ to $Q_{\text{ideal}}$. Therefore,

$$\eta = \frac{Q}{Pb(T_b - T_a)} = \int_0^1 \theta(\zeta)d\zeta. \tag{7}$$

## 3 Green's function fixed point iteration method

Consider the second order nonlinear boundary value problem

$$\begin{aligned} &F(t,u,u',u'') = f(t), \ \ a \le t \le b, \\ &B_1[u] \equiv \alpha u(a) + \beta u'(a) = \gamma_1, \\ &B_2[u] \equiv \alpha u(b) + \beta u'(b) = \gamma_2. \end{aligned} \tag{8}$$

The proposed method begins by expressing equation (8) in the form

$$Lu + Nu = f(t), \tag{9}$$

where $L$ is a linear operator and $N$ is a nonlinear operator. Assuming that $u_0(t)$ is the solution to $Lu = 0$, we construct the iterative solution to (9) using Picard's fixed point iteration in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t G(t,s)\left(Lu + Nu - f(s)\right)ds, \tag{10}$$

where $G(t,s)$ is the Green's function satisfying

$$\begin{aligned} L[G(t,s)] &= \delta(t-s), \\ B_1[G(t,s)] &= 0, \\ B_2[G(t,s)] &= 0, \end{aligned} \tag{11}$$

in which $\delta(t-s)$ is the Dirac delta function, which can be constructed using basic properties of Green's function. It is to be mentioned that there is flexibility in choosing $Lu$ so that $u_0$ can be found and $G(t,s)$ can be constructed. We shall apply this procedure to solve (4)

Letting

$$F(\zeta,\theta,\theta',\theta'') = \beta\theta\frac{d^2\theta}{d\zeta^2} + \beta\left(\frac{d\theta}{d\zeta}\right)^2, \tag{12}$$

then, equation (4) is expressed in the form

$$\Lambda[\theta] \equiv \frac{d^2\theta}{d\zeta^2} - \psi^2\theta = -F(\zeta,\theta,\theta',\theta''). \tag{13}$$

The general solution of equation (13) is expressed in the form

$$\theta = \theta_h + \theta_p = \theta_h - \int_0^1 G(\zeta,s)F(s,\theta,\theta',\theta'')ds, \tag{14}$$

where $G(\zeta,s)$ is Green's function that satisfies the equation

$$\begin{aligned}\Lambda[G(\zeta,s)] &= \delta(\zeta-s),\\ G(1,s) &= 0,\\ \frac{dG}{d\zeta}(0,s) &= 0,\end{aligned} \tag{15}$$

In equation (14), $\theta_h$ is solution to $\Lambda[\theta]=0$, subject to boundary conditions (5), that is

$$\theta_h = C_1e^{-\psi\zeta}+C_2e^{\psi\zeta}.$$

Using $\theta_h$, the boundary conditions in (15), and the properties of Green's function, $G(\zeta,s)$ can be found. As Green's function exhibits a singular behavior at $\zeta=s$, then it is natural to express it in the form

$$G(\zeta,s)=\begin{cases} C_1e^{-\psi\zeta}+C_2e^{\psi\zeta} \text{ if } 0<\zeta<s\\ C_3e^{-\psi\zeta}+C_4e^{\psi\zeta} \text{ if } 1>\zeta>s\end{cases}, \tag{16}$$

where $C_1,C_2,C_3$,and $C_4$ are constants. From the boundary condition given in (15), we obtain

$$-C_1+C_2=0, \tag{17}$$
$$C_3e^{-\psi}+C_4e^{\psi}=0. \tag{18}$$

The continuity of Green's function for all $0<\zeta<1$, that is $G(\zeta,s)|_{\zeta\to s^+}-G(\zeta,s)|_{\zeta\to s^-}=0$ implies that

$$C_1e^{-\psi s}+C_2e^{\psi s}-C_3e^{-\psi s}+C_4e^{\psi s}=0. \tag{19}$$

Integrating (15) over the domain $(s^-,s^+)$ yields

$$\int_{s^-}^{s^+}\left[\frac{d^2}{d\zeta^2}G(\zeta,s)-\psi^2G(\zeta,s)\right]d\zeta=\int_{s^-}^{s^+}\delta(\zeta-s)d\zeta, \tag{20}$$

and using the jump condition at $s=\zeta$, we have

$$\frac{d}{d\zeta}G(s^+,s)-\frac{d}{d\zeta}G(s^-,s)=1, \tag{21}$$

or equivalently,

$$-\psi C_3e^{-\psi s}+\psi C_4e^{\psi s}+\psi C_1e^{-\psi s}-\psi C_2e^{\psi s}=1. \tag{22}$$

Solving the linear system (17)-(19) and (22) for $C_1,C_2,C_3$ and $C_4$ implies that

$$\begin{aligned}C_1 &= \alpha(e^{-\psi(s-1)}-e^{\psi(s-1)}),\\ C_2 &= C_1,\\ C_3 &= \alpha(e^{-\psi(-s+1)}-e^{\psi(-s+1)}),\\ C_4 &= C3,\end{aligned} \tag{23}$$

where

$$\alpha=\frac{-1}{2\psi(e^{-\psi}+e^{\psi})}. \tag{24}$$

Now consider the following linear integral operator

$$L[\theta]=\theta_h+\int_0^1 G(\zeta,s)\left(\theta''(s)-\psi^2\theta(s)\right)ds. \tag{25}$$

Adding and subtracting $F(\zeta,\theta,\theta',\theta'')$ within the integrand gives

$$\begin{aligned}L[\theta]=\theta_h+\int_0^1 G(\zeta,s)\Big[&\theta''(s)-\psi^2\theta(s)+\beta\theta(s)\theta''(s)\\ &+\beta\left(\theta'(s)\right)^2\Big]ds\\ &-\int_0^1 G(\zeta,s)F(\zeta,\theta,\theta',\theta'')ds.\end{aligned}$$

From equation (14), we obtain

$$\begin{aligned}L[\theta]=\theta+\int_0^1 G(\zeta,s)\Big[&\theta''(s)-\psi^2\theta(s)+\beta\theta(s)\theta''(s)\\ &+\beta\left(\theta'(s)\right)^2\Big]ds\end{aligned}$$

Applying Picard's fixed point iteration method on the operator $L$, gives the iterative formula

$$\begin{aligned}\theta_{n+1}=\theta_n+\int_0^1 G(\zeta,s)\Big[&\theta_n''(s)-\psi^2\theta_n(s)+\beta\theta_n(s)\theta_n''(s)\\ &+\beta\left(\theta_n'(s)\right)^2\Big]ds.\end{aligned} \tag{26}$$

## 4 Numerical experiments and discussion

In this section, we present examples that cover both the constant and variable thermal conductivity cases. The approximate solution obtained by the proposed method (GFIM) will be compared with exact solution, if exists, and approximate solutions obtained by the differential transformation method (DTM) and the fourth order Runge Kutta method (RK4) presented in [5].

*Example 1.*In equation (4), let $\beta=0$ (constant thermal conductivity). A comparison between the GFIM, the DTM, and the exact results for the dimensionless temperature $\theta$ at various values of $\zeta$ for the thermo-geometric parameters $\psi=0.5$ is given in Table 1 and for $\psi=1$ os given in Table 2.

*Example 2.*In equation (4), let $\beta\neq 0$ (variable thermal conductivity). A comparison between the GFIM, the DTM, and the RK4 method for the dimensionless temperature $\theta$ at various values of $\zeta$ for $\beta=0.4$ and $\psi=1$ is given in Table 3 and for $\beta=0.2$ and $\psi=0.5$ is given in Table 4.

**Table 1:** Comparison between the GFIM, the DTM, and the exact results for the case $\beta = 0$ and $\psi = 0.5$

| $\zeta$ | Exact | GFIM | DTM [5] | GFIM Error | DTM Error |
|---|---|---|---|---|---|
| 0.0 | 0.8868188840 | 0.8868188840 | 0.8868188841 | 0 | $1 \times 10^{-10}$ |
| 0.1 | 0.8879276385 | 0.8879276385 | 0.8879276383 | 0 | $2 \times 10^{-10}$ |
| 0.2 | 0.8912566747 | 0.8912566747 | 0.8912566748 | 0 | $1 \times 10^{-10}$ |
| 0.3 | 0.8968143168 | 0.8968143168 | 0.8968143173 | 0 | $3 \times 10^{-10}$ |
| 0.4 | 0.9046144618 | 0.9046144618 | 0.9046144623 | 0 | $5 \times 10^{-10}$ |
| 0.5 | 0.9146766141 | 0.9146766141 | 0.9146766135 | 0 | $6 \times 10^{-10}$ |
| 0.6 | 0.9270259345 | 0.9270259345 | 0.9270259345 | 0 | 0 |
| 0.7 | 0.9416933025 | 0.9416933025 | 0.9416933025 | 0 | 0 |
| 0.8 | 0.9587153943 | 0.9587153943 | 0.9587153946 | 0 | $3 \times 10^{-10}$ |
| 0.9 | 0.9781347739 | 0.9781347739 | 0.9781347735 | 0 | $4 \times 10^{-10}$ |
| 1.0 | 1.0000000000 | 1.0000000000 | 0.9999999999 | 0 | $1 \times 10^{-10}$ |

**Table 2:** Comparison between the GFIM, the DTM, and the exact results for the case $\beta = 0$ and $\psi = 1$

| $\zeta$ | Exact | GFIM | DTM [5] | GFIM Error | DTM Error |
|---|---|---|---|---|---|
| 0.0 | 0.6480542737 | 0.6480542737 | 0.6480542737 | 0 | 0 |
| 0.1 | 0.6512972462 | 0.6512972462 | 0.6512972462 | 0 | 0 |
| 0.2 | 0.6610586204 | 0.6610586204 | 0.6610586207 | 0 | $3 \times 10^{-10}$ |
| 0.3 | 0.6774360915 | 0.6774360915 | 0.6774360915 | 0 | 0 |
| 0.4 | 0.7005935707 | 0.7005935707 | 0.7005935709 | 0 | $2 \times 10^{-10}$ |
| 0.5 | 0.7307628258 | 0.7307628258 | 0.7307628258 | 0 | 0 |
| 0.6 | 0.7682458010 | 0.7682458010 | 0.7682458015 | 0 | $5 \times 10^{-10}$ |
| 0.7 | 0.8134176383 | 0.8134176383 | 0.8134176386 | 0 | $3 \times 10^{-10}$ |
| 0.8 | 0.8667304327 | 0.8667304327 | 0.8667304332 | 0 | $5 \times 10^{-10}$ |
| 0.9 | 0.9287177566 | 0.9287177566 | 0.9287177570 | 0 | $4 \times 10^{-10}$ |
| 1.0 | 1.0000000000 | 1.0000000000 | 1.0000000001 | 0 | $1 \times 10^{-10}$ |

**Table 3:** Green's function iterative method solution for problem (4)-(5) with $\beta = 0.4$ and $\psi = 1$

| $\zeta$ | GFIM | DTM [5] | RK4 |
|---|---|---|---|
| 0.0 | 0.7160464622 | 0.7160464622 | 0.7160464718 |
| 0.1 | 0.7188301611 | 0.7188301611 | 0.7188301796 |
| 0.2 | 0.7271884199 | 0.7271884199 | 0.7271884325 |
| 0.3 | 0.7411425954 | 0.7411425952 | 0.7411426043 |
| 0.4 | 0.7607278542 | 0.7607278540 | 0.7607278639 |
| 0.5 | 0.7859925455 | 0.7859925453 | 0.7859925543 |
| 0.6 | 0.8169973499 | 0.8169973500 | 0.8169973564 |
| 0.7 | 0.8538142334 | 0.8538142337 | 0.8538142400 |
| 0.8 | 0.8965252354 | 0.8965252359 | 0.8965252366 |
| 0.9 | 0.9452211296 | 0.9452211304 | 0.9452211232 |
| 1.0 | 1.0000000000 | 0.9999999996 | 1.0000000000 |

**Table 4:** Green's function iterative method solution for problem (4)-(5) with $\beta = 0.2$ and $\psi = 0.5$

| $\zeta$ | GFIM | DTM [5] | RK4 [5] |
|---|---|---|---|
| 0.0 | 0.9034471796 | 0.9034471796 | 0.9034471816 |
| 0.1 | 0.9044037536 | 0.9044037536 | 0.9044037555 |
| 0.2 | 0.9072745703 | 0.9072745703 | 0.9072745722 |
| 0.3 | 0.9120629117 | 0.9120629117 | 0.9120629135 |
| 0.4 | 0.9187742375 | 0.9187742374 | 0.9187742391 |
| 0.5 | 0.9274161701 | 0.9274161700 | 0.9274161715 |
| 0.6 | 0.9379984742 | 0.9379984741 | 0.9379984755 |
| 0.7 | 0.9505330295 | 0.9505330293 | 0.9505330304 |
| 0.8 | 0.9650337980 | 0.9650337981 | 0.9650337986 |
| 0.9 | 0.9815167861 | 0.9815167860 | 0.9815167865 |
| 1.0 | 1.0000000000 | 0.9999999996 | 1.0000000000 |

It is noted from Tables 1 and 2 that the current method gives the exact solution whereas the DTM leaves a maximum error up to $6 \times 10^{-10}$. But in Example 2, where no exact solution exists, we notice from Tables 3 and 4, that the approximations obtained by the proposed method highly agree with those obtained by other methods.

The proposed method was further employed for the following studies:

1. The variation in temperature along the fin surface in the case of constant thermal conductivity ($\beta = 0$). It is noted from Fig. 2 that the dimensionless temperature increases as the thermo-geometric fin parameter increases.

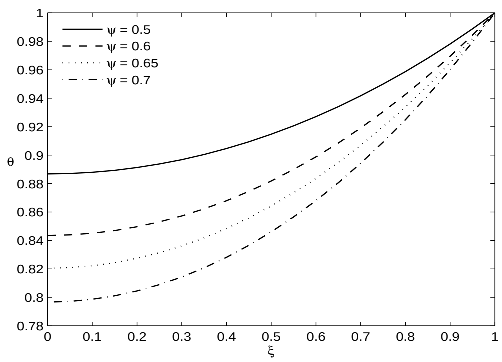


**Fig. 2:** Variation in dimensionless temperature in the case of constant thermal conductivity.

2. The variation in temperature along the fin surface in the case of variable thermal conductivity ($\beta \neq 0$). Fig. 3 shows a plot of the dimensionless temperature along the fin surface under the assumption that the the thermo-geometric parameter is fixed ($\psi = 0.5$). It is noted that temperature distribution increases as thermal conductivity increases.
3. Determining the fin efficiency. In Fig. 4, the fin efficiency, given in equation (7), is plotted against the thermo-geometric fin parameter for various thermal conductivity parameters.

## 5 Conclusion

In this study, a Green's function fixed point iterative method was formulated and applied for finding the temperature distribution along the fin surface and for determining the fin efficiency of convective straight fins with temperature-dependent thermal conductivity. The method is used to show the variation in temperature distribution in the cases of constant and variable thermal conductivity and to determine the fin efficiency over a given domain of the thermo-geometric fin parameter as the thermal conductivity parameter varies. Results obtained by this method were very convincing when compared to exact solutions and other numerical methods.

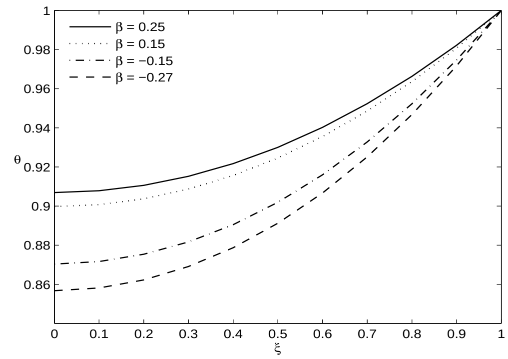


**Fig. 3:** Variation in dimensionless temperature in the case of variable thermal conductivity.

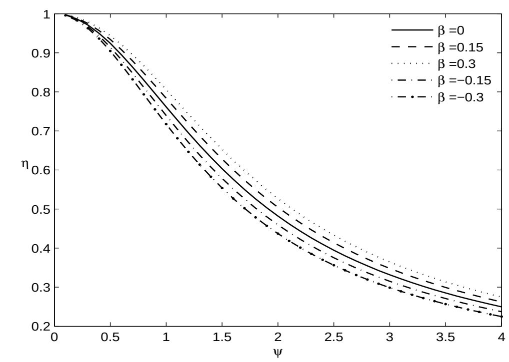


**Fig. 4:** Fin efficiency is plotted against the thermo-geometric fin parameter for various values of thermal conductivity.

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