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Ostrowski Type Inequalities for Functions Whose Derivatives are (m,h_1,h_2)-Convex

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Abstract: We prove new inequalities of Ostrowski type for functions with (m,h_1,h_2)-convex derivative. These generalize previous results on convex, s-convex and h-convex functions.

Keywords: Convex functions, convexity, m convexity, m h1 h2 convex functions

1 Introduction.

Let $I\subset\mathbb{R}$ and let $f:I\to\mathbb{R}$ be a function differentiable in I°, the interior of I, such that $f'\in\mathscr{L}[a,b]$, where $a,b\in I$ and $a<b$. If $|f'(x)|\le M$ for all $x\in I$, then the inequality

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|\le\frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]$$

holds, for all $x\in I$. This result is known in the literature as the *Ostrowski inequality*.
Recently, there have been generalizations of this inequality for several classes of functions such as bounded variation functions, lipschitzian, monotones, absolutely continuous, convex, s-convex and h-convex functions among others [2,5,1,8].
In this paper we present new inequalities of Ostrowski type for functions whose derivatives are (m,h_1,h_2)-convex.

Recall (c.f. [2,9]) that a real-valued function f defined in a real interval J is said to be be convex if for all $x,y\in J$ and for any $t\in(0,1)$

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y). \quad (1)$$

If inequality (1) is strict hen we say that f is *strictly convex*, and if inequality (1) is reversed the function f is said to be *concave*.

Definition 1([2,3,9]). *Let $0<s\le 1$. A function $f:[0,+\infty)\to\mathbb{R}$ is said to be s-convex in the first sense, or s_1-convex if*

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y),$$

for all $x,y\in[0,+\infty)$ and $\alpha,\beta\in(0,1)$ with $\alpha^s+\beta^s=1$. The function f is called s-convex in the second sense, or s_2-convex if

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y),$$

for all $x,y\in[0,+\infty)$ and $\alpha,\beta\in(0,1)$ with $\alpha+\beta=1$.

Throughout this paper, I and J denote intervals in $\mathbb{R}$ with $(0,1)\subset J$.

Definition 2.*[9,12] Let $h:J\to\mathbb{R}$ be a non negative function, not identically zero. A function $f:J\to\mathbb{R}$ is said to be h-convex if for all $x,y\in J$ and $t\in(0,1)$*

$$f(tx+(1-t)y)\le h(t)f(x)+h(1-t)f(y).$$

For the sake of completeness in our exposition and for the reader convenience we will present several results (without proofs) which will be needed to understand the setting of our main results. The primary references are [2] and [12].

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Lemma 1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ a function differentiable in I° where $a,b \in I$ with $a<b$. If $f' \in \mathscr{L}[a,b]$, then*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)du = (b-a)\int_0^1 p(t) f'(ta+(1-t)b)dt$$

holds, for all $t \in [0,1]$ where

$$p(t) = \begin{cases} t, & t \in [0, \frac{b-x}{b-a}] \,, \\ t-1, & t \in (\frac{b-x}{b-a}, 1] \end{cases}$$

for all $x \in [a,b]$.

Lemma 2. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ a function differentiable in I° where $a,b \in I$ and $a<b$. If $f' \in \mathscr{L}[a,b]$, then*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)du = \frac{(x-a)^2}{b-a}\int_0^1 t f'(tx+(1-t)a)dt - \frac{(b-x)^2}{b-a}\int_0^1 t f'(tx+(1-t)b)dt.$$

for all $x \in [a,b]$.

Theorem 1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable in I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a<b$. If $|f'|$ is convex in $[a,b]$, then*

$$|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \le \frac{b-a}{6}\Bigg[\left(4\left(\frac{b-x}{b-a}\right)^3 - 3\left(\frac{b-x}{b-a}\right)^2 + 1\right)|f'(a)| + \left(9\left(\frac{b-x}{b-a}\right)^2 - 4\left(\frac{b-x}{b-a}\right)^3 - 6\left(\frac{b-x}{b-a}\right) + 2\right)|f'(b)|\Bigg]$$

for all $x \in [a,b]$. The constant $\frac{1}{6}$ is the best possible, in the sense that it can not be replace for a smaller one.

Theorem 2. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable in I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a<b$. If $|f'|$ is s_2-convex for some $s \in (0,1]$ and $|f'(x)| \le M$, $x \in [a,b]$, then*

$$|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \le \frac{M}{b-a}\Big[\frac{(x-a)^2+(b-x)^2}{s+1}\Big],$$

for all $x \in [a,b]$.

Now we state Ostrowski type inequalities for certain h-convex functions.

Theorem 3. *Let $h: J$ be a non-negative and super-multiplicative[1] function, such that $h(\alpha) \ge \alpha$ for all $\alpha \in (0,1)$ and let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable in I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a<b$. If $|f'|$ is h-convex on I and $|f'(x)| \le M$, $x \in [a,b]$, then*

$$|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \le \frac{M[(x-a)^2+(b-x)^2]}{b-a}\int_0^1 [h(t^2)+h(t-t^2)]$$

for all $x \in [a,b]$.

The proof of this lemma can be found in [12].

Theorem 4. *Let $h: J$ be a non-negative and super-multiplicative[2] function and let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable in I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a<b$. If $|f'|^q$ is an h-convex function on $[a,b]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $h(t) \ge t$ and $|f'(x)| \le M$, $x \in [a,b]$, then*

$$|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \le \frac{Mh^{\frac{1}{q}}(1)}{b-a}\left(\int_0^1 (h(t^p)dt\right)^{\frac{1}{p}}((x-a)^2+(b-x)^2)$$

for all $x \in [a,b]$.

The proof of this lemma can be found in [12].

2 Basic properties of generalized convex functions.

In 2011 Maksa and Palés, [7], introduced a wider class of generalized convex functions as follows:
A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α, β, a, b)-convex if it satisfies the inequality

$$f(\alpha(t)x+\beta(t)y) \le a(t)f(x)+b(t)f(y), \quad x,y \in I,$$

where $\alpha, \beta, a, b: J \subseteq [0,1] \to \mathbb{R}$ are given functions.

Then in 2014 Shi, Xi and Qi gave the following particular definition

Definition 3([10]). *Let $h_1, h_2: [0,1] \to \mathbb{R}_+$ and $m \in (0,1]$. A function $f: I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be (m,h_1,h_2)-convex if*

$$f(tx+m(1-t)y) \le h_1(t)f(x)+mh_2(t)f(y) \qquad (2)$$

holds, for all $x,y \in I$ and $t \in [0,1]$. If the inequality is reversed f is said to be (m,h_1,h_2)-concave.

Remark.

(1) If $m=1$, $h_1(t)=t$ and $h_2(t)=1-t$ in (2) then we get the classical notion of convexity.

[1] $h(xy) \ge h(x)h(y)$ all $x,y \in J$ such that $xy \in J$

[2] $h(x+y) \ge h(x)+h(y)$ all $x,y \in J$ such that $x+y \in J$

(2)If $h_1(t) = t$ and $h_2(t) = 1-t$ in (2) we get the notion of m-convex function, c.f. [11, 6].
(3)If $m = 1$, $h_1(t) = t^s$ y $h_2(t) = 1-t^s$ in (2) then we obtain the notion of s_1-convex function.
(4)If $m = 1$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ in (2) then we obtain the notion of s_2-convex function.
(5)If $h_1(t) = 1$ y $h_2(t) = \dfrac{1}{m}$ in (2) then we obtain the notion of P-convex function.
(6)if $m = 1$ and $h_2(t) = h_1(1-t)$ in (2) then we obtain the notion of h-convex function.
(7)If $m = 1$, $h_1(t) = t^{-s}$ and $h_2(t) = (1-t)^{-s}$ in (2) then we have the notion of s-Godunova-Levin convex function.

*Example 1.*All MT-convex functions, defined in [13], are (m,h_1,h_2)-convex for $m = 1$, $h_1(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $h_2(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$.

Proposition 1.*Let $h_1,h_2,h_3,h_4 : [0,1] \to \mathbb{R}_+$ be functions and let $m \in (0,1]$. If $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is (m,h_1,h_2)-convex and $g : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is (m,h_3,h_4)-convex, then $f+g$ is (m,h_5,h_6)-convex where $h_5 = \max\{h_1,h_3\}$ and $h_6 = \max\{h_2,h_4\}$.*

*Proof.*If $x,y \in I$ and $t \in [0,1]$ then:

$$\begin{aligned}
&(f+g)(tx+m(1-t)y),\\
&= f(tx+m(1-t)y)+g(tx+m(1-t)y)\\
&\le h_1(t)f(x)+mh_2(t)f(y)+h_3(t)g(x)+mh_4(t)g(y)\\
&\le h_5(t)f(x)+mh_6(t)f(y)+h_5(t)g(x)+mh_6(t)g(y)\\
&= h_5(t)(f(x)+g(x))+mh_6(t)(f(y)+g(y))\\
&= h_5(t)(f+g)(x)+mh_6(t)(f+g)(y)
\end{aligned}$$

which proves the result.

Proposition 2.*Let $h_1,h_2 : [0,1] \to \mathbb{R}_+$ be functions and let $m \in (0,1]$. If $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is (m,h_1,h_2)-convex, then αf is (m,h_1,h_2)-convex for all $\alpha \in \mathbb{R}_+$.*

*Proof.*If $x,y \in I$ y $t \in [0,1]$ then:

$$\begin{aligned}
(\alpha f)(tx+m(1-t)y) &= \alpha f(tx+m(1-t)y)\\
&\le \alpha(h_1(t)f(x)+mh_2(t)f(y))\\
&= h_1(t)(\alpha f)(x)+mh_2(t)(\alpha f)(y)).
\end{aligned}$$

The proof is complete.

□

Proposition 3.*Let $q_n,r_n : [0,1] \to \mathbb{R}_+$ two sequences of functions that converge pointwise to h_1 y h_2 respectively, and let $m \in (0,1]$. If f_n is a sequence of functions that converge pointwise to f and each $f_k : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is (m,q_k,r_k)-convex, for all $k > n_0$, with $n_0 \in \mathbb{Z}_+$, then f is (m,h_1,h_2)-convex.*

*Proof.*Indeed, if $x,y \in I$ y $t \in [0,1]$ then:

$$\begin{aligned}
&f(tx+m(1-t)y)\\
&= \lim_{n\to+\infty} f_n(tx+m(1-t)y)\\
&= \lim_{k\to+\infty} f_{n_0+k}(tx+m(1-t)y)\\
&= \lim_{k\to+\infty} f_k(tx+m(1-t)y)\\
&\le \lim_{k\to+\infty} q_k(t)f_k(t)+mr_k(t)f_k(y)\\
&= h_1(t)f(x)+mh_2(t)f(y).
\end{aligned}$$

proving the result.

Theorem 5.*Let $f : [0,+\infty) \to \mathbb{R}$ a finite function on $\left[ma, \dfrac{b}{m}\right] \subseteq [0,+\infty)$, $(m,h_1,h_2)-$convex with $m \in (0,1]$ and there is a M such that $|h_i(t)| \le M$. Then f is bounded on any closed interval $[a,b]$.*

*Proof.*Let $x \in [a,b]$, then there is a $t \in [0,1]$ such that $x = ta+(1-t)b$, we have

$$\begin{aligned}
f(x) &= f(ta+(1-t)b)\\
&\le h_1(t)f(a)+mh_2(t)f\left(\frac{b}{m}\right)\\
&\le M\left(f(a)+f\left(\frac{b}{m}\right)\right).
\end{aligned}$$

Thus, f is upper bounded on $[a,b]$.

Now we notice that any $x \in [a,b]$ can be written as $\dfrac{a+b}{2}+t$ for $|t| \le \dfrac{b-a}{2}$,

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right)\\
&\le h_1\left(\frac{1}{2}\right)f\left(\frac{a+b}{2}+t\right)+mh_2\left(\frac{1}{2}\right)f\left(\frac{\frac{a+b}{2}-t}{m}\right).
\end{aligned}$$

Applying the first part of this theorem , we obtain that

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)-mh_2\left(\frac{1}{2}\right)M\\
&\le f\left(\frac{a+b}{2}\right)-mh_2\left(\frac{1}{2}\right)f\left(\frac{\frac{a+b}{2}-t}{m}\right)\\
&\le h_1\left(\frac{1}{2}\right)f\left(\frac{a+b}{2}+t\right)=h_1\left(\frac{1}{2}\right)f(x).
\end{aligned}$$

Which we obtain that

$$\left[h_1\left(\frac{1}{2}\right)\right]^{-1}\left(f\left(\frac{a+b}{2}\right)-mh_2\left(\frac{1}{2}\right)M\right) \le f(x)$$

3 Main results.

In this section we establish some Ostrowski type inequalities for functions with (m,h_1,h_2)-convex derivative. These, generalize previous ones.

Theorem 6. *Let $f : I \to \mathbb{R}$ be a function differentiable on I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a < b$. If $|f'|$ is (m,h_1,h_2)-convex in $[a,b]$ then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq (b-a)|f'(a)|\Big[\int_0^{\frac{b-x}{b-a}} th_1(t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t)h_1(t)dt\Big]$$
$$+\, m(b-a)|f'\left(\frac{b}{m}\right)|\Big[\int_0^{\frac{b-x}{b-a}} th_2(t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t)h_2(t)dt\Big]$$

for all $x \in [a,b]$.

Proof. Using Lemma (1) and definition (2) we have que:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq (b-a)\int_0^{\frac{b-x}{b-a}} t|f'(ta+(1-t)b)|dt$$
$$+(b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)|f'(ta+(1-t)b)|dt$$
$$= (b-a)\int_0^{\frac{b-x}{b-a}} t|f'\left(ta+m(1-t)\frac{b}{m}\right)|dt$$
$$+(b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left|f'\left(ta+m(1-t)\frac{b}{m}\right)\right|dt$$
$$\leq (b-a)\int_0^{\frac{b-x}{b-a}} t\left(h_1(t)|f'(a)|+mh_2(t)\left|f'\left(\frac{b}{m}\right)\right|\right)dt$$
$$+(b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)\left(h_1(t)|f'(a)|+mh_2(t)\left|f'\left(\frac{b}{m}\right)\right|\right)dt$$
$$= (b-a)\left[|f'(a)|\int_0^{\frac{b-x}{b-a}} th_1(t)dt+m\left|f'\left(\frac{b}{m}\right)\right|\int_0^{\frac{b-x}{b-a}} th_2(t)dt\right]$$
$$+\,(b-a)\left[|f'(a)|\int_{\frac{b-x}{b-a}}^1 (1-t)h_1(t)dt\right.$$
$$\left.+m\left|f'\left(\frac{b}{m}\right)\right|\int_{\frac{b-x}{b-a}}^1 (1-t)h_2(t)dt\right]$$
$$= (b-a)|f'(a)|\left[\int_0^{\frac{b-x}{b-a}} th_1(t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t)h_1(t)dt\right]$$
$$+m(b-a)\left|f'\left(\frac{b}{m}\right)\right|\left[\int_0^{\frac{b-x}{b-a}} th_2(t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t)h_2(t)dt\right]$$

which proves the result.

Theorem 7. *Let $h_1,h_2 : [0,1] \to \mathbb{R}_+$ be non-negative functions and let $f : I \to \mathbb{R}$ be a function differentiable in I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a < b$. Suppose $m \in (0,1]$. If $|f'|$ is a (m,h_1,h_2)-convex function on I and $|f'(x)| \leq M$, $x \in [a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(h_1(t)+mh_2(t))dt$$

holds for all $x \in [a,b]$.

Proof. Using Lemma (2) and definition (2) we have:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 t|f'(tx+(1-t)a)|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 t|f'(tx+(1-t)b)|dt$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 t|f'\left(tx+m(1-t)\frac{a}{m}\right)|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 t|f'\left(tx+m(1-t)\frac{b}{m}\right)|dt$$
$$\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left(h_1(t)|f'(x)|\right.$$
$$\left.+\,mh_2(t)|f'\left(\frac{a}{m}\right)|\right)dt$$
$$+\,\frac{(b-x)^2}{b-a}\int_0^1 t\left(h_1(t)|f'(x)|+mh_2(t)|f'\left(\frac{b}{m}\right)|\right)dt$$
$$\leq M\frac{(x-a)^2}{b-a}\int_0^1 t(h_1(t)+mh_2(t))dt$$
$$+M\frac{(b-x)^2}{b-a}\int_0^1 t(h_1(t)+mh_2(t))dt$$
$$= \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(h_1(t)+mh_2(t))dt,$$

which proves the result.

Theorem 8. *Let $h_1,h_2 : [0,1] \to \mathbb{R}_+$ be non-negative functions and let $f : I \to \mathbb{R}$ be a function differentiable in I° and such that $f' \in \mathscr{L}[a,b]$ where $a,b \in I$ and $a < b$. Suppose $m \in (0,1]$. If $|f'|^q$ is a (m,h_1,h_2)-convex function on $[a,b]$, $p,q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [a,b]$, then*

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\int_0^1 (h_1(t)+mh_2(t))dt\right]^{\frac{1}{q}}\frac{((x-a)^2+(b-x)^2)}{b-a}.$$

for all $x \in [a,b]$.

*Proof.*Suppose $p > 1$. Using Lemma (2) and Hölders inequality we have:

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$=\frac{(x-a)^2}{b-a}\int_0^1 t|f'(tx+(1-t)a)|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 t|f'(tx+(1-t)b)|dt$$
$$\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}.$$

On the other hand, since $|f'|^q$ is (m,h_1,h_2)-convex:

$$\int_0^1 |f'(tx+(1-t)a)|^q dt$$
$$=\int_0^1 |f'\left(tx+m(1-t)\frac{a}{m}\right)|^q dt$$
$$\le \int_0^1 \left(h_1(t)|f'(x)|^q+mh_2(t)|f'\left(\frac{a}{m}\right)|^q\right)dt$$
$$\le M^q\int_0^1 (h_1(t)+mh_2(t))dt.$$

Similarly, we obtain:

$$\int_0^1 |f'(tx+(1-t)b)|^q dt \le M^q\int_0^1 (h_1(t)+mh_2(t))dt.$$

Hence

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left[M^q\int_0^1 (h_1(t)+mh_2(t))dt\right]^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left[M^q\int_0^1 (h_1(t)+mh_2(t))dt\right]^{\frac{1}{q}}$$
$$=M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\int_0^1 (h_1(t)+mh_2(t))dt\right]^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{b-a}$$

and the proof is complete.

*Proof.*by Theorem 6 we have:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\le (b-a)|f'(a)|\left[\int_0^{\frac{b-x}{b-a}} th_1(t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t)h_1(t)dt\right]$$
$$+m(b-a)\left|f'\left(\frac{b}{m}\right)\right|\left[\int_0^{\frac{b-x}{b-a}} th_2(t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t)h_2(t)dt\right]$$
$$\le (b-a)|f'(a)|\left[\int_0^{\frac{b-x}{b-a}} t.tdt+\int_{\frac{b-x}{b-a}}^1 (1-t).tdt\right]$$
$$+m(b-a)|f'\left(\frac{b}{m}\right)|\left[\int_0^{\frac{b-x}{b-a}} t.(1-t)dt+\int_{\frac{b-x}{b-a}}^1 (1-t).(1-t)dt\right]$$
$$=(b-a)|f'(a)|\left[\int_0^{\frac{b-x}{b-a}} t^2dt+\int_{\frac{b-x}{b-a}}^1 (t-t^2)dt\right]$$
$$+m(b-a)|f'\left(\frac{b}{m}\right)|\left[\int_0^{\frac{b-x}{b-a}} (t-t^2)dt+\int_{\frac{b-x}{b-a}}^1 (1-2t+t^2)dt\right]$$
$$=(b-a)|f'(a)|\left[\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3+\frac{1}{6}-\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2+\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right]$$
$$+m(b-a)|f'\left(\frac{b}{m}\right)|\left[\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2-\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right.$$
$$\left.+\frac{1}{3}-\frac{b-x}{b-a}+\left(\frac{b-x}{b-a}\right)^2-\frac{1}{3}\left(\frac{b-x}{b-a}\right)^3\right]$$
$$=(b-a)|f'(a)|\left[\frac{2}{3}\left(\frac{b-x}{b-a}\right)^3-\frac{1}{2}\left(\frac{b-x}{b-a}\right)^2+\frac{1}{6}\right]$$
$$+m(b-a)|f'\left(\frac{b}{m}\right)|\left[-\frac{2}{3}\left(\frac{b-x}{b-a}\right)^3+\frac{3}{2}\left(\frac{b-x}{b-a}\right)^2-\frac{b-x}{b-a}+\frac{1}{3}\right]$$
$$=\frac{(b-a)}{6}|f'(a)|\left[4\left(\frac{b-x}{b-a}\right)^3-3\left(\frac{b-x}{b-a}\right)^2+1\right]$$
$$+m\frac{(b-a)}{6}|f'\left(\frac{b}{m}\right)|\left[9\left(\frac{b-x}{b-a}\right)^2-4\left(\frac{b-x}{b-a}\right)^3-6\left(\frac{b-x}{b-a}\right)+2\right]$$
$$=\frac{(b-a)}{6}\left[|f'(a)|\left(4\left(\frac{b-x}{b-a}\right)^3-3\left(\frac{b-x}{b-a}\right)^2+1\right)\right.$$
$$\left.+m\left|f'\left(\frac{b}{m}\right)\right|\left(9\left(\frac{b-x}{b-a}\right)^2-4\left(\frac{b-x}{b-a}\right)^3-6\left(\frac{b-x}{b-a}\right)+2\right)\right]$$

and the proof is complete.

4 Applications

Theorem 9.*If* $\max\{h_1(t),h_2(t)\}\le\min\{t,1-t\}\ \forall t\in[0,1]$, *then the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\le \frac{b-a}{6}\left[|f'(a)|\left(4\left(\frac{b-x}{b-a}\right)^3-3\left(\frac{b-x}{b-a}\right)^2+1\right)\right.$$
$$\left.+m|f'\left(\frac{b}{m}\right)|\left(9\left(\frac{b-x}{b-a}\right)^2-4\left(\frac{b-x}{b-a}\right)^3-6\left(\frac{b-x}{b-a}\right)+2\right)\right].$$

holds, for all $x\in[a,b]$.

*Remark.*If $m=1$ then we obtain the result of Theorem 1.

Theorem 10.*Let* $s\in(0,1]$. *If* $\max\{h_1(t),h_2(t)\}\le\min\{t^s,(1-t)^s\}\ \forall t\in[0,1]$, *then*

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\le \frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\frac{s+m+1}{(s+1)(s+2)}\right).$$

*Proof.*By Theorem (7):

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(h_1(t)+mh_2(t))dt$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(t^s+m(1-t)^s)dt$$
$$=\frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\int_0^1 t^{s+1}dt+m\int_0^1 t(1-t)^s\right)dt$$
$$=\frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\frac{1}{s+2}+\frac{m}{(s+1)(s+2)}\right)$$
$$=\frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\frac{s+1+m}{(s+1)(s+2)}\right).$$

The proof is complete.

*Remark.*If $m=1$ then we get the result of Theorem 2.

Theorem 11.*Let $s\in(0,1]$. If $\max\{h_1(t),h_2(t)\}\leq\min\{t^s,1-t^s\}$ $\forall t$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\frac{2+ms}{2(s+2)}\right).$$

*Proof.*By Theorem (7):

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(h_1(t)+mh_2(t))dt$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(t^s+m(1-t^s))dt$$
$$=\frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\int_0^1 t^{s+1}dt+m\int_0^1 t(1-t^s)\right)dt$$
$$=\frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\frac{1}{s+2}+\frac{ms}{2(s+2)}\right)$$
$$=\frac{M}{b-a}((x-a)^2+(b-x)^2)\left(\frac{2+ms}{2(s+2)}\right)$$

finishing the proof.

*Remark.*If $m=1$ we get the classical Ostrowski inequality.

Theorem 12.*If $h_2(t)$ is a super-multiplicative function and $h_2(t)\geq t$ $\forall t\in[0,1]$, then*

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{M[(x-a)^2+(b-x)^2]}{b-a}\int_0^1(h_1(t)h_2(t)+mh_2(t^2))dt.$$

*Proof.*By Theorem (7) we have:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 t(h_1(t)+mh_2(t))dt$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1 h_2(t)(h_1(t)+mh_2(t))dt$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1(h_2(t)h_1(t)+mh_2(t)h_2(t))dt$$
$$\leq \frac{M}{b-a}((x-a)^2+(b-x)^2)\int_0^1(h_2(t)h_1(t)+mh_2(t^2))dt.$$

which proves the result.

Theorem 13.*If, in addition, $h_1(t)=h_2(1-t)$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{M[(x-a)^2+(b-x)^2]}{b-a}\int_0^1(h_2(t-t^2)+mh_2(t^2))dt$$

*Proof.*By Corollary (12) and the fact that $h_2(t)$ is super-multiplicative, we have:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{M[(x-a)^2+(b-x)^2]}{b-a}\int_0^1(h_1(t)h_2(t)+mh_2(t^2))dt.$$
$$=\frac{M[(x-a)^2+(b-x)^2]}{b-a}\int_0^1(h_2(1-t)h_2(t)+mh_2(t^2))dt.$$
$$=\frac{M[(x-a)^2+(b-x)^2]}{b-a}\int_0^1(h_2(t-t^2)+mh_2(t^2))dt.$$

The proof is complete.

*Remark.*If $m=1$ the we get the result of Theorem 3.

Theorem 14.*If h_2 is super-additive, $h_1(t)=h_2(1-t)$ and $m=1$, then*

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{Mh_2^{\frac{1}{q}}(1)}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{q}}[(x-a)^2+(b-x)^2].$$

*Proof.*By Theorem 8 we have:

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq M\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left[\int_0^1(h_1(t)+mh_2(t))dt\right]^{\frac{1}{q}}\frac{((x-a)^2+(b-x)^2)}{b-a},$$

On the other hand, using the fact that $h_1(t)=h_2(1-t)$, that h_2 is a super-additive and $m=1$, we have

$$\int_0^1(h_1(t)+mh_2(t))dt=\int_0^1 h_2(1)dt=h_2(1),$$

therefore:

$$|f(x)-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{Mh_2^{\frac{1}{q}}(1)}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{q}}[(x-a)^2+(b-x)^2]$$
$$=\frac{Mh_2^{\frac{1}{q}}(1)}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{q}}[(x-a)^2+(b-x)^2],$$

finishing the proof.

Theorem 15. *If in addition $h_2(t)\geq t\ \forall t\in[0,1]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{Mh_2^{\frac{1}{q}}(1)}{b-a}\left(\int_0^1 h_2(t^p)dt\right)^{\frac{1}{q}}[(x-a)^2+(b-x)^2].$$

Proof. It readily follows from Corollary 14 and the inequality $h(t^p)\geq t^p$.

Remark. Notice that the result of the last corollary is the same of Theorem 4.

5 Conclusions

The main contributions of this paper has been the introduction of a new class of function of generalized convexity, functions with (m,h_1,h_2)-convex derivative,we have shown that these classes contain some previously known classes as special cases as well as Ostrowski type inequalities for these functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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