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Connected Edge Fixed Monophonic Number of a Graph

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Abstract: For an edge xy in a connected graph G of order $p \geq 3$, a set $S \subseteq V(G)$ is an xy-monophonic set of G if each vertex $v \in V(G)$ lies on either an $x-u$ monophonic path or a $y-u$ monophonic path for some element u in S. The minimum cardinality of an xy-monophonic set of G is defined as the xy-monophonic number of G, denoted by $m_{xy}(G)$. An xy-monophonic set of cardinality $m_{xy}(G)$ is called a m_{xy}-set of G. A connected xy-monophonic set of G is an xy-monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected xy-monophonic set of G is the connected xy-monophonic number of G and is denoted by $cm_{xy}(G)$. A connected xy-monophonic set of cardinality $cm_{xy}(G)$ is called a cm_{xy}-set of G. We determine bounds for it and find the same for some special classes of graphs. If d, n and $p \geq 4$ are positive integers such that $2 \leq d \leq p-2$ and $1 \leq n \leq p-1$, then there exists a connected graph G of order p, monophonic diameter d and $cm_{xy}(G) = n$ for some edge xy in G. Also, we give some characterization and realization results for the parameter $cm_{xy}(G)$.

Keywords: monophonic path, edge fixed monophonic set, edge fixed monophonic number, connected edge fixed monophonic set, connected edge fixed monophonic number

1 Introduction

By a graph $G = (V,E)$ we mean a finite undirected connected graph without loops and multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to [1,2]. For vertices x and y in a connected graph G, the *distance* $d(x,y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x,y)$ is called an $x-y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete. A $leaf$ of a graph is a bridge with the degree of one of its vertex is one. If G and H are two graphs, then the *join* $G+H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. The *closed interval* $I_m[x,y]$ consists of all vertices lying on some $x-y$ monophonic path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v,u) : u \in V(G)\}$. The *monophonic radius*, $\text{rad}_m(G)$ of G is $\text{rad}_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $\text{diam}_m(G)$ of G is $\text{diam}_m(G) = \max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [3] and further studied in [4].

The edge fixed concept of a graph was introduced by Santhakumaran and Titus in 2009. Let xy be any edge of G. A set S of vertices of G is an *xy-geodominating set* if every vertex of G lies on either an $x-u$ geodesic or a $y-u$ geodesic for some element u in S. The minimum cardinality of an xy-geodominating set of G is defined as the *xy-geodomination number* of G and is denoted by $g_{xy}(G)$. An xy-geodominating set of cardinality $g_{xy}(G)$ is called a *g_{xy}-set* of G. The edge fixed geodomination number was introduced and studied in [5].

The concept of edge fixed monophonic number was introduced by Titus and Eldin Vanaja [6]. A set S of vertices of G is an *xy-monophonic set* if every vertex of G lies on either an $x-u$ monophonic path or a $y-u$ monophonic path for some element u in S. The minimum cardinality of an xy-monophonic set of G is defined as the *xy-monophonic number* of G and is denoted by $m_{xy}(G)$. An xy-monophonic set of cardinality $m_{xy}(G)$ is called a *m_{xy}-set* of G. Edge fixed monophonic sets have interesting

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applications in channel assignment problem in radio technologies, molecular problems in theoretical chemistry and designing the channel for a communication network.

The following theorems will be used in the sequel.

Theorem 1.[2] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i)*The vertex v is a cut vertex of G.*
(ii)*There exist vertices u and w distinct from v such that v is on every $u-w$ path.*
(iii)*There exists a partition of the set of vertices $V-\{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u-w$ path.*

Theorem 2.[2] *Let G be a connected graph with at least three vertices. Then G is a block if and only if every two vertices of G lie on a common cycle.*

Theorem 3.[6] *Let xy be any edge of a connected graph G of order at least three.*

(i)*Every simplicial vertex of G other than the vertices x and y belongs to every m_{xy}-set.*
(ii)*No cut vertex of G belongs to any m_{xy}-set.*

Throughout this paper G denotes a connected graph with at least three vertices.

2 Connected edge fixed monophonic number

Definition 1. *Let xy be any edge of a connected graph G of order at least three. A connected xy-monophonic set of G is an xy-monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected xy-monophonic set of G is the connected xy-monophonic number of G and is denoted by $cm_{xy}(G)$. A connected xy-monophonic set of cardinality $cm_{xy}(G)$ is called a cm_{xy}-set of G.*

Example 1. For the graph G given in Figure 1, the minimum edge fixed monophonic sets, the edge fixed monophonic numbers, the minimum connected edge fixed monophonic sets and the connected edge fixed monophonic numbers are given in Table 1.

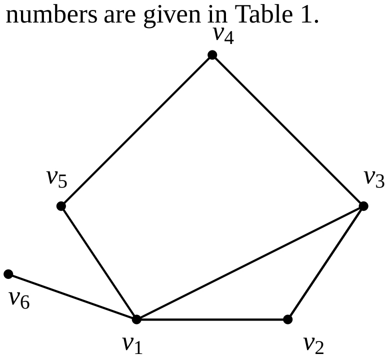

Figure 1: A graph G for connected edge fixed monophonic numbers

Table 1: The connected edge fixed monophonic numbers of the graph G given in Figure 1

Edge e	m_e-sets	$m_e(G)$	cm_e-sets	$cm_e(G)$
v_1v_2	$\{v_4,v_6\},\{v_5,v_6\}$	2	$\{v_5,v_1,v_6\}$	3
v_2v_3	$\{v_4,v_6\},\{v_5,v_6\}$	2	$\{v_5,v_1,v_6\}$	3
v_3v_4	$\{v_2,v_6\}$	2	$\{v_2,v_1,v_6\}$	3
v_4v_5	$\{v_2,v_6\}$	2	$\{v_2,v_1,v_6\}$	3
v_5v_1	$\{v_2,v_6\}$	2	$\{v_2,v_1,v_6\}$	3
v_1v_6	$\{v_2,v_4\}$	2	$\{v_2,v_3,v_4\}$	3
v_1v_3	$\{v_2,v_6,v_4\}$, $\{v_2,v_6,v_5\}$	3	$\{v_2,v_1,v_6,v_5\}$	4

We observe that for any edge xy in G, the vertices x and y do not belong to any m_{xy}-set of G, where as x or y may belong to a cm_{xy}-set of G. For the graph G given in Figure 1, the vertex v_1 is an element of a $cm_{v_1v_2}$-set.

In the following theorem we establish the relationship between the edge fixed monophonic number and the connected edge fixed monophonic number of a graph G.

Theorem 4. *For any edge xy in G, $m_{xy}(G) \leq cm_{xy}(G)$.*

Theorem 5. *Let xy be any edge of a connected graph G. If $z \notin \{x,y\}$ is a simplicial vertex of G, then z belongs to every connected xy-monophonic set of G.*

Proof. Since every connected xy-monophonic set is an xy-monophonic set, the result follows from Theorem 3 (i).

Theorem 6. (i)*For the complete graph K_p $(p \geq 3)$, $cm_{xy}(K_p) = p-2$ for any edge xy in K_p.*
(ii)*For any cycle C_p, $cm_{xy}(C_p) = 1$ for every edge xy in C_p.*
(iii)*For the wheel $W_p = K_1 + C_{p-1} (p \geq 5)$, $cm_{xy}(W_p) = 1$ for any edge xy in W_p.*
(iv)*For any edge xy in the complete bipartite graph $K_{m,n} (2 \leq m \leq n)$, $cm_{xy}(K_{m,n}) = 1$ or 2 according as $m = 2$ or $m > 2$.*

Proof.(i) For any edge xy in K_p, let $S = V(K_p) - \{x,y\}$. Since every vertex of K_p is a simplicial vertex, it follows from Theorem 5 that $cm_{xy}(K_p) \geq |S| = p-2$. It is clear that S is the connected xy-monophonic set of K_p and so $cm_{xy}(K_p) = p-2$.
(ii) Let xy be any edge of a cycle C_p. If $p = 3$, then $C_3 = K_3$ and so by (i), $cm_{xy}(C_p) = 1$. Let z be a vertex different of x and y. Clearly every vertex of C_p lies on an $x-z$ monophonic path or a $y-z$ monophonic path and so $\{z\}$ is a connected xy-monophonic set of C_p. Hence $cm_{xy}(C_p) = 1$.
(iii) Let xy be any edge in W_p. Then either x or y is a vertex of C_{p-1}. Let $x \in V(C_{p-1})$ and let z be a non-adjacent vertex of x in C_{p-1}. It is clear that every vertex of W_p lies on an $x-z$ monophonic path. Hence $\{z\}$ is a connected xy-monophonic set of W_p and so $cm_{xy}(W_p) = 1$.
(iv) Let xy be any edge in $K_{m,n}$. Then x and y belong to different partitions, say $x \in V_1$ and $y \in V_2$.
Case 1. $m = 2$. Let $z \neq x$ be the other vertex in V_1. Then any vertex of V_2 lies on an $x-z$ monophonic path and so

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$\{z\}$ is a connected xy-monophonic set of $K_{2,n}$. Hence $cm_{xy}(K_{2,n}) = 1$.
Case 2. $m > 2$. Let $z_1 \neq x$ be a vertex in V_1 and let $z_2 \neq y$ be a vertex in V_2. Then any vertex in V_2 lies on an $x - z_1$ monophonic path and any vertex in V_1 lies on an $y - z_2$ monophonic path. It is clear that $\{z_1, z_2\}$ is a connected xy-monophonic set of $K_{m,n}$ and so $cm_{xy}(K_{m,n}) = 2$.

In Theorem 6 (iv), if $m = 1$, then $K_{m,n} = K_{1,n}$ is a star. Since a star is a tree, we can find the connected edge fixed monophonic number using the following theorem.

Theorem 7. *(i)If T is any tree of order p, then $cm_{xy}(T) = p$ for any non-leaf xy of T.*
(ii)If T is any tree of order p which is not a path, then for a leaf xy with end vertex x, $cm_{xy}(T) = p - d_m(x,u)$, where u is the vertex of T with $\deg(u) \geq 3$ such that $d_m(x,u)$ is minimum.
(iii)If T is a path, then $cm_{xy}(T) = 1$ for a leaf xy of T.

Proof.(i) Let xy be a non-leaf of T and let S be any connected xy-monophonic set of T. By Theorem 5, every connected xy-monophonic set of T contains all simplicial vertices. If $S \neq V(T)$, then there exists a cut vertex v of T such that $v \notin S$. Let u and w be two end vertices belonging to different components of $T - \{v\}$. Since v lies on the unique path (monophonic path) joining u and w, it follows that the subgraph $G[S]$ induced by S is not connected, which is a contradiction. Hence $cm_{xy}(T) = p$.
(ii) Let T be a tree which is not a path and let xy be a leaf of T with end vertex x. Also let u be the vertex of T with $\deg(u) \geq 3$ such that $d_m(x,u)$ is minimum. Let $S = (V(T) - I_m[x,u]) \cup \{u\}$. Clearly S is a connected xy-monophonic set of T and hence $cm_{xy}(T) \leq |S| = p - d_m(x,u)$. We claim that $cm_{xy}(T) = p - d_m(x,u)$. Otherwise, there is a connected xy-monophonic set M of T with $|M| < p - d_m(x,u)$. By Theorem 5, every connected xy-monophonic set of T contains all simpilical vertices other than x and y, and hence there exists a cut vertex v of T such that $v \in S$ and $v \notin M$. Let $B_1, B_2, \ldots, B_l$ $(l \geq 3)$ be the components of $T - \{u\}$. Assume that x belongs to B_1.
Case 1. Suppose that $v = u$. Let $z \in B_2$ and $w \in B_3$ be two end vertices of T. By Theorem 1, v lies on the $z - w$ monophonic path. Since z and w belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.
Case 2. Suppose that $v \neq u$. Let $v \in B_i$ $(i \neq 1)$. Now we can choose an end vertex $s \in B_i$ such that v lies on the $u - s$ monophonic path. Let $a \in B_j$ $(j \neq i, 1)$ be an end vertex of T. By Theorem 1, u lies on an $s - a$ monophonic path. Since s and a belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.
(iii) Let T be a path. For a leaf xy in T with end vertex x, let z be the other end vertex of T. Clearly every vertex of T lies on the $x - z$ monophonic path and so $\{z\}$ is a connected xy-monophonic set of T. Hence $cm_{xy}(T) = 1$.

Theorem 8. *For any edge xy in a connected graph G, $1 \leq cm_{xy}(G) \leq p$.*

Proof. Since $V(G)$ induces a connected xy-monophonic set of G, we have $cm_{xy}(G) \leq p$. Also, it is clear that $cm_{xy}(G) \geq 1$. Hence $1 \leq cm_{xy}(G) \leq p$.

The following theorem is clear from the definition of connected edge fixed monophonic number and Theorem 4.

Theorem 9. *For any edge xy in a connected graph G, $cm_{xy}(G) = 1$ if and only if $m_{xy}(G) = 1$.*

Definition 2. *An edge xy in a connected graph G is called an extreme connected edge if $G - \{x,y\}$ is connected.*

Theorem 10. *There is no graph G of order p with $cm_{xy}(G) = p$ for every edge xy in G.*

Proof. Every connected graph G contains either a leaf or an extreme connected edge. If xy is a leaf with x is an end vertex of G, then $S = V(G) - \{x\}$ is a connected xy-monophonic set of G and so $cm_{xy}(G) \leq |S| = p - 1$. If xy is an extreme connected edge of G, then $S = V(G) - \{x,y\}$ is a connected xy-monophonic set of G and so $cm_{xy}(G) \leq |S| = p - 2$. Hence there is no graph G with $cm_{xy}(G) = p$ for any edge xy in G.

Theorem 11. *Let G be a connected graph of order p with at most one cut vertex. Then $cm_{xy}(G) = p - 1$ or $p - 2$ for every edge xy in G if and only if $G = K_1 + \cup m_j K_j$ with $\sum m_j \neq 2$ and $K_j \neq K_2$.*

Proof. Let $G = K_1 + \cup m_j K_j$ with $\sum m_j \neq 2$ and $K_j \neq K_2$. Suppose that G has no cut vertex. Then $G = K_p$ and hence by Theorem 6 (i), $cm_{xy}(G) = p - 2$ for any edge xy in G. Suppose that G has exactly one cut vertex, say z. Then we have two cases.
Case 1. $\sum m_j = 2$ and $K_j \neq K_2$. Then G is either $K_1 + (K_1 \cup K_1) = P_3$ or $K_1 + (K_1 \cup K_s)$, or $K_1 + (K_r \cup K_s)$, where $r, s \geq 3$. If $G = P_3$, then $cm_{xy}(G) = 1 = p - 2$ for any edge xy in G. If $G = K_1 + (K_1 \cup K_s)$, then G has one leaf incident with z, $'s'$ non-leafs incident with z, and all the remaining edges are not incident with z. It is easily verified that, if xy is either a leaf or an edge not incident with z, then $cm_{xy}(G) = p - 2$ and if xy is an edge incident with z but not a leaf, then $cm_{xy}(G) = p - 1$. Hence $cm_{xy}(G) = p - 1$ or $p - 2$ for any edge xy in G.
Case 2. $\sum m_j > 2$. Then $G - z$ has at least three components. If xy is an edge incident with z, then $cm_{xy}(G) = p - 1$ and if xy is an edge not incident with z, then $cm_{xy}(G) = p - 2$. Hence $cm_{xy}(G) = p - 1$ or $p - 2$ for any edge xy in G.
Conversely, suppose that $cm_{xy}(G) = p - 1$ or $p - 2$ for any edge xy in G. Since G has at most one cut vertex, we have two cases.
Case 1. G has no cut vertex. Then G is a block. If $p = 3$, then $G = K_3 = K_1 + K_2$. If $p \geq 4$, we claim that G is complete. Suppose that G is not complete. Then there exit two vertices x and y in G such that $d(x,y) \geq 2$. By Theorem 2, x and y lie on a common cycle and hence lie on a smallest cycle, say $C : x, x_1, x_2, \ldots, y, \ldots, x_n, x$, of

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length at least 4. Then x, x_1 and x_n do not belong to any cm_{xx_1}-set of G and so $cm_{xx_1}(G) \leq p-3$, which is a contradiction. Hence G is the complete graph and so $G = K_p = K_1 + K_{p-1}$.

Case 2. G has one cut vertex, say z. If $p = 3$, then $G = P_3 = K_1 + 2K_1$. If $p \geq 4$, we claim that $G = K_1 + \cup m_j K_j$ with $\sum m_j \neq 2$ and $K_j \neq K_2$. For that first we claim that every block of G is complete. Suppose there exists a block B, which is not complete. Then there exist two vertices u and v in B such that $d(u,v) \geq 2$. By Theorem 2, both u and v lie on a common cycle and hence u and v lie on a smallest cycle $C : u, u_1, u_2, \ldots, v, \ldots, u_n, v$ of length at least 4. Let u_i, u_{i+1}, u_{i+2} be the consecutive vertices of C distinct from z, then u_i, u_{i+1} and u_{i+2} do not belong to any $cm_{u_i u_{i+1}}$-set of G and hence $cm_{u_i u_{i+1}}(G) \leq p-3$, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \cup m_j K_j$, where K_1 is the vertex z and $\sum m_j \geq 2$.

If $\sum m_j = 2$ and $K_j = K_2$, then G has two complete blocks and one of it is K_3. Let xy be an edge of K_3 with x and y are simplicial vertices of G. It is clear that $S = V(G) - V(K_3)$ is the cm_{xy}-set of G and so $cm_{xy}(G) = p-3$, which is a contradiction. Hence the result.

Theorem 12. *Let G be a connected graph of order $p \geq 4$. Then $cm_{xy}(G) = p-1$ for every edge xy in G if and only if $G = K_{1,p-1}$.*

Proof. Let $G = K_{1,p-1}$. Then by Theorem 7 (ii), $cm_{xy}(G) = p-1$ for every edge xy in G. Conversely, suppose that $cm_{xy}(G) = p-1$ for every edge xy in G. Claim that $G = K_{1,p-1}$. Since $cm_{xy}(G) = p-1$ for every edge xy in G, by Theorem 11, we have $G = K_1 + \cup m_j K_j$. Now, it is enough to prove that every K_j is K_1. If not, $K_j = K_n$ $(n \geq 2)$ for some j. Then G contains a complete block K_{n+1}. Let xy be an edge in K_{n+1}, which is not incident with the cut vertex of G. Then $cm_{xy}(G) = p-2$, which is a contradiction. Hence $G = K_1 + (p-1)K_1 = K_{1,p-1}$.

Note 1. If G is a connected graph of order 3, then G is either P_3 or K_3. Then $cm_{xy}(G) = 1 = p-2$ for any edge xy in G.

Theorem 13. *If a, b and p are positive integers such that $2 \leq a \leq b \leq p-3$, then there exists a connected graph G of order p, $m_{xy}(G) = a$ and $cm_{xy}(G) = b$ for some edge xy in G.*

Proof. **Case 1**. $2 \leq a = b \leq p-3$.

Let $C_{p-a} : v_1, v_2, \ldots, v_{p-a}, v_1$ be the cycle of order $p-a$ and let K_{a+2} be the complete graph of order $a+2$. Let G be the graph obtained from the cycle C_{p-a} and the complete graph K_{a+2} by identifying the edge $v_1 v_{p-a}$ in C_{p-a} with an edge in K_{a+2}. The graph G is shown in Figure 2.

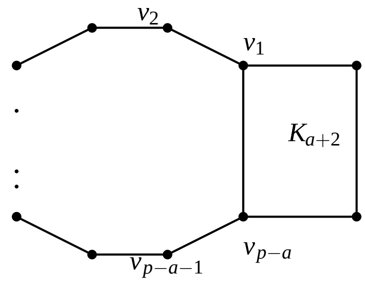

Figure 2: The graph G in Case 1 of Theorem 13

The graph G is of order p and has $'a'$ simplicial vertices $S = V(K_{a+2}) - \{v_1, v_{p-a}\}$. Then by Theorem 3 (i), for the edge $xy = v_1 v_2$, every m_{xy}-set of G contains S and hence $m_{xy}(G) \geq a$. It is clear that every vertex of G lies on an $x-z$ monophonic path or a $y-z$ monophonic path for some z in S, it follows that S is an xy-monophonic set of G and so $m_{xy}(G) = a$. Also, since $G[S]$ is connected, $cm_{xy}(G) = a$.

Case 2. $2 \leq a < b \leq p-3$.

Let $P : v_1, v_2, \ldots, v_{b-a}$ be a path of order $b-a$ and let $C_{p-b+1} : w_1, w_2, \ldots, w_{p-b+1}, w_1$ be a cycle of order $p-b+1$. Let G be the graph obtained from the path P, the wheel $W = K_1 + C_{p-b+1}$ and the complete graph K_a by identifying the vertex v_1 of P with the central vertex K_1 of W and identifying the vertex v_{b-a} of P with any one vertex, say z, of K_a. Then G has order p and it is shown in Figure 3.

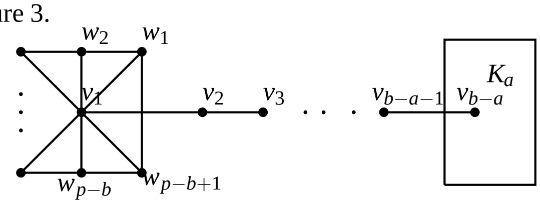

Figure 3: The graph G in Case 2 of Theorem 13

Let $S = V(K_a) - \{z\}$ be the set of all simplicial vertices of G. Let $xy = w_1 w_2$. By Theorem 3 (i), every xy-monophonic set of G contains S. It is clear that S is not an xy-monophonic set of G and so $m_{xy}(G) > |S| = a-1$. Let $S' = S \cup \{w_{p-b}\}$. Then S' is an xy-monophonic set of G and so $m_{xy}(G) = a$. Also, since the induced subgraph $G[S']$ is not connected, $cm_{xy}(G) > a$. To connect a vertex in C_{p-b+1} to a vertex in K_a, we need a path of length $b-a$. Hence $cm_{xy}(G) = a + b - a = b$.

Theorem 14. *For any three positive integers d, n and $p \geq 4$ with $2 \leq d \leq p-2$ and $1 \leq n \leq p-1$, there exists a connected graph G such that its order is p, monophonic diameter is d and the connected xy-monophonic number is n for some edge xy in G.*

Proof. We prove this theorem by considering two cases.

Case 1. $d = 2$. If $n = p-1$, then let $G = K_{1,p-1}$. By Theorem 12, $cm_{xy}(G) = p-1$ for any edge xy in G. If $n = 1$, then let $G = K_{2,p-2}$. Then by Theorem 6 (iv),

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$cm_{xy}(G) = 1$ for any edge xy in G. Now we consider two subcases.

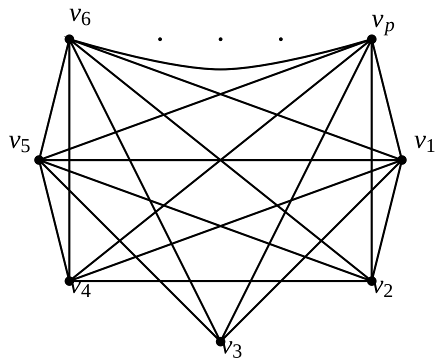

Figure 4: The graph G in Case 1 of Theorem 14 with $n = 2$

Subcase (i) $n = 2$. Let $V = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of the complete graph K_p. The graph G is obtained by removing the edges v_2v_3 and v_3v_4 from the complete graph K_p. Then G has order p, monophonic diameter $d = 2$ and it is shown in Figure 4. Let $S = \{v_2, v_3, v_4\}$ be the set of all simplicial vertices of G. By Theorem 5, every connected xy-monophonic set of G contains all the simplicial vertices other than x and y. Then for an edge $xy = v_3v_p$, $S_1 = \{v_2, v_4\}$ is the minimum connected xy-monophonic set of G and so $cm_{xy}(G) = 2$.

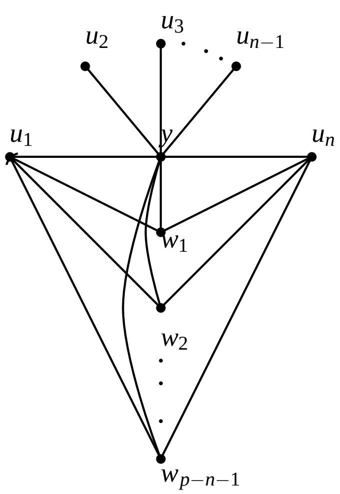

Figure 5: The graph G in Case 1 of Theorem 14 with $3 \leq n \leq p-2$

Subcase (ii) $3 \leq n \leq p-2$. Let $K_{1,n}$ be a star with end vertices $u_1, u_2, \ldots, u_n$ and cut vertex y. Let G be the graph obtained from $K_{1,n}$ by adding $p-n-1$ new vertices $w_1, w_2, w_3, \ldots, w_{p-n-1}$ and joining each w_i $(1 \leq i \leq p-n-1)$ to the vertices u_1, u_n and y. The graph G has order p, monophonic diameter $d = 2$ and it is shown in Figure 5.

Let $xy = u_1y$ and let $S = \{u_1, u_2, \ldots, u_n\}$. If $p-n-1 = 1$, then S is the set of all simplicial vertices of G. Then by Theorem 5, every connected xy-monophonic set contains $S_1 = S - \{u_1\}$. Since the induced subgraph $G[S_1]$ is not connected, $cm_{xy}(G) > |S_1| = n-1$. It is clear that $S_1' = S_1 \cup \{y\}$ is the connected xy-monophonic set of G and so $cm_{xy}(G) = n$. If $p-n-1 > 1$, then $S_2 = S - \{u_1, u_n\}$ is the set of all simplicial vertices of G. Then by Theorem 5, every connected xy-monophonic set of G contains S_2. It is clear that S_2 is not a connected xy-monophonic set of G and so $cm_{xy}(G) > n-2$. Clearly, $S_2' = S_2 \cup \{u_n, y\}$ is a minimum connected xy-monophonic set of G and so $cm_{xy}(G) = n$.

Case 2. $3 \leq d \leq p-2$. Let $P_d : u_0, u_1, u_2, \ldots, u_d$ be a path of length d.

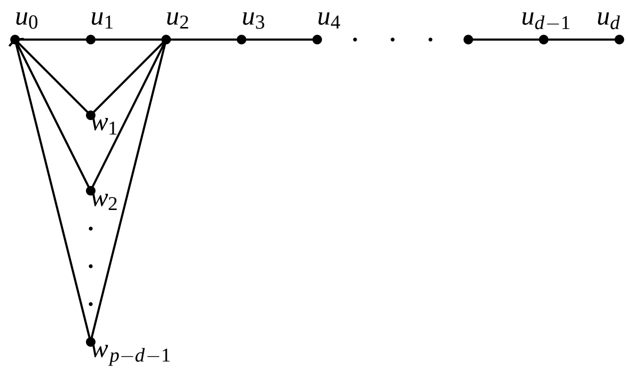

Figure 6: The graph G in Case 2 of Theorem 14 with $n = 1$

Subcase (i) $n = 1$. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-1}$ to P_d and join these to both u_0 and u_2, there by producing the graph G of Figure 6. Then G has order p and monophonic diameter d. For the edge $xy = u_0u_1$, clearly $\{u_d\}$ is the minimum connected xy-monophonic set of G so that $cm_{xy}(G) = 1$.

Subcase (ii) $n = 2$. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-2}, v$ to P_d and join each w_i $(1 \leq i \leq p-d-2)$ to both u_0 and u_2; and join v to both u_{d-1} and u_d, there by producing the graph G of Figure 7. Then G has order p and monophonic diameter d. For the edge $xy = u_0u_1$, clearly $\{u_d, v\}$ is the cm_{xy}-set of G so that $cm_{xy}(G) = 2$.

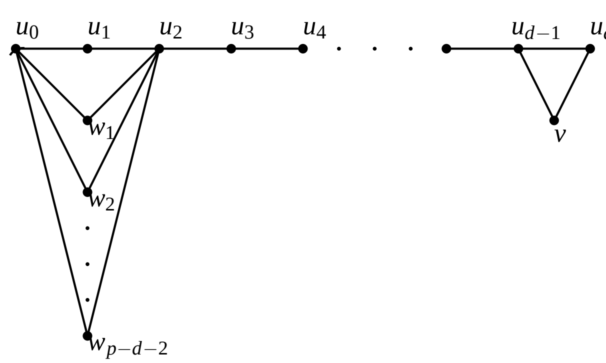

Figure 7: The graph G in Case 2 of Theorem 14 with $n = 2$

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Subcase (iii) $3 \leq n \leq p-1$. We consider two cases. If $n \leq p-d$, then add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-n+1}, v_1, v_2, \ldots, v_{n-2}$ to P_d and join each w_i $(1 \leq i \leq p-d-n+1)$ to both u_0 and u_2; and join each v_j $(1 \leq j \leq n-2)$ to u_{d-1}, there by producing the graph G of Figure 8. Then G has order p and monophonic diameter d. Clearly, $S = \{u_d, v_1, v_2, \ldots, v_{n-2}\}$ is the set of all simplicial vertices of G. Let $xy = u_0u_1$. Then by Theorem 5, every connected xy-monophonic set of G contains S. It is clear that S is not a connected xy-monophonic set of G and so $cm_{xy}(G) > |S| = n-1$. Let $S' = S \cup \{u_{d-1}\}$. Then S' is an xy-monophonic set of G and $G[S']$ is connected so that $cm_{xy}(G) = |S'| = n$.

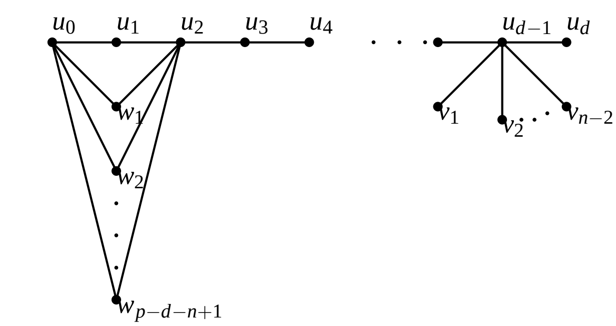

Figure 8: The graph G in Case 2 of Theorem 14 with $3 \leq n \leq p-1$ and $n \leq p-d$

If $n > p-d$, then add $p-d-1$ new vertices $v_1, v_2, \ldots, v_{p-d-1}$ to P_d and join each v_i $(1 \leq i \leq p-d-1)$ to u_{p-n}, there by producing the graph G of Figure 9. Since G is a tree, by Theorem 7 (ii), $cm_{xy}(G) = p-(p-n) = n$ for the edge $xy = u_0u_1$.

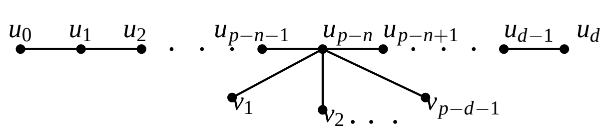

Figure 9: The graph G in Case 2 of Theorem 14 with $3 \leq n \leq p-1$ and $n > p-d$

For any connected graph G, $\text{rad}_m(G) \leq \text{diam}_m(G)$. It is shown in [3] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph G. This theorem can also be extended so that the connected edge fixed monophonic number can be prescribed under some conditions.

Theorem 15. *For any three positive integers a, b and $n \geq 4$ with $1 \leq a < b$, there exists a connected graph G such that its monophonic radius is a, monophonic diameter is b and the connected xy-monophonic number is n for some edge xy in G.*

Proof. **Case 1**. $a = 1$. Then $b \geq 2$. Let $C : u_1, u_2, \ldots, u_{b+2}, u_1$ be a cycle of order $b+2$. Let G be the graph obtained by adding $n-2$ new vertices $v_1, v_2, \ldots, v_{n-2}$ to C and joining each of the vertices $v_1, v_2, \ldots, v_{n-2}, u_3, u_4, \ldots, u_{b+1}$ to the vertex u_1. The graph G is shown in Figure 10. It is easily verified that $1 \leq e_m(x) \leq b$ for any vertex x in G and $e_m(u_1) = 1$, $e_m(u_2) = b$. Then $\text{rad}_m(G) = 1$ and $\text{diam}_m(G) = b$. The set $S = \{v_1, v_2, \ldots, v_{n-2}, u_2, u_{b+2}\}$ is the set of all simplicial vertices of G.

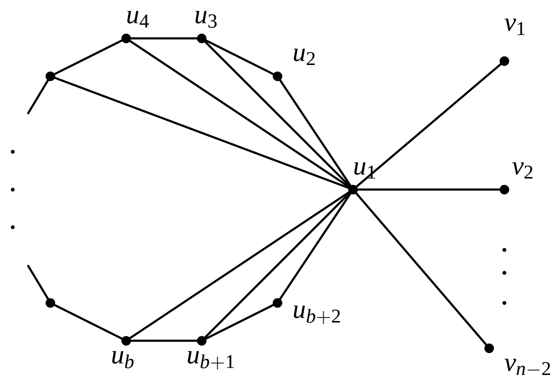

Figure 10: The graph G in Case 1 of Theorem 15

Let $xy = u_2u_3$. It is clear that $S' = S - \{u_2\}$ is an xy-monophonic set of G and so $cm_{xy}(G) \geq n-1$. Since the induced subgraph $G[S']$ is not connected, $cm_{xy}(G) > n-1$. Let $S'' = S' \cup \{u_1\}$. Then S'' is an xy-monophonic set of G and $G[S'']$ is connected and so $cm_{xy}(G) = |S''| = n$.

Case 2. $a \geq 2$.

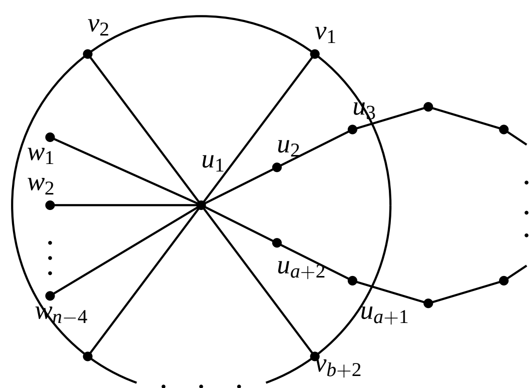

Figure 11: The graph G in Case 2 of Theorem 15

Let $C : u_1, u_2, \ldots, u_{a+2}, u_1$ be a cycle of order $a+2$ and let $W = K_1 + C_{b+2}$ be the wheel with $V(C_{b+2}) = \{v_1, v_2, \ldots, v_{b+2}\}$ and $V(K_1) = \{u_1\}$. Let G be the graph obtained from C and W by adding $n-4$ new vertices $w_1, w_2, \ldots, w_{n-4}$ and joining each w_i $(1 \leq i \leq n-4)$ to the vertex u_1. The graph G is shown in Figure 11. It is easily verified that $a \leq e_m(x) \leq b$ for any vertex x in G and $e_m(u_1) = a$, $e_m(v_1) = b$. Thus $\text{rad}_m(G) = a$ and $\text{diam}_m(G) = b$. The set $S = \{w_1, w_2, \ldots, w_{n-4}\}$ is the set of all simplicial vertices of G. Let $xy = v_1v_2$. It is clear that every xy-monophonic set of G contains all simplicial vertices, at least one non-adjacent

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vertex of u_1 in C and at least one non-adjacent vertex of either v_1 or v_2 in C_{b+2}. Let $S' = S \cup \{u_3, v_4\}$. It is clear that S' is an xy-monophonic set of G and its induced subgraph $G[S']$ is not connected. Let $S'' = S' \cup \{u_1, u_2\}$. Since S'' is a minimum connected xy-monophonic set of G, we have $cm_{xy}(G) = n$.

3 Connected edge fixed monophonic subgraph

Definition 3. *A graph H is a cm_{xy}-subgraph if there exists a connected graph G with H is an induced subgraph of G and $V(H)$ is a cm_{xy}-set of G.*

Theorem 16. *Every connected graph is the cm_{xy}-subgraph of some connected graph having an edge xy.*

Proof. Let H be a connected graph and let x, y be the vertices of K_2. Let G be the graph obtained from $H \cup K_2$ by joining the vertices x and y to every vertex of H. The resulting graph is shown in Figure 12. Claim that H is the cm_{xy}-subgraph of G. It is clear that $e_m(x) = e_m(y) = 1$ and so no vertex of H is an internel vertex of any monophonic path starting from the vertices x and y. Hence $V(H)$ is the cm_{xy}-set of G and so H is the cm_{xy}-subgraph of G.

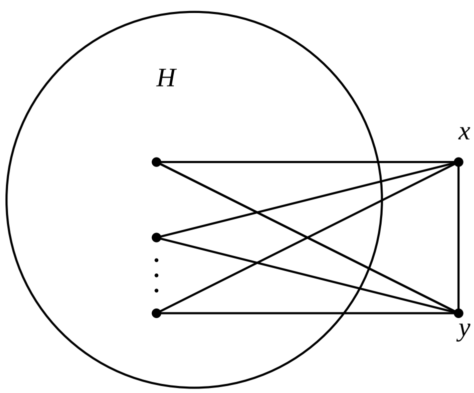

Figure 12: The graph G of Theorem 16

Theorem 17. *Let xy be an edge of a connected graph H. If H is a cm_{xy}-subgraph, then xy is a bridge but not a leaf of H.*

Proof. Suppose that H is a cm_{xy}-subgraph of a connected graph G for an edge xy in H. Then xy is also an edge of G. Now we claim that xy is a bridge but not a leaf of H. If not, xy is either a leaf or a non-bridge of H. Hence at most one vertex of the edge xy is a cut vertex of H. If $G = H$, then $cm_{xy}(G) \leq |V(G)| - 1$ and so H is not a cm_{xy}-subgraph of G. If $G \neq H$, then either $V(H) = V(G)$ or $V(H) \subset V(G)$.
Case 1. $V(H) = V(G)$. Then at most one vertex of the edge xy is a cut vertex of G. If x is a cut vertex of G, then y is not an element of any minimum connected xy-monophonic set of G. Hence $V(H)$ is not a cm_{xy}-set of G and so H is not a cm_{xy}-subgraph of G, which is a contradiction. If both x and y are non-cut vertices of G, then $S = V(H) - \{x, y\}$ is a connected xy-monophonic set of G and so $V(H)$ is not a cm_{xy}-set of G. Hence H is not a cm_{xy}-subgraph of G, which is a contradiction.
Case 2. $V(H) \subset V(G)$. Then at most two vertices on the edge xy are cut vertices of G.

Subcase (i) No vertex of xy is a cut vertex of G. Then x and y do not belong to any minimum connected xy-monophonic set of G and so $V(H)$ is not a cm_{xy}-set of G, which is a contradiction.

Subcase (ii) Exactly one vertex of xy is a cut vertex of G. If x is a cut vertex of G, then y is not an element of any minimum connected xy-monophonic set of G. Hence $V(H)$ is not a cm_{xy}-set of G and so H is not a cm_{xy}-subgraph of G, which is a contradiction.

Subcase (iii) Both vertices of xy are cut vertices of G. Then $G - \{x, y\}$ is disconnected and it has two or more components. Since at most one vertex of the edge xy is a cut vertex of H, at least one component of $G - \{x, y\}$ has no elements from H. It is clear that any connected xy-monophonic set of G contains at least one element from each component of $G - \{x, y\}$. Hence $V(H)$ is not a cm_{xy}-set of G and so H is not a cm_{xy}-subgraph of G, which is a contradiction.

Theorem 18. *Let H be a connected graph with every vertex of H is either a cut vertex or a simplicial vertex. Then H is a cm_{xy}-subgraph if and only if xy is a bridge but not a leaf of H.*

Proof. Let H be a connected graph with every vertex of H is either a cut vertex or a simplicial vertex. If H is a cm_{xy}-subgraph, then by Theorem 17, xy is a bridge but not a leaf of H.

Conversely, let xy be a bridge but not a leaf. By Theorem 5, the set of all simplicial vertices is a subset of every connected xy-monophonic set of H. Let S be a connected xy-monophonic set of H. Now, claim that $S = V(H)$. Otherwise, there exists a cut vertex v of H such that $v \notin S$. Let u and w be two simplicial vertices belonging to different components of $H - \{v\}$. Since v lies on the unique path joining u and w, it follows that the subgraph $G[S]$ induced by S is not connected, which is a contradiction. Hence $V(H)$ is the unique connected xy-monophonic set of H and so H is a cm_{xy}-subgraph.

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