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Structural Properties of Absorption Cayley Graphs

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Abstract: Let R be a commutative ring with two binary operators addition $(+)$ and multiplication $(.)$. Then Z_n is a ring of integers modulo n, where n is a positive integer. An *Absorption Cayley graph* denoted by $\Omega(Z_n)$ is a graph whose vertex set is Z_n, the integer modulo n and edge set $E = \{ab : a+b \in S\}$, where $S = \{a \in Z_n : ab = ba = a$ for some $b \in Z_n, b \neq a\}$. Here $ab = a$ is the *Absorption property* as b is absorbed in a. We study the characterization of Absorption cayley graphs along with its properties such as connectedness, degree, hamiltoniacity, diameter, planarity, girth, regularity etc.

Keywords: Cayley graph, absorption cayley graph, addition cayley graph, planar graph, diameter, girth.

1 Introduction

Cayley graphs are widely studied in the literature as one can approach them to solve specific problems such as rearrangement and design of parallel CPU's [1] [2] [3]. Recent studies show the use of cayley graphs in exploratory analysis on family of trivalent cayley graphs associated with $PSL_2(p)$ [4]. These graphs may also be used to solve the problems which were previously too large, such as the diameter of Rubik's $2*2*2$ cube [5]. Here we discuss Absorption cayley graphs extensively with many of its properties so that it can be applied in the designing of the networks and parallel computing.

For standard terminology and notation in graph theory we refer Harary [6], West [7] and for algebra we consult Gallian [8], Dummit and Foote [9] respectively. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

The *integral ring* Z_n is a ring of integers modulo n with respect to addition and multiplication. The set U_n denotes the set of *units*, $\{a \in U_n : ab = ba = 1, for\ some\ b \in Z_n\}$. The *Euler phi function* represented by $\phi(n)$, is the number of non negative integers less than n that are co-prime to n. A *zero divisor* of a commutative ring is a non-zero element r such that $rs = 0$ for some other non-zero element s of the ring. If the ring R is commutative, then $rs = 0 \Leftrightarrow sr = 0$.

The *Cayley table* of a group G is a table whose rows/columns corresponds to elements of G and whose entries on row a and column b is $a*b$, where $*$ is the operation on G.

The *Cayley graph* was introduced in year 1878 by Cayley for finite groups. Let G be a finite group and S be a subset of G such that $S = S^{-1}$ and $1_G \in S$. Then *Cayley graph*, denoted by $\Gamma = Cay(G,S)$ relative to S is a graph with vertex set G and edge set $E(\Gamma,S) = \{gh | hg^{-1} \in S\}$. Substantial research has been done on cayley graphs in [10], [11], [12], [13], [14] and [15].

Given an integer n, one defines the *Unitary Cayley graph*, denoted $Cay(Z_n, Z_n^*)$, to be the graph whose vertex set is Z_n, the integers modulo n, with an edge between two vertices $x,\ y$ if $x-y$ is a unit in (the ring) Z_n. Many properties of unitary cayley graphs are discussed in [16] [17] [18] [19].

The next kind of graphs defined are *Unitary addition graphs*. For a subset S of the abelian group G, we denote by $Cay^+(G,S)$ the *Addition Cayley graph* induced by S on G, this is the graph with the vertex set G and the edge set $\{(g_1,g_2) \in G*G : g_1+g_2 \in S\}$ [20] [21]. For a positive integer $n > 1$, the unitary addition Cayley graph G_n is the graph whose vertex set is Z_n, the integers modulo n and if

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U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a+b \in U_n$ refer [22]. Recently [23] has intoduced new type of unitary graphs.

An *Absorption Cayley graph* denoted by $\Omega(Z_n)$ is a graph whose vertex set is Z_n, the integer modulo n and edge set $E = \{ab : a+b \in S\}$, where $S = \{a \in Z_n : ab = ba = a$ for some $b \in Z_n, b \neq a\}$.

Some examples of the set S for different Z_n
For Z_1, Z_2, Z_3, Z_5, Z_7
$S = \{0\}$.

For Z_4
$S = \{0, 2\}$.

For Z_6
$S = \{0, 2, 3\}$.

For Z_8
$S = \{0, 2, 4, 6\}$.

For Z_9
$S = \{0, 3, 6\}$.

The *chromatic number* of a graph G is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. Chromatic number of a graph G is denoted by $\chi(G)$. The *clique number* is the size of the largest complete subgraph in a graph. A graph G is called *perfect* if and only if $\chi(H) = \omega(H)$, for every induced subgraph H of G.

An *independent set*, is a set of vertices of which no pair is adjacent. Independence number $\beta(G)$ of a graph G is the size of the largest independent set of G.

A graph G is *embedded* in a surface S when its vertices are represented by points in S, and each edge by a curve joining corresponding points in S, in such a way that no curve intersects itself, and two curves intersect each other only at a common vertex. A graph which can be embedded in the plane is called *planar*. A planar graph is called *outerplanar* if it can be embedded in the plane in such a way that all of its vertices are in the same face.

A graph is said to be *regular* if degree of each vertex is same. A graph is called *(r_1, r_2)-semiregular* if its vertex set can be partitioned into two subsets V_1 and V_2 such that all the vertices in V_i are of degree r_i for $i = 1, 2$.

A cycle in a graph that contains every vertex of graph is called a Hamiltonian cycle. A Hamiltonian graph is a graph that contains a Hamiltonian cycle.

The *eccentricity* of a vertex v in a connected graph is the maximum distance of v from any other vertex in the graph. The *radius* of a graph is the minimum eccentricity of any vertex. The *diameter* of a graph is the maximum eccentricity of any vertex in the graph. The *girth* of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e. it's an acyclic graph), its girth is defined to be infinity.

2 Properties of the set S

Proposition 2.1. For a positive n, and ring Z_n the subset S does not contain 1.

Proof. Let if possible $1 \in S$. Then there exist an element $b \in Z_n$ such that $1.b = b.1 = 1$ but then $b = 1$ which is a contradiction to the fact that $b \neq 1$. Hence $1 \notin S$.

Theorem 2.2. Let Z_n be the ring modulo n. If n is such that $n = 2m$ then $(m+1) \notin S$.

Proof. Let if possible $(m+1) \in S$. Then there exist $b \neq m+1$ or $b-1 \neq m$ and $b \neq 1$ such that
$b.(m+1) = (m+1) mod (2m)$
$\Rightarrow 2m/(m+1).(b-1)$.
Then two case arise:

(1) If m is odd.
Then $(m+1)$ is even. Clearly m does not divide $(m+1)$. Thus $m/(b-1)$.
$\Rightarrow (b-1) = km$ for some positive integer k.
Which is not possible.

(2) If m is even.
Clearly m does not divide $(m+1)$ and 2 also does not divide $(m+1)$. Then
$2m/(b-1)$
$\Rightarrow \ b = k(2m)+1$ for some positive integer k.
$\Rightarrow \ b = 1 mod (2m)$
which is a contradiction.

Thus $(m+1) \notin S$.

Theorem 2.3. For ring Z_n, n being a positive integer. S is equal to the set of zero divisor if and only if $n \neq 2m$, m being odd.

Proof. Let us suppose $n = 2m$, m being odd. Clearly $S \cap U_n = \phi$, since if $S \cap U_n \neq \phi$ then there exist $a \in Z_n$ such that $a \in S$ and $a \in U_n$, also $\exists\ b_1, b_2 \in Z_n$ such that $ab_1 = a$ and $ab_2 = 1$ which is a contradiction, since an integer in Z_n is either a unit or a zero divisor.
Thus $S \subset Z_n \setminus U_n$.
Hence $S \subset V_n$. Where V_n is the set of zero divisor.
To show $V_n \subset S$. Let $a \in V_n$. Then $\exists\ b \neq 0$ such that $ab = ba = 0$. To show $\exists\ c$ such that $ac = ca = a$, $c \neq 1$, $c \neq a$.
$\Rightarrow \ a(c-1) = 0$
Clearly $c = b+1$ and $c \neq 1$ as $b \neq 0$.
Also $c \neq a$, let if possible $c = a$.
$\Rightarrow \ (b+1) = a$
$\Rightarrow \ b(b+1) = 0$
$\Rightarrow$ either $b = 0$ or $b+1 = c = 0$
Which is a contradiction. Hence $c \neq a$.

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Therefore, $V_n = S$, if $n \neq m$, m being odd.
Conversely, let $S = V_n$. To show $n \neq 2m,\ m$ being odd.
Let if possible $n = 2m,\ m$ being odd thus (m+1) is even.
Then by Theorem 2.2 $(m+1) \notin S$, clearly $(m+1) \in U_n$.
But HCF$(m+1, 2m) = 2$, Thus $(m+1) \notin U_n$.
$\Rightarrow\ (m+1) \in V_n = S$. This is a contradiction. Hence $n \neq 2m$, m being odd.

Corollary 2.4. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $n \neq 2m,\ m$ being odd. Then

$$Z_n = U_n \cup S,\ such\ that\ Z_n \cap S = \phi.$$

Proof. The proof follows from the Theorem 2.2.

Corollary 2.5. If $n = 2m,\ m$ being odd. Then

$$Z_n = U_n \cup S \cup \{m+1\},\ such\ that\ Z_n \cap S \cap \{m+1\} = \phi.$$

Proof. The proof follows from the Theorem 2.2.

Corollary 2.6. S forms a subgroup of $(Z_n, +)$, if $n = p^{\alpha}$.
Proof. Let $n = p^{\alpha}$, then by definition of subset S of Z_n and for $a \in S$ there exist b such that $p^{\alpha}/a(b-1),\ b \neq a$ and $b \neq 1$. To show S is a subgroup of Z_n. Thus it is enough to show
(i) $0 \in S$.
(ii) For every $a_1, a_2 \in S,\ a_1 + a_2 \in S$.
Clearly if $a = p^{\alpha}$ then $a = 0$ and $0 \in S$. Thus (i) holds.
Next if $p^{\alpha}/a(b-1)$ and $a \neq p^{\alpha}$ then if $p^{\alpha}/(b-1)$
$\Rightarrow (b-1) = kp^{\alpha}$
$\Rightarrow b = 1\ mod\ p^{\alpha}$
which is not possible. Thus p^{α_1}/a and $p^{\alpha_2}/(b-1)$ where $\alpha = \alpha_1.\alpha_2$. Thus $a = p^{\alpha_1}$, where $\alpha_1 < \alpha$.
Let $a_1, a_2 \in S$ then to show $a_1 + a_2 \in S$.
$a_1 = k_1 p_1^{\alpha}$ and $a_2 = k_2 p_2^{\alpha}$ for some positive integers k_1 and k_2.
$a_1 + a_2 = k_1 p^{\alpha_1} + k_2 p^{\alpha_2}$ then clearly $a_1 + a_2 = kp^{\beta}$ where $\beta = min\{\alpha_1, \alpha_2\}$, for some k.Thus $a_1 + a_2 \in S$. Thus (ii) holds true.
Thus S is a subgroup of Z_n.

Theorem 2.7.

$$|S| = \begin{cases} n - \phi(n) - 1, & n = 2m,\ m\ is\ odd \\ n - \phi(n), & otherwise. \end{cases} \quad (1)$$

Proof. If $n \neq 2m$ where m is an odd integer.
We know that every non zero element in Z_n is either an unit or a zero divisor. Also if $n \neq 2m$ then set S is the set of all zero divisors of Z_n.
Since $|U_n| = \phi(n)$, and
$|S| = |Z_n| - |U_n|$
$|S| = n - \phi(n)$ by Corollary 2.4.
If $n = 2m$, where m is an odd integer.
Clearly $m+1$ is even. Let $a = m+1$ and $2/a$. Thus $a \notin U_n$.
Thus a is a zero divisor, $\exists\ b$ such that $ab = 0$.
Then $b - 1 = m$
$\Rightarrow\ b = m+1$
$\Rightarrow\ b = a$. Thus $a \notin S$.
Hence $|S| = n - \phi(n) - 1$.

3 Observations of some graphs

When n = p, where p is a prime, we have the following observations:
Observation 3.1. Chromatic number of $\Omega(Z_p)$ is 2.

Observation 3.2. Edge chromatic number of $\Omega(Z_p)$ is 1.

Observation 3.3. Clique number of $\Omega(Z_p) = 2$.

Observation 3.4. Independence number of $\Omega(Z_p) = (p+1)/2$ and edge independence number = $(p-1)/2$.

When $n = 2^{\alpha}$, we have the following observations:
Observation 3.5. Chromatic number of $\Omega(Z_{2^{\alpha}}) = 2^{\alpha-1}$.

Observation 3.6. Edge chromatic number of $\Omega(Z_{2^{\alpha}}) = 2^{\alpha-1} - 1$.

Observation 3.7. Clique number of $\Omega(Z_{2^{\alpha}}) = 2^{\alpha-1}$.

Observation 3.8. Independence number of $\Omega(Z_{2^{\alpha}}) = 2$ and edge independence number = $2^{\alpha-1}$.

4 Relation between Absorption cayley graph's adjacency matrix and cayley table

As in the cayley table **(1)** for Z_6, We know that cayley table is symmetric with each entry coming in each row and each column exactly ones. Also given is the adjacency matrix of $\Omega(Z_6)$.
A very interesting relation can be observed between adjacency matrix of Absorption cayley graph and its cayley table. For $n = 6$, we know that $S = \{0,\ 2,\ 3\}$. If we place zero at the diagonal elements in cayley table and give 1 for each element $a \in S$ and zero for all other elements. Then we obtain the adjacency matrix for $\Omega(Z_6)$.

5 Degree of a vertex in Absorption cayley graph

Theorem 5.1. Degree of a vertex in Absorption cayley graph $\Omega(Z_n)$ is either $|S|$ or $|S| - 1$.
Proof. It is clear from the cayley table of $(Z_n, +)$ that for every $m \in S,\ a \in Z_n$

$$|\{b : a + b = m\ mod\ n\}| = |S|.$$

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But the degree of a vertex $a \in Z_n$ is due to its adjacencies to a vertex $b \in Z_n,\ b \neq a$ such that $a+b=m \ mod \ n$. Hence the degree of vertex is either $|S|$ or $|S|-1$.

Theorem 5.2.
The number of edges of $\Omega(Z_n) =$

$$k(\lceil\frac{(n-1)}{2}\rceil)+(|S|-k)(\lceil\frac{(n-1)}{2}\rceil-1)$$

where k is the number of odd elements in $|S|$.

Proof. Let $m \in S$, then m appears n times in cayley table once in each row and column. Here two cases arise:

(i)If m is odd. Then $\lceil\frac{(n-1)}{2}\rceil$ is the number of distinct appearance of m for $a,b \in Z_n$ such that $a+b=m \ mod \ n$. Since cayley table is symmetric thus m occurs twice for each pair a and b. Thus total number of edges for all k odd elements in $S = k(\lceil\frac{(n-1)}{2}\rceil)$.

(ii)If m is even. Thus $a=m/2$ does not constitute for an edge as a is the diagonal element in cayley table of Z_n. Thus the number of edges reduce in this case and there are $|S|-k$ even elements in S. So the total number of edges due to even elements $= (|S|-k)(\lceil\frac{(n-1)}{2}\rceil-1)$.

Thus total number of edges in $\Omega(Z_n) = k(\lceil\frac{(n-1)}{2}\rceil)+(|S|-k)(\lceil\frac{(n-1)}{2}\rceil-1)$.

6 Characterization of Absorption cayley graph

Theorem 6.1. A given graph G of order n is isomorphic to an Absorption cayley graph $\Omega(Z_n)$ if and only if there are $|S|$ number of edge disjoint subgraphs $G_{m_1},\ldots,G_{|S|}$ whose union is G such that

(i)$ab \in E(G_{m_i})$ if and only if $a+b = m_i(mod\ n), i \neq j$.

(ii)$|E(G_{m_i})| = \begin{cases} \lceil\frac{(n-1)}{2}\rceil, \ m_i \ is \ odd \\ \lceil\frac{(n-1)}{2}\rceil-1, \ m_i \ is \ even. \end{cases}$

Proof. Neccesity: Let us suppose G is isomorphic to an Absorption cayley graph $\Omega(Z_n)$.
To show that there exist $|S|$ number of edge disjoint subgraphs whose union is G and satisfies properties (i) and (ii). The following cases arise:

(1) If n is odd.
Then S will contain $n-\phi(n)$ odd integers. Let $S = \{0,m_2,\ldots,m_{n-\phi(n)}\}$. For elements in S we will show that there exist corresponding subgraphs in G which are edge disjoint and whose union will be G.
For $0 \in S$, if $a+b=0 \ mod \ n$ then ab is an edge in G. Clearly b is an inverse of a. Thus $b=(n-a) \ mod \ n$. Also each non zero element has an inverse since Z_n is a group with respect to addition. Thus there are $(n-1)/2$ such pairs and hence edges in G.
For other non zero elements say m_i in S, ab is an edge if $a+b=m_i \ mod \ n$.
Clearly $a+b \in Z_n$ then by cayley table for finite commutative group Z_n under addition we know that m_i appears exactly n number of times, coming exactly once in each row and column. If m_i appears in jth row and kth column then clearly $j+k=m_i$. Also it appears on diagonal of cayley table where $j=k$ which is not considered. So there will be $(n-1)$ such pairs. but since cayley tables are symmetric the total number of such edges would be $(n-1)/2$. Thus for each element in S there would be a corresponding subgraph whose union is G.
Clearly by construction (i) holds. Let if possible ab be an edge corresponding to two elements in S say m_i and m_j. Thus this means $a+b=m_i$ and $a+b=m_j$. Which means that corresponding to one position there are two values in cayley table. Which is a contradiction. Hence (ii) holds.

(2) Let n be even.
Clearly as in (1). For $0 \in S$ there are $\lceil\frac{(n-1)}{2}\rceil$ pairs such that $a+b=0 \ mod \ n$ but $a=b$ for one pair. Thus there are $\lceil\frac{n-1}{2}\rceil-1$ number of edges.
Let $m_i \in S$. If m_i is even. Then it will be a diagonal element in cayley table. Thus corresponding to m_i there will be $\lceil\frac{(n-1)}{2}\rceil-1$ edges in G. Else for odd integers in S there will be $\lceil\frac{(n-1)}{2}\rceil$ edges.

Thus in both the cases G is the union of edge disjoint subgraphs generated by elements of S, satisfying the two properties.

Sufficiency: Let G be a graph which is the union of $|S|$ number of edge disjoint subgraphs say $G_{m_1},\ldots,\ G_{|S|}$ satisfying both the properties. To show G is isomorphic to an Absorption cayley graph $\Omega(Z_n)$.
Clearly there are same number of vertices in G and Absorption cayley graphs.
Now we will show that the adjacency is preserved. Let uv be an edge in G. Then $uv \in G_k$, for some $1 \leq k \leq |S|$.
Clearly by (i) $u+v \in S$.
$\Rightarrow uv \in \Omega(Z_n)$.
Thus G is a subgraph of $\Omega(Z_n)$.
Let if possible ab be an edge in $\Omega(Z_n)$ which is not an edge in G. Clearly, $a+b \in S$. Also $\exists\ ij$ an edge in G_k such that $a+b=i+j$. If $i+j$ is even then there are $\lceil\frac{(n-1)}{2}\rceil-1$ number of edges. Thus ab should coincide with some ij in G_k. If $i+j$ is odd then there are $\lceil\frac{n-1}{2}\rceil$ number of edges. Thus again ab should coincide with some ij in G_k, which is a contradiction. Thus ab is an edge in G.
Thus sufficiency holds.

Corollary 6.2. A given graph G of order n is isomorphic to an Absorption cayley graph $\Omega(Z_n)$ if and only if

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(i) $n = p$, where p is a prime and the number of edges is $(n-1)/2$ and degree of each vertex is less than or equal to 1.
(ii) $n = p^\alpha$ and $\alpha > 1$ the graph G is disconnected with two components C_1 and C_2 such that for a vertex m if p/m then $m \in C_1$ otherwise in C_2, C_1 is complete and C_2 is such that it is p-regular.
(iii) $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ such that $p_i \neq 2$ for all $i = 1 \; to \; k$, then number of edges in G are $n(n-1)/2 - |\Gamma(U_n)|$. Degree of each vertex is $|S|$ or $|S|-1$.

7 Connectedness of a Absorption graph

Theorem 7.1. An Absorption graph $\Omega(Z_n)$ is connected if and only if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ such that $p_i \neq p_j$ for $i \neq j$, p_i is prime and $k \geq 2$.

Proof. We will prove this result by contraposition. That is we will show that Absorption graph $\Omega(Z_n)$ is disconnected if and only if $n = p_1^\alpha$.
Let $n = p^\alpha$. Then clearly S contains the multiples of p and it forms a group with respect to addition. Thus by closure property all vertices in S form a complete subgraph. Also there does not exist an element b such that $a \in S$ and $b \in Z_n$ such that $a + b \in S$, so that a and b will never form an edge. Clearly the graph will be disconnected.
Conversely, let $\Omega(Z_n)$ be disconnected. If possible let $n = pq \; p \neq q$, p and q being prime. Then clearly S will not form a group as S is a group if and only if $n = p^\alpha$. Now $p, q \in S$, implies p and 0 , q and 0 will form edges in $\Omega(Z_n)$, $p-1$ and 1 will be an edge and so will 1 and $q-1$ provided $p \neq q \neq 2$ and so on. Thus because of primes p and q there will be a path between every pair of vertices in Absorption graph $\Omega(Z_n)$. Hence $\Omega(Z_n)$ will be connected which is a contradiction. Thus $n \neq pq$, for $p \neq q$ p and q being prime.

Theorem 7.2. The number of component in a disconnected Absorption graph $\Omega(Z_n)$ is

(i) $(n-1)/2$ if $n = p$.
(ii) 2 if $n = p^\alpha$, $\alpha > 1$.

p being prime.

Proof. Let us consider the following two cases.

(1) Let $n = p$
As p is a prime thus set S is a singleton set containing zero. Then $\Omega(Z_n)$ will contain an edge ab if $a + b = 0 \; mod \; n$ i.e. ab is an edge if b is an inverse of a which is unique. Thus for each a there is a unique b and vice-versa, which is an edge and no other vertex c is there such that ac is an edge in G. Therefore, there are total $(n-1)/2$ such unique pairs. Hence there are $(n-1)/2$ number of disjoint edges.
(2) Let $n = p^\alpha$
Then S contains multiples of p less than p^α. As discussed in the previous theorem S forms a group. Thus elements in S forms a complete subgraph of order $|S|$, which is one component. Next we show that all elements in $Z_n - S$ are connected and forms another component. We know that a simple graph with n vertices will be connected if the degree of each vertex is greater than $1/2(n-1)$. Clearly we know that $\phi(p^\alpha) = p^\alpha - p^{\alpha-1}$.

By Theorem 5.1, we know that least possible degree of a vertex is $|S|-1$. Thus we claim that $n - \phi(n) - 1 \geq \frac{\phi(n)-1}{2}$
or $n - \phi(n) \geq \frac{\phi(n)-1}{2} + 1$
or $n - \phi(n) \geq \frac{\phi(n)+1}{2}$
or $p^\alpha - [p^\alpha - p^{\alpha-1}] \geq \frac{p^\alpha - p^{\alpha-1}+1}{2}$
or $2p^{\alpha-1} \geq p^\alpha - p^{\alpha-1} + 1$
or $3p^{\alpha-1} \geq p^\alpha + 1$.
We will prove the above equation by induction on p and α.
Let $p = 2$ then for $\alpha = 1$ and 2 the result holds. Next let the result be true for α that is $3 * 2^{\alpha-1} \geq 2^\alpha + 1$. We will prove that $3 * 2^\alpha \geq 2^{\alpha+1} + 1$. Clearly $3 * 2^\alpha = 3 * 2 * 2^\alpha$
$= 2 * (3 * 2^{\alpha-1})$
$\geq 2 * (2^\alpha + 1)$
$= 2^{\alpha+1} + 2$
$= (2^{\alpha+1} + 1) + 1$
$\geq 2^{\alpha+1} + 1$.
Hence the result holds for each α. In the same way the result is true for each prime p.

8 Regularity of Absorption cayley graphs

Theorem 8.1. The Absorption cayley graphs $\Omega(Z_n)$ are either regular or $(|S|, |S|-1)$-semiregular.

Proof. Let us consider the Absorption graph $\Omega(Z_n)$ for different value of n.

(i) For $n = 2^\alpha$, $\alpha \geq 1$.
The graph is a disconnected with two components (by Theorem 7.2). Since, here $S = \{0, 2, 4, 6 \dots, 2^{\alpha-1}\}$, $|S| = n/2$. Then clearly one component will have these elements of S as complete subgraph, thus each vertex having degree $|S|-1$. The second component will have all odd integers as its vertices which again will form a complete subgraph, since $a, b \in Z_n \setminus S$ implies $a + b = 0 \; mod \; 2$. Again degree of each vertex is $|S|-1$, thus graph is $|S|-1$ regular.

(ii) For $n = p$, p being prime.
$\Omega(Z_p)$ contains $(p-1)/2$ number of K_2 for $p \neq 2$ and vertex 0 as an isolated vertex. Also $|S| = 1$. Thus the graph is $(0,1)-$semiregular.
(iii) For $n = p^\alpha$, $p \neq 2$ and $\alpha > 1$.
The graph is disconnected with one component having multiples of p as vertices and being complete subgraph

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it has degree $|S|-1$. The other component consist of vertices coprime to p, each having $|S|$ degree.

(iv) For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ such that $p_i \neq 2$ for all $i = 1\ to\ k$.
The graph is connected. The degree as proved in Theorem 5.1, is either $|S|$ or $|S|-1$.

Corollary 8.2. The Absorption cayley graph $\Omega(Z_n)$ is never Eulerian.

Proof. Clearly by previous theorem for $n \neq 2^\alpha$, the graph is $(|S|,|S|-1)-$semiregular. If $|S|$ is even then $|S|-1$ will be odd and vice versa. Thus all vertices can never be of even degree. Also if $n = 2^\alpha,\ \alpha > 1$ then $\Omega(Z_n)$ is $|S|-1$ regular. But $|S| = 2^{\alpha-1}$ which is again even, thus $|S|-1$ is odd. Thus for any value of $n,\ \Omega(Z_n)$ can never have all the vertices with even degree. Thus $\Omega(Z_n)$ is never Eulerian.

9 Hamiltonian cycle in Absorption cayley graph

Theorem 9.1. [24] Let G be a graph of order $n \geq 3$. If

$$deg(u) + deg(v) \geq n$$

for each pair u,v of non adjacent vertices of G, then G is Hamiltonian.

Theorem 9.2. [24] Let G be a graph of order $n \geq 3$. If $deg(v) \geq n/2$ for each vertex v of G, then G is Hamiltonian.

Theorem 9.3. An Absorption cayley graph $\Omega(Z_n)$ is Hamiltonian if $|S| > n/2$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $n \neq 2m,\ m$ being odd, and $p_i \neq p_j$ for $i \neq j$.

Proof. Clearly we can discuss the Hamiltonian property only for connected graphs and absorption cayley graph is connected if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},\ n \neq 2m,\ m$ being odd, and $p_i \neq p_j$ for $i \neq j$. Let $|S| > n/2$, then $|S|-1 \geq n/2$ and we know that the degree of $\Omega(Z_n)$ is either $|S|$ or $|S|-1$. Hence by Theorem 9.2, $\Omega(Z_n)$ is Hamiltonian.

10 Planarity of Absorption cayley graphs

Theorem 10.1. An Absorption cayley graph $\Omega(Z_n)$ is planar if

$$n = \begin{cases} p \text{ where } p \text{ is prime.} \\ 2^\alpha, \alpha \leq 3. \\ 6. \end{cases} \tag{2}$$

Proof. We know that a simple planar connected graph has a vertex with degree less than six. Clearly for $n = pq,\ n > 10,\ n = p^\alpha,\ p > 3$ and $n = 3^\alpha,\ \alpha \geq 3$ the degree of every vertex is > 6 as $|S| > 6$ and by Theorem 5.1, degree of each vertex is either $|S|$ or $|S|-1$. Thus graph is non-planar.
If $n = p$ then the graph is always disconnected with $(p-1)/2$ copies of K_2 and one K_1. Thus planar. Also for $n = 2^\alpha,\ \alpha \leq 3$ the graph is disconnected with two components being complete subgraphs each being $(2)^{\alpha-1}$ regular. Thus for $\alpha = 2$, the graph has K_2 as two components and for $\alpha = 3$ it has K_4 as two components which is again planar. For $\alpha > 3$ the components contain K_5 making it non planar. Again if $n = 6$, it can be seen in Figure 1 that the graph is planar.
For $n = 3^\alpha,\ \alpha = 2$, one of the component containing integers co-prime to 3 is isomorphic to $K_{3,3}$ as in Figure 1. Thus making the graph non-planar. Also for $n = 10$ the graph in Figure 1 has an subgraph homeomorphic to K_5, making the graph non-planar. Hence the theorem.

11 Representation of Absorption graphs as factor graphs

One of the most striking feature of Absorption cayley graphs is that they can be seen as the union of subgraphs generated by primes.

Theorem 11.1. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $n \neq 2m,\ m$ being odd. Then Absorption graph $\Omega(Z_n)$ can be expressed as union of cliques generated by multiples of primes $p_1, p_2, \dots, p_k\ < n$.

Proof. By Theorem 2.6, we know that S forms a group if $n = p^\alpha$. Clearly if $n \neq 2m,\ m$ being odd, then $S = S_1 \cup S_2 \cdots \cup S_k$, each generated by p_i for $i = 1,\ \dots,\ k$. These S_i will be groups with respect to addition. Thus for each subgroup S_i, its elements will form a clique in $\Omega(Z_n)$. Hence $\Omega(Z_n)$ can be expressed as the union of cliques generated by multiples of prime.

Corollary 11.2. If $n = 2m,\ m$ being odd, then $\Omega(Z_n)$ consists of cliques generated by primes $p_1, p_2, \dots, p_k\ < n$, where $m = p_1 p_2 \dots p_k$.

Theorem 11.3. Absorption cayley graph $\Omega(Z_n)$ is bipartite if and only if $n = p$, where p is prime.

Proof. Let us consider $\Omega(Z_n)$, where $n = p,\ p$ is prime. Clearly $S = \{0\}$, thus a,b in Z_p forms an edge in $\Omega(Z_p)$ if and only if $a+b = 0$. Now we can place a and b in different sets say V_1 and V_2. Similarly for all such edges we can place these adjacent vertices in V_1 and V_2 since each vertex will have degree one. And then 0 can be placed in any of the sets. Thus $\Omega(Z_p)$ is bipartite.
Conversely, let $\Omega(Z_n)$ be bipartite. Let if possible $n \neq p$. Then clearly $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, for some $k > 1$ where p_i are primes or simply some power of prime.
If $n = p^\alpha,\ \alpha > 1$ then by Theorem 7.2, $\Omega(Z_n)$ will have two components atleast one of them complete and thus containing odd cycles of length 3.

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If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ then clearly two cases arise:
If $n = 2m,\ m$ being odd then by Corollary 11.2, the graph consists of clique and hence cycle of odd length. Thus $n \neq 2m,\ m$ being odd. Then also by Theorem 11.1, it will consist cliques and hence $\Omega(Z_n)$ will not be bipartite.

Thus for $n \neq p$ the Absorption graph $\Omega(Z_n)$ is not bipartite.

12 Girth, radius and diameter of Absorption cayley graph

Theorem 12.1. The girth for connected Absorption cayley graph $\Omega(Z_n)$ is four for $n = 6$ and three for $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},\ n > 6,\ p_i$ being prime for each $i = 1, \dots, k$.

Proof. By Theorem 7.1, we know that $\Omega(Z_n)$ is connected if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $k > 2$. Let $n = 6$ clearly we can see in Figure 1 that the graph has girth four. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},\ n > 6$, we that for atleast one prime $p_1, \dots, p_k$ in n there are three or more multiples in Z_n. For example, if $n = 10 = 2 * 5$ then $S = \{0, 2, 4, 5, 8\}$ containing four multiples of 2, which will form a complete subgraph in $\Omega(Z_n)$. Hence for $n > 6$ there will be a three cycle in $\Omega(Z_n)$. Thus the theorem.

Theorem 12.2. The diameter for connected Absorption cayley graph $\Omega(Z_n)$ is two.

Proof. By Theorem 11.1 and Corollary 11.2, $\Omega(Z_n)$ can be represented as cliques generated by primes. Then clearly each composite number belongs to more than one clique and each clique consist of vertices belonging to more than one clique. Also every pair of clique has atleast one vertex common. Now each prime forms an edge with element zero and thus any two primes have distance two. For a composite integer $m = pq$ where p and q are prime, the distance with any other composite $k = p'q'$(where p and q is not a factor) the distance again remains two as there would be an element in clique of p' and q' which is present in clique generated by p and q thus m and k would be at a distance two with each other. Similarly for any vertex in $\Omega(Z_n)$ will have a maximum distance two with any of the vertex. Thus the diameter of $\Omega(Z_n)$ is two.

Lemma 12.3. The eccentricity of each vertex in Absorption cayley graph $\Omega(Z_n)$ is two.

Proof. Let $a \in Z_n$ then a belongs to atleast one of the cliques generated by some prime say p_i. Let b be any vertex. Also b belongs to a clique generated by some prime say p_j, If d represents the distance then two cases arise:

(i) $p_i = p_j$, then $d(a, b) = 1$.
(ii) $p_i \neq p_j$. Then there exist $c = p_i p_j$ in Z_n which belongs to both the cliques. Hence $d(a, b) = 2$.

Since, no connected Absorption cayley graph is complete. Thus for each a there exist a b such that $d(a, b) = 2$. This is true for each vertex $a \in Z_n$. Hence eccentricity of each vertex in $\Omega(Z_n) = 2$.

Theorem 12.4. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $n \neq 2m,\ m$ being odd, the radius of absorption cayley graph $\Omega(Z_n)$ is two.

Proof. By Lemma 12.3, eccentricity of each vertex in $\Omega(Z_n)$ is two. Thus radius of $\Omega(Z_n)$ is two.

Corollary 12.5. Every Absorption graph $\Omega(Z_n)$ such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $n \neq 2m,\ m$ being odd is self centered.

Proof. By Lemma 12.3, eccentricity of all the vertices is two. Hence the Absorption cayley graph $\Omega(Z_n)$ is self centered.

13 Relation of Absorption cayley graphs with unitary addition graphs

Theorem 13.1. The Absorption cayley graphs $\Omega(Z_n)$ are compliment of unitary addition graphs if $n \neq 2m,\ m$ is odd.

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $n \neq 2m,\ m$ being odd. By corollary 2.4, each vertex in Z_n belongs to one of the two sets either U_n or S. And since Z_n is a group with respect to addition then $a + b \in Z_n,\ \forall a, b \in Z_n$. Also by definition of U_n and S, no element can be in the intersection of these two subsets of Z_n. Thus if $a + b \in U_n$ then $a + b \notin S$ and vice-versa. Hence the two graphs formed by U_n and S are compliment of each other for $n \neq 2m$, m being odd(as shown in Figure 2).

14 Perfectness of Absorption cayley graph

Theorem 14.1. [25] Strong Perfect Graph Theorem(SPGT). A graph G is perfect if and only if G and its complement G have no induced cycles of odd length atleast 5.

Theorem 14.2. [26] The unitary addition Cayley graph $G_n,\ n \geq 2$, is perfect if and only if n is even or $n = p^m;\ m \geq 1$.

Theorem 14.3. The Absorption cayley graph $\Omega(Z_n)$ is perfect if and only if n is even or $n = p^m;\ m \geq 1$.

Proof. By Theorem 13.1, 14.1 and 14.2, the Absorption cayley graph $\Omega(Z_n)$ is perfect if and only if G_n is perfect, that is when n is even or $n = p^m;\ m \geq 1$. Since, if $n = 2m,\ m$ being odd.

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15 Edge connectivity of Absorption cayley graph

Theorem 15.1. [27] Let G be a graph with diameter ≥ 2. Then the edge connectivity $\lambda(G)$ is equal to the minimum degree $\delta(G)$.

Theorem 15.2. The edge connectivity of a connected Absorption cayley graph $\Omega(Z_n)$, represented by $\lambda(\Omega(Z_n))$ is equal to $|S|-1$.

Proof. By Theorem 12.2, the diameter of a connected Absorption cayley graph $\Omega(Z_n)$ is two, thus the edge connectivity is equal to the minimum degree of $\Omega(Z_n)$. Thus edge connectivity is equal to $|S|-1$.

16 Examples of Cayley table and adjacency matrix

Cayley table 1 and adjacency matrix of Z_6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$$A = \begin{bmatrix} 0&0&1&1&0&0 \\ 0&0&1&0&0&1 \\ 1&1&0&0&1&0 \\ 1&0&0&0&0&1 \\ 0&0&1&0&0&1 \\ 0&1&0&1&1&0 \end{bmatrix}$$

Caley table 2 and adjacency matrix of $\Omega(Z_n)$ for $n=8$

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

$$\begin{bmatrix} 0&0&1&0&1&0&1&0 \\ 0&0&0&1&0&1&0&1 \\ 1&0&0&0&1&0&1&0 \\ 0&1&0&0&0&1&0&1 \\ 1&0&1&0&0&0&1&0 \\ 0&1&0&1&0&0&0&1 \end{bmatrix}$$

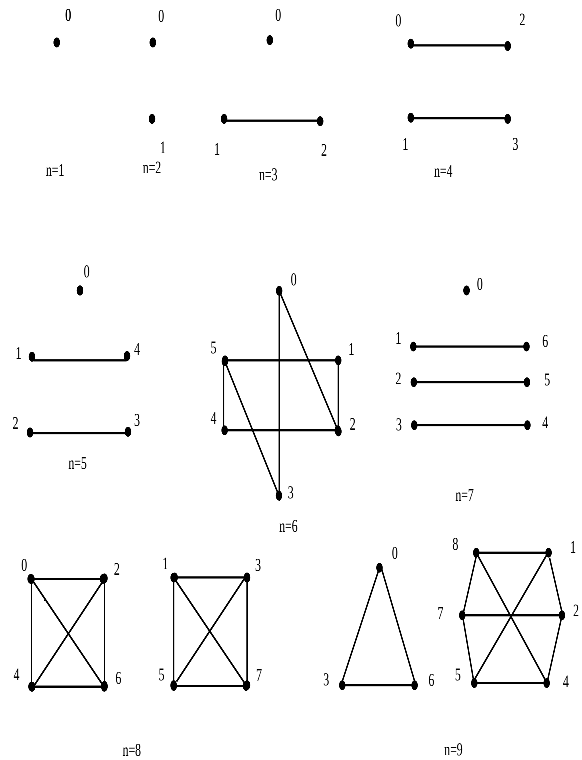

Figure 1: Examples of Absorption cayley graphs

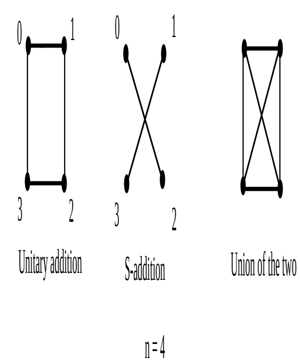

Figure 2: Example of union of Absorption cayley graphs and unitary addition graph being as complete graph

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