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The Zweier Ideal Convergent Sequence Spaces Over $p-$ Metric Spaces and de la Vall$\acute{E}$e-poussin Mean of Order α Defined by Musielak-Orlicz Functions

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Abstract: In this paper we introduce the $I-$ of χ^2 sequence spaces over $p-$ metric spaces using Zweier transform and defined by Musielak function. We also examine some topological properties and prove some inclusion relation between these spaces and the Fatou property and the Banach-Saks property of the new space and $\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$.

Keywords: Zweier operator, analytic sequence, χ^2 space, difference sequence space,Musielak - Orlicz function, $p-$ metric space, Lacunary sequence, ideal.

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \,(m,n = 1,2,3,\ldots)\,.$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m,n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = sup_{m,n}\left\{|x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m,n : 1,2,3,\ldots\right\}, \quad (1)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . \\ . \\ . \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

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A double sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.

Let M and Φ are mutually complementary Orlicz functions. Then, we have:
(i) For all $u,y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[1]] \quad (2)$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \quad (3)$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \quad (4)$$

Lindenstrauss and Tzafriri [2] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-Orlicz function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mn})(u) : u \geq 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty \right\},$$

where $\widetilde{M}_f$ is a need not convex modular defined by

$$\widetilde{M}_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \widetilde{f}_{mn}(|x_{mn}|)^{(1/m)+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{ inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \widetilde{f}_{mn}\left(\frac{|x_{mn}|^{(1/m)+n}}{mn}\right)\right) \leq 1 \right\}$$

If X is a sequence space, we give the following definitions:

(i)X' = the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\ for\, each\, x \in X \right\}$;

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\ foreach\, x \in X \right\}$;

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn \geq 1}\left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X \right\}$;

(v)$let\, X\, be\, an\, FK - space \supset \phi; then\, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn}|a_{mn}x_{mn}|^{1/m+n} < \infty,\ foreach\, x \in X \right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of\, X, \gamma - dual\, of\, X,\ \delta - dual\, of X\, respectively.X^{\alpha}$ is defined by Gupta and Kamptan [1] . It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [4] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar [3]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1}|\Delta x_k| \text{ and}$$
$$\|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m,n \in \mathbb{N}$.

Let λ and μ be two sequence spaces and $A = \left(a_{k\ell}^{mn}\right)$ is an infinite four dimensional matrix of real or complex numbers $a_{k\ell}^{mn}$, where $k,\ell,m,n \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ, and we denot it by writing $A : \lambda \to \mu$.

If for every sequence $x = (x_{mn}) \in \lambda$ the sequence $Ax = \{A(Ax)_{mn}\} \in \mu$, then A transform of $x \in \mu$, where $(Ax)_{mn} = \sum_m \sum_n a_{k\ell}^{mn} x_{mn}, (k,\ell \in \mathbb{N})$.

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \to \mu$. Thus $a \in (\lambda,\mu)$ if and only if $(Ax) \in \mu$

for every $x \in \lambda$.

The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method.

The Z^p- transformation of the sequence $x = (x_{mn})$, (i.e) $y_{ij} = p x_{ij} + (1-p) x_{i-1j-1}$.

Z^p denotes the matrix $Z^p = \left(Z^{ij}_{mn}\right)$ defined by

$$Z^{ij}_{mn} = \begin{cases} p, & \text{if } (mn = ij); \\ 1-p, & \text{if } (i-1 = m, j-1 = n); \\ 0, otherwise & \end{cases}$$

The Zweier sequence spaces Z_{Λ^2} and Z_{χ^2} as follows: $Z_{\Lambda^2} = \left\{x = (x_{mn}) \in w^2 : Z^p x \in \Lambda^2\right\}$ and $Z_{\chi^2} = \left\{x = (x_{mn}) \in w^2 : Z^p x \in \chi^2\right\}$.

2 Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,

(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ $for 1 \leq p < \infty$;

(or)

(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,

for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ & . & & \\ & . & & \\ & . & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if

(1) $\rho(x) \geq 0$, for all $x \in X$;

(2) $\rho(-x) = \rho(x)$, for all $x \in X$;

(3) $\rho(x+y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$;

(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $\rho(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $\rho(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [5], Theorem 10.4.2, p.183).

The notion of deal convergence was introduced first by Kostyrko et al.[8] as a generalization of statistical convergence which was further studied in toplogical spaces by Kumar et al.[6,7] and also more applications of ideals can be deals with various authors by B.Hazarika [9-21] and B.C.Tripathy and B. Hazarika [22-24].

A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if

(1) $\phi \in I$

(2) $A, B \in I$ imply $A \bigcup B \in I$

(3) $A \in I, B \subset A$ imply $B \in I$.

A sequence of positive integers $\theta = (k_{rs})$ is called double lacunary if $k_{00} = 0, 0 < k_{rs} < k_{r+1,s+1}$ and $\varphi_{rs} = k_{rs} - k_{r-1s-1} \to \infty$ as $r, s \to \infty$. The intervals determined by θ will be denoted by $J_{rs} = (k_{r-1s-1}, k_{rs})$ and $q_{rs} = \frac{k_{rs}}{k_{r-1s-1}}$.

Let $\lambda = (\lambda_{rs})$ be an increasing sequence of positive real numbers tending to ∞ such that $\lambda_{rs} \leq \lambda_{rs} + 1, \lambda_{11} = 1$. The generalized de la Vall$\acute{e}$e-Poussin mean is defined by $t_{rs} = \frac{1}{\lambda_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} x_{mn}$, where $I_{rs} = [(r,s) - \lambda_{rs} + 1, r, s]$ for $r, s = 1, 2, 3, \cdots$. A sequence $x = (x_{mn})$ is said to be $(V, \lambda)-$ summable to a number L if $t_{rs}(x) \to L$ as $r, s \to \infty$. If $\lambda_{rs} = r, s$ then $(V, \lambda)-$ summability is reduced to Ces$\acute{a}$ro summability.

We denote Λ the set of all increasing sequences of positive real numbers tending to ∞ such that $\lambda_{rs} \leq \lambda_{rs} + 1, \lambda_{11} = 1$.

Let I be an admissible ideal of $\mathbb{N} \times \mathbb{N}$, $f = (f_{mn})$ be a Musielak-Orlicz function, $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a $p-$metric space. By $w^2(p-X)$ we denote the space of all sequences defined over $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$.

The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq sup q_{mn} = H, K = max\left(1, 2^{H-1}\right)$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K\left\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\right\} \quad (5)$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \leq max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.

3 χ^2 sequence space of order α

In this section let $\alpha \in (0,1]$ be any real number, let $\lambda = (\lambda_{rs})$ be an increasing sequence of positive real numbers tending to ∞ such that $\lambda_{rs} \leq \lambda_{rs}+1, \lambda_{11}=1$, and q be a positive real number such that $1 \leq q < \infty$. Now we define the following class of sequence spaces:

$$\left[Z^{\alpha}_{\chi^2 f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I = \left\{x=(x_{mn}): \left\{\frac{1}{\lambda^{\alpha}_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \varepsilon\right\} \in I\right\},$$

where $\mu_{mn}(x) = \left[(m+n)!\left(Z^i x\right)_{mn}\right]^{(1/m)+n}$

$$\left[Z^{\alpha}_{\Lambda^2 f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I = \left\{x=(x_{mn}): \exists K>0, \left\{sup_{r,s}\frac{1}{\lambda^{\alpha}_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[\|\eta_{mn}(x), (d(x_1),\ d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq K\right\} \in I\right\},$$

where $\eta_{mn}(x) = \left[\left(Z^i x\right)_{mn}\right]^{(1/m)+n}$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces. $\left[Z^{\alpha}_{\chi^2 f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ and $\left[Z^{\alpha}_{\Lambda^2 f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ which we shall discuss in this paper.

4 Main Results

4.1 Theorem

Let $\widetilde{f} = \left(\widetilde{f_{mn}}\right)$ be a Zweier Musielak-Orlicz function, the sequence spaces $\left[Z^{\alpha}_{\chi^2 \widetilde{f}\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ and $\left[Z^{\alpha}_{\Lambda^2 \widetilde{f}\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

4.2 Theorem

Let $\widetilde{f} = \left(\widetilde{f_{mn}}\right)$ be a Zweier Musielak-Orlicz function, the sequence space $\left[Z^{\alpha}_{\chi^2 \widetilde{f}\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ is a paranormed space with respect to the paranorm defined by

$g(x) = inf\left\{\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\}.$

Proof: Clearly $g(x) \geq 0$ for $x = (x_{mn}) \in \left[Z^{\alpha}_{\chi^2 \widetilde{f}\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$

Since $\widetilde{f_{mn}}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then $inf\left\{\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\}$. Suppose that $\mu_{mn}(x) \neq 0$ for each $m,n \in \mathbb{N}$. Then $\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \to \infty$. It follows that $\left(\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)^{1/H} \to \infty$ which is a contradiction. Therefore $\mu_{mn}(x) = 0$. Let $\left(\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)^{1/H} \leq 1$ and $\left(\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)^{1/H} \leq 1$

Then by using Minkowski's inequality, we have

$$\left(\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)^{1/H} \leq \left(\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)^{1/H} + \left(\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)^{1/H}.$$

So we have

$$g(x+y) = inf\left\{\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\} \leq inf\left\{\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\} + inf\left\{\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\}.$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition, $g(\lambda x) = inf\left\{\left[\widetilde{f_{mn}}\left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\}$. Then $g(\lambda x) = inf \left\{((|\lambda| t)^{1/H}: \left[\widetilde{f_{mn}}\left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \leq 1\right\}$ where $t = \frac{1}{|\lambda|}$. Since $|\lambda| \leq max(1, |\lambda|)$, we have $g(\lambda x) \leq max(1, |\lambda|)\, inf \left\{t^{1/H}: \left[\widetilde{f_{mn}}\left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] q1\right\}$

This completes the proof.

4.3 Theorem

The $\beta-$ dual space of $\left[Z^{\alpha}_{\chi^{2}\widetilde{f}\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I} = \left[Z^{\alpha}_{\Lambda^{2}\widetilde{f}\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$

Proof: First, we observe that

$$\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I} \subset \left[Z^{\alpha}_{\Gamma^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}.$$

Therefore

$$\left[Z^{\alpha}_{\Gamma^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta} \subset \left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta}.$$

Hence

$$\left[Z^{\alpha}_{\Lambda^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I} \tag{6}$$

$$\subset \left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta}. \tag{7}$$

Next we show that $\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta} \subset \left[Z^{\alpha}_{\Lambda^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$

Let $y = (y_{mn}) \in \left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta}$

Consider $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with

$x = (x_{mn}) \in \left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$

$x = [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})]$

$$= \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0 & \ldots & 0 \\ . \\ . \\ . \\ 0 & 0 & \ldots \frac{\lambda_{rs}}{(m+n)!} & \frac{-\lambda_{rs}}{(m+n)!} & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0 & \ldots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0 & \ldots & 0 \\ . \\ . \\ . \\ 0 & 0 & \ldots \frac{\lambda_{rs}}{(m+n)!} & \frac{-\lambda_{rs}}{(m+n)!} & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] =$$

$$\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0, & \ldots & 0 \\ . \\ . \\ . \\ 0 & 0 & \ldots f_{mn}\left(\frac{\lambda_{rs}}{(m+n)!}\right) & f_{mn}\left(\frac{\lambda_{rs}}{(m+n)!}\right) & \ldots & 0 \\ 0 & 0 & \ldots f_{mn}\left(\frac{-\lambda_{rs}}{(m+n)!}\right) & f_{mn}\left(\frac{\lambda_{rs}}{(m+n)!}\right) & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0, & \ldots & 0 \end{pmatrix}. \quad \text{Hence}$$

converges to zero.

Therefore $[(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \in \left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$.

Hence $d((\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) = 1$. But

$|y_{mn}| \leq \|f\| d((\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) \leq \|f\| \cdot 1 < \infty$ for each m, n. Thus (y_{mn}) is a double analytic sequence and hence an $p-$ metric space of Zweier Musielak Orlicz function is a double analytic sequence. In other words $y \in \left[Z^{\alpha}_{\Lambda^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$. But $y = (y_{mn})$ is arbitrary in $\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta}$.

Therefore

$$\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta} \subset \left[Z^{\alpha}_{\Lambda^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I} \tag{8}$$

hTo prove the inclusion From (4.1) and (4.2) we get

$$\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta} = \left[Z^{\alpha}_{\Lambda^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta}.$$

4.4 Theorem

The dual space of $\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I}$ is $\left[Z^{\alpha}_{\Lambda^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I\beta}$. In other words $\left[Z^{\alpha}_{\chi^{2}f\lambda}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I^{*}} =$

$\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$

Proof: We recall that $\delta_{mn} = \begin{pmatrix} 0\;0 & \ldots 0 & 0 \ldots \\ 0\;0 & \ldots 0 & 0 \ldots \\ . & & \\ . & & \\ . & & \\ 0\;0 & \ldots \frac{\lambda_{rs}}{(m+n)!} & 0 \ldots \\ 0\;0 & \ldots 0 & 0 \ldots \end{pmatrix}$

with $\frac{\lambda_{rs}}{(m+n)!}$ in the $(m,n)th$ position and zero's else where, $\left[Z_{\chi^{2f}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I =$

$$\begin{pmatrix} 0. & . & .\;0 \\ . & & \\ . & & \\ . & & \\ 0 & f\left(\frac{\lambda_{rs}}{(m+n)!}\right)^{1/m+n} & .\;0 \\ & (m,n)^{th} & \\ 0 & . & .\;0 \end{pmatrix}$$

which is a $p-$ metric of double gai sequence of Zweier Musielak Orlicz function. Hence, $x \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$. $f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$ with $x \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ and $f \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$, where $\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I^*}$ is the dual space of $\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ Take $x = (x_{mn}) \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$. Then,

$$|y_{mn}| \leq \|f\| \, d(\varphi_{rs}, 0) < \infty \qquad \forall m,n \tag{9}$$

Thus, (y_{mn}) is a double analytic sequence and hence an $p-$ metric is a Zweier Musielak Orlicz function of double analytic sequence. In other words, $y \in \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$. Therefore $\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I^*} = \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$. This completes the proof.

4.5 Theorem

(i) If the sequence (f_{mn}) satisfies Δ_2- condition, then $\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I = \left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$.
(ii) If the sequence (g_{mn}) satisfies Δ_2- condition, then $\left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I = \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$
Proof: Let the sequence (f_{mn}) satisfies Δ_2- condition, we get

$$\left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I \subset \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I \tag{10}$$

To prove the inclusion $\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I \subset \left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$, let $a \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$. Then for all $\{x_{mn}\}$ with $(x_{mn}) \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty. \tag{11}$$

Since the sequence (f_{mn}) satisfies Δ_2- condition, then $(y_{mn}) \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$, we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left|\frac{y_{mn}a_{mn}\lambda_{rs}}{(m+n)!}\right| < \infty$. by (4.5). Thus $(a_{mn}) \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I = \left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ and hence $(a_{mn}) \in \left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$. This gives that

$$\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I \subset \left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I \tag{12}$$

we are granted with (4.4) and (4.6) $\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I = \left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$
(ii) Similarly, one can prove that $\left[Z^{\alpha}_{\chi^{2g\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I \subset \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^I$ if the sequence (g_{mn}) satisfies Δ_2- condition.

4.6 Proposition

If $0 < q_{mn} < r_{mn} < \infty$ for each m and m, then $\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^q\right]^I \subseteq$

$\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{r}_{p}\right]^{I}$

Proof: Let $x = (x_{mn}) \in \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I}$. We have

$$sup_{rs}\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I} < \infty$$

for sufficiently large value of m and n. Since f_{mn}'s are non-decreasing, we get

$$sup_{rs}\left[Z_{\chi^{2f}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{r}_{p}\right]^{I} \subset \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I}$$

Thus, $x = (x_{mn}) \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{r}_{p}\right]^{I}$. This completes the proof.

4.7 Proposition

(i) If $0 < inf q_{mn} \leq q_{mn} < 1$ then $\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I} \subset \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$.

(ii) If $1 \leq q_{mn} \leq sup q_{mn} < \infty$, then $\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I} \subset \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I}$

Proof: Let $x = (x_{mn}) \in \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I}$. Since $0 < \inf q_{mn} \leq 1$, we have $sup_{rs}\left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I} \subset \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I}$, and hence $x = (x_{mn}) \in \left[Z^{\alpha}_{\chi^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$.

(ii) Let q_{mn} for each (m,n) and $sup_{rs}q_{mn} < \infty$. Let $x = (x_{mn}) \in \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$. Then for each $0 < \varepsilon < 1$, there exists a positive integer N such that

$$sup_{rs}\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I} \leq \varepsilon < 1,$$

for all $m,n \geq N$. This implies that $sup_{rs}\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I} \subset \sup_{rs}\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$. Thus $x = (x_{mn}) \in \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{q}_{p}\right]^{I}$.This completes the proof.

4.8 Theorem

The space $\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$ is not Fatou property.

Proof: Let $x = (x_{mn})$ be a real sequence and (x_{mn}) be any non-decreasing sequence of non-negative elements form $\left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$ such that $x_{mn}(ij) \to x(ij)$ as $i,j \to \infty$ coordinatewisely and $d(x,y) = sup_{i,j}\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\}$. Let us denote $T = d(x,y) = sup_{i,j}\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\}$. Since the supremum is homogeneous, then we have $\frac{1}{T} sup_{r,s}\frac{1}{\lambda^{\alpha}_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[\|\eta_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right] \leq sup_{r,s}\frac{1}{\lambda^{\alpha}_{rs}}\left(\frac{\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[\|\eta_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_{p}\right]}{d(x,y)}\right) = \frac{d(x,y)}{d(x,y)} = 1$ Also by the assumptions that (x_{mn}) is non-dreceasing and convergent to x coordinatewisely and by the Beppo-Levi theorem, we have $\frac{1}{T} lim_{mn\to\infty} sup_{r,s}\frac{1}{\lambda^{\alpha}_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[\|\eta_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right] = sup_{r,s}\frac{1}{\lambda^{\alpha}_{rs}}\left(\frac{\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[\|\eta_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_{p}\right]}{T}\right) \leq 1.$ whence $d(x,y) \leq T = sup_{i,j}\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\} = lim_{i,j\to\infty}\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\} < \infty.$

Therefore $x \in \left[Z^{\alpha}_{\Lambda^{2f\lambda}}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p}\right]^{I}$. On the other hand, since $0 \leq x$ for any natural number i,j and the sequence (x_{ij}) is non-decreasing, we obtain the sequence $\left(\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\}\right)$ is bounded form above by $d(x,y)$. Therefore $lim_{i,j\to\infty}\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\} \leq d(x,y)$ which contradicts the above inequality proved already, yields that $d(x,y) = lim_{i,j\to\infty}\left\{\left|x_{ij} - 0\right|^{1/i+j}_{\alpha} : i,j = 1,2,3,\cdots\right\}$. This completes the proof.

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