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Statistical Convergence of Bernstein Operators

A. Esi[1], *S. Araci*[2,*] *and M. Acikgoz*[3]

[1] Department of Mathematics, Science and Art Faculty, Adiyaman University, Adiyaman, Turkey
[2] Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey
[3] Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, TR-27310 Gaziantep, Turkey

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Abstract: The Bernstein operator is one of the important topics of approximation theory in which it has been studied in great details for a long time. The aim of this paper is to study the statistical convergence of sequence of Bernstein polynomials. In this paper, we introduce the concepts of statistical convergence of Bernstein polynomials and V_B-summability and related theorems. We also study Korovkin type-convergence of Bernstein polynomials.

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1 Introduction

The notion of statistical convergence was introduced by Fast [4] and Schoenberg [12] independently in the same year 1951. Actually the idea of statistical convergence was formerly given under the name "almost convergence". The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition of celebrated monograph by Zygmund published in Warsaw in 1935 [16]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by Fridy [17], Šalát [11], topological groups (Prullage, [8]), topological spaces (Cakalli and Khan [19], Di Maio and Kočinac [6]), locally convex spaces (Maddox [5]), measure theory (Miller [18]), fuzzy mathematics in sequence spaces ([15]), free spaces ([7]), normed spaces (Reddy [10]) and probabilistic normed space (Rahmat [9]).

Let $K \subseteq N$. Then $\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|$ is said to be natural density of the set K.

As known, the Bernstein operator of order n is given by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$. $B_n(f;x)$ was introduced in 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem cf. [1], [2], [3].

In 1912, Bernstein showed the following Theorem ([13], [14]).

Theorem 1. *Given a function $f \in C[0,1]$ and any $\varepsilon > 0$, there exists an integer N such that*

$$|f(x) - B_n(f;x)| < \varepsilon$$

for all $n \geq N$ and $0 \leq x \leq 1$.

By Theorem 1, we note that if f is merely bounded on $[0,1]$, the sequence $(B_n(f,x))_{n=1}^{\infty}$ converges to $f(x)$ at any point in which f is continuous. As a remarkable property, we note further that the derivatives of Bernstein operator converge to the derivatives of the function cf. [2].

* Corresponding author e-mail: mtsrkn@hotmail.com

2 Statistical convergence of Bernstein polynomials

The aim of this paper is to study the statistical convergence of sequence of Bernstein polynomials. In this part, we now give two definitions which will be played an important role in order to prove Theorem 2.

Definition 1.*Let f be a continuous function defined on the closed interval $[0,1]$. A sequence of Bernstein polynomials $(B_n(f,x))$ is said to be statistically convergent or s_B-convergent to f if, for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |B_k(f,x) - f(x)|\}$ has natural density zero, i.e., $\delta(K_\varepsilon) = 0$. That is,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |B_k(f,x) - f(x)| \ge \varepsilon\}| = 0.$$

In this case, we write $\delta\text{-}\lim B_k(f,x) = f(x)$ or $B_k(f,x) \underset{\rightarrow}{s_B} f(x)$.

Definition 2.*Let f be a continuous function defined on the closed interval $[0,1]$. A sequence of Bernstein polynomials $(B_k(f,x))$ is said to be strongly Cesáro summable of V_B-summable to $f(x)$ if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |B_k(f,x) - f(x)| = 0.$$

Theorem 2.*Let f be a continuous function defined on the closed interval $[0,1]$. Then*

(i) If a sequence of Bernstein polynomials $(B_k(f,x))$ is V_B-Summable to $f(x)$ then $(B_k(f,x))$ is s_B-convergent to $f(x)$.

(ii) If $|B_k(f,x) - f(x)| \le M$ for all $x \in [0,1]$ and a sequence of Bernstein polynomials $(B_k(f,x))$ is s_B-convergent to $f(x)$ then it is V_B-summable to $f(x)$.

*Proof.*Let $\varepsilon > 0$ and a sequence of Bernstein polynomials $(B_k(f,x))$ is V_B-summable to $f(x)$. We have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} |B_k(f,x) - f(x)| &\ge \frac{1}{n} \sum_{\substack{k=1 \\ |B_n(f,x)-f(x)|\ge\varepsilon}}^{n} |B_k(f,x) - f(x)| \\ &\ge \frac{\varepsilon}{n} |\{k \le n : |B_k(f,x) - f(x)| \ge \varepsilon\}|. \end{aligned}$$

Therefore $(B_n(f,x))$ is s_B-convergent to $f(x)$.

(ii) Suppose that $(B_n(f,x))$ is s_B-convergent and $|B_n(f,x) - f(x)| \le M$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} |B_k(f,x) - f(x)| &= \frac{1}{n} \sum_{\substack{k=1 \\ |B_n(f,x)-f(x)|>\varepsilon}}^{n} |B_k(f,x) - f(x)| \\ &\quad + \frac{1}{n} \sum_{\substack{k=1 \\ |B_n(f,x)-f(x)|}}^{n} |B_k(f,x) - f(x)| \\ &\le \frac{M}{n} |\{k \le n : |B_k(f,x) - f(x)|\}| + \varepsilon \end{aligned}$$

which implies that the sequence of Bernstein polynomials is V_B-summable.

3 Statistical Korovkin type approximation theorem

In this section, we prove an analogue of classical Korovkin theorem by using the concepts of statistical convergence of sequences of Bernstein polynomials. Recently, such types of approximation theorems are proved such as in [20] and [21] by using the notion of statistical convergence.

The classical Korovkin approximation theorem can be found in [22] and [23] as folllows:

Let $C[a,b]$ be the space of all functions f continuous on the interval $[a,b]$. Suppose that (T_n) is a sequence of positive linear operators from $C[a,b]$ to $C[a,b]$ such that

$$\lim_n \|T_n(f,x) - f(x)\|_\infty = 0 \quad (f \in C[a,b])$$

if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty$ for $i = 0,1,2,\cdots$, where $f_0(x) = 1$, $f_1(x) = x$ and $f(x) = x^2$.

We know that $C[a,b]$ is a Banach space with the norm

$$\|f\|_\infty = \sup_{a\le x\le b} |f(x)| \quad (f \in C[a,b]).$$

We write $T_n(f,x)$ for $T_n(f(t),x)$ and T is a positive operator if $T_n(f,x) \ge 0$, $(\forall f(x) \ge 0)$.

Theorem 3.*Suppose that $(B_n(f,x))$ is a sequence of Bernstein polynomials. Then for any function $f \in C[0,1]$*

$$\delta - \lim_n \|B_n(f,x) - f(x)\|_\infty = 0 \tag{2}$$

if and only if

$i)$ $\delta - \lim_n \|B_n(1,x) - 1\|_\infty = 0$

$ii)$ $\delta - \lim_n \|B_n(t,x) - x\|_\infty = 0$

$iii)$ $\delta - \lim_n \left\|B_n(t^2,x) - x^2\right\|_\infty = 0.$

*Proof.*The conditions (i), (ii) and (iii) follow immediately from condition (2), since each of the functions 1, x, x^2 belong to $C[0,1]$. Now we prove the converse part: By the continuity of f on $[0,1]$, we can write

$$|f(x)| \le M \quad (0 < x < 1).$$

Therefore

$$|f(t) - f(x)| \le 2M \quad (0 < x < 1). \tag{3}$$

Also, since $f \in [0,1]$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \tag{4}$$

for all $0 < |t - x| < \delta < 1$. Using (3) and (4), we get

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2} a \qquad (0 < |t - x| < \delta < 1)$$

where $a = (t - x)^2$. This means that

$$-\left(\varepsilon + \frac{2M}{\delta^2} a\right) < f(t) - f(x) < \varepsilon + \frac{2M}{\delta^2} a.$$

Now we may apply $B_n(1,x)$ to this inequality since $B_n(f,x)$ is monotone and linear. So

$$B_n(1,x)\left(-\varepsilon-\tfrac{2M}{\delta^2}a\right)<T_n(1,x)(f(t)-f(x))<B_n(1,x)\left(\varepsilon+\tfrac{2M}{\delta^2}a\right).$$

Note that x is fixed and therefore $f(x)$ is a constant number. So, from the last inequality, we have

$$-B_n(1,x)\varepsilon-\tfrac{2M}{\delta^2}B_n(a,x)<B_n(f,x)-f(x)B_n(1,x)<\varepsilon B_n(1,x)+\tfrac{2M}{\delta^2}B_n(a,x). \tag{5}$$

But

$$\begin{aligned} B_n(f,x)-f(x) &= B_n(f,x)-f(x)B_n(1,x)+f(x)B_n(1,x)-f(x)\\ &= B_n(f,x)-f(x)B_n(1,x)+f(x)(B_n(1,x)-1). \end{aligned} \tag{6}$$

From (5) and (6), we get

$$B_n(f,x)-f(x)<\varepsilon B_n(1,x)+\tfrac{2M}{\delta^2}B_n(a,x)+f(x)(B_n(1,x)-1).$$

Let us estimate $B_n(a,x)$ as follows:

$$\begin{aligned} B_n(a,x) &= B_n\left((t-x)^2,x\right)=B_n\left(t^2-2tx+x^2,x\right)\\ &= B_n\left(t^2,x\right)-2xB_n(t,x)+x^2B_n(1,x)\\ &= \left(B_n\left(t^2,x\right)-x^2\right)-2x(B_n(t,x)-x)+x^2B_n(1,x). \end{aligned}$$

Now using (6), we get

$$\begin{aligned} B_n(f,x)-f(x) &< \varepsilon B_n(1,x)+\frac{2M}{\delta^2}\left(B_n\left(t^2,x\right)-x^2\right)\\ &\quad -2x(B_n(t,x)-x)+x^2(B_n(1,x)-1)\\ &\quad +f(x)(B_n(1,x)-1)\\ &= \varepsilon(B_n(1,x)-1)+\varepsilon+\frac{2M}{\delta^2}\left(B_n\left(t^2,x\right)-x^2\right)\\ &\quad -2x(B_n(t,x)-x)+x^2(B_n(1,x)-1)+f(x)(B_n(1,x)-1). \end{aligned}$$

Since ε is arbitrary, we may write

$$\begin{aligned} \|B_n(f,x)-f(x)\|_\infty &\leq \left(\varepsilon+\frac{2M}{\delta^2}+M\right)\|B_n(1,x)-1\|_\infty\\ &\quad +\frac{4M}{\delta^2}\|B_n(t,x)-x\|_\infty+\frac{2M}{\delta^2}\left\|B_n\left(t^2,x\right)-x^2\right\|_\infty\\ &\leq A\left(\|B_n(1,x)-1\|_\infty+\|B_n(t,x)-x\|_\infty+\left\|B_n\left(t^2,x\right)-x^2\right\|_\infty\right) \end{aligned}$$

where $A=\max\left(\varepsilon+\frac{2M}{\delta^2}+M,\frac{4M}{\delta^2}\right)$.

Now, for $\varepsilon'>0$, we write

$$K=\left\{\begin{array}{c} n\in\mathbb{N}:\|B_n(1,x)-1\|_\infty+\|B_n(t,x)-x\|_\infty\\ +\left\|B_n\left(t^2,x\right)-x^2\right\|_\infty\geq\frac{\varepsilon'}{A}\end{array}\right\},$$

$$K_1=\left\{n\in\mathbb{N}:\|B_n(1,x)-1\|_\infty\geq\frac{\varepsilon'}{3A}\right\},$$

$$K_2=\left\{n\in\mathbb{N}:\|B_n(t,x)-x\|_\infty\geq\frac{\varepsilon'}{3A}\right\},$$

$$K_3=\left\{n\in\mathbb{N}:\left\|B_n\left(t^2,x\right)-x^2\right\|_\infty\geq\frac{\varepsilon'}{3A}\right\}.$$

Then $K\subset K_1\cup K_2\cup K_3$ and then

$$\delta(K)\leq\delta(K_1)+\delta(K_2)+\delta(K_3).$$

Therefore, using (2)-(4), we get

$$\delta-\lim_n\|B_n(f,x)-f(x)\|_\infty=0.$$

This completes the proof.

4 Conclusion

In this paper, we have studied the statistical convergence of sequence of Bernstein polynomials. By this consideration, we have introduced the concepts of statistical convergence of Bernstein polynomials and V_B-summability and related theorems. We have also given Korovkin type-convergence of Bernstein polynomials.

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Ayhan Esi was born in Istanbul, Turkey, on March 5, 1965. Ayhan Esi got his B.Sc. from Inonu University in 1987 and M. Sc. and Ph.D. degree in pure mathematics from Elazig University, Turkey in 1990 and 1995, respectively. His research interests include Summability Theory, Sequences and Series in Analysis and Functional Analysis. In 2000, Esi was appointed to Education Faculty in Gaziantep University. In 2002, Esi was appointed as the head of Department of Mathematics in Science and Art Faculty in Adiyaman of the Inonu University. In 2006, Esi joined the Department of Mathematics of Adiyaman University.

Serkan Araci was born in Hatay, Turkey, on October 1, 1988. He has published over than 80 papers in reputed international journals. His research interests include p-adic analysis, theory of analytic numbers, q-series and q-polynomials, p-adic analysis, and theory of umbral calculus. Currently, he works as a Lecturer at Hasan Kalyoncu University, Sahinbey, Turkey. On the other hand, Araci is an editor and a referee for several international journals

Mehmet Acikgoz received M. Sc. And Ph. D. From Cukurova University, Turkey. He is currently associate Professor at University of Gaziantep. His research interests are approximation theory, functional analysis, p-adic analysis and analytic numbers theory.