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Ikeda Map and Phase Conjugated Ring Resonator Chaotic Dynamics

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Abstract: The dynamical behavior of a laser beam confined within a ring-phase-conjugated resonator modeled with a chaos generating element based on the Ikeda map is presented. Using the matrix ray optics, explicit expressions are obtained for the elements of the 2x2 dynamical matrix, which is found in terms of the specific map parameters, state variables and resonator parameters. The phase diagrams obtained show that the so called Ikeda map beams present complex dynamics, including critical points and metastability.

Keywords: Ikeda map, Chaos, Resonator, Laser.

1 Introduction

A key property stemming from the Ikeda map is a bistable behavior; in the past decades bi-stable optical elements have generated a large interest from its applicability as an optical device which may be used to obtain variable length pulses, infinite pulse trains [1], optical amplifiers, memory functions [3] and logical gate arrays [4], furthermore it has also been reported that intracavity non-linear elements can improve significantly the output power performance of a phase-conjugated laser oscillator [2]. On the other hand, bistable behavior can be observed on the transmitted light within a Fabry-Pérot cavity with a two-level absorber; this phenomenon known as optical bistability can be considered as a first order phase transition in a system far from thermal equilibrium [6]. It is known that optical bistability can be obtained in two different ways; one comes from the saturation of light absorption by the two-level absorber and it is called absorptive bistability [5], whilst the other can establish through the interaction between the cavity mistuning and the nonlinear dispersion of an absorbing unit, thus called dispersive bistability [3]. The usual approach to study this behavior is the mean field theory, an approximation helping to avoid certain complexities that arise in the mathematical treatment of large-scale and complex systems. While studying optical bistability, this is achieved by taking the average of the spatial variation of the electric and the polarization fields, with the target being to obtain a set of difference equations which don't contain the spatial coordinate as has been shown by Bonifacio and Lugiato, this treatment allows absorptive bistability to be fully treated analytically while taking into account the propagation effects. Ikeda then used these results where Bonifacio and Lugiato proposed a simplified system, implying ring cavity to be used as the feedback mechanism instead of the Fabry-Pérot one [7]. Finally the Maxwell-Bloch equations were applied to the ring cavity to then take the fast limit of the longitudinal relaxation in order to obtain the Ikeda map equations [8]. Once the Ikeda map equations are introduced, an idea arises to obtain a theoretical" optical element exhibiting the same behavior as the Ikeda map, which is the main scope of the present paper. Note that similar ideas have been previously explored and discussed elsewhere [10], [11], [20]. Which allowed several other maps to be treated; the logistic map [10], Hénon map[12], Standard map[13], Duffing map[14], [15] and the Tinkerbell map [16], [17].

This paper is organized as follows; in Section 2, after briefly discussing the previous work done by Ikeda and others about the Ikeda map, the relevant equations inherent to the specific case we deal with are introduced

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and formulated in terms of a real two-dimensional mapping. Then, in Section 3, is where the matrix framework used in this paper is outlined, as it is known that in paraxial optics any optical element can be described by a 2x2 matrix; we proceed this by means of a spatial mode analysis of a laser beam confined within a ring phase-conjugated optical resonator, via manipulating with transfer, or ABCD matrices. Notice that this way is quite relevant at studying the propagation of light rays through complex optical systems, being a measure to obtain the final key variables of the ray (position and angle). Subsequently, in Section 4, we introduce an intracavity chaos generating element associated to the Ikeda map, whose posterior state is determined by its prior state, and obtain finally the ABCD matrix for optical rays that follow the Ikeda map'behavior; these rays - for the sake of simplicity - will be referred further to as 'Ikeda beams'. Afterwards, Section 5, in continuation of the former section, we proceed the treatment to obtain Ikeda beams which remained the same but now with the thickness of the intracavity element being taken into account, thus allowing to obtain a more general case for the Ikeda beams. At the end, results obtained from numerical calculations are presented and overall map behavior is discussed in Section 6, uncovering the complex dynamics, critical points, and meta-stability, present in the proposed model.

2 Ikeda Map

The complex Ikeda map was obtained by Kensuke Ikeda in 1987. There it is shown that under the appropriate conditions the dynamics of the transmitted light inside a cavity, which are described by the Maxwell-Bloch equations, can be described by a set of difference-differential equations which do not involve the spatial coordinate.

$$\varepsilon_{0n} = \sqrt{T}\varepsilon_{in} + R\varepsilon_{0n-1}e^{\alpha L\phi(x)}e^{i(\alpha L\Delta(\phi(x)+\frac{1}{2})-\delta_0)} \quad (1)$$

Equation (1) is the original Ikeda map equation, as it was originally reported in reference [8]. With these results Ikeda later has carried out a linear stability analysis [9] of it; afterwards, a few years later, S. M. Hammel, et al, have simplified the original Ikeda map through ignoring saturable absorption and showed that the overall bifurcation structure of the map remains unaffected after this action being made [18]. The mentioned simplification allows equation (1) to be expressed as

$$g_n = E + Rexp\left[i\left(\phi - \frac{p}{1+|g_{n-1}|^2}\right)\right]g_{n-1} \quad (2)$$

With Eq.(2) being the simplified complex representation of the Ikeda map where

$$E = \sqrt{T}\frac{E_{in}}{\Delta}, \qquad \phi = kL$$
$$p = \frac{\alpha_0 L}{2\Delta}, \qquad g_n = \frac{E_n}{\Delta}$$

The two-dimensional, real-valued Ikeda map can be expressed in the form

$$y_{n+1} = E + R\left(y_n cos(\tau_n) + \theta_n sin(\tau_n)\right) \quad (3)$$

$$\theta_{n+1} = R\left(y_n sin(\tau_n) + \theta_n cos(\tau_n)\right) \quad (4)$$

where $\tau_n = \phi - \frac{p}{1+y_n^2+\theta_n^2}$.

In this system, E represents the electric field of the incident laser beam immediately after the first mirror, with the dimensionless state variables y_n, θ_n representing, correspondingly, the distance that separates the ray from the optical axis and the angle measured from the axis to the ray; accordingly y_{n+1}, θ_{n+1} denote the values of these variables after one passage. Furthermore, τ_n is the phase shift of the optical signal during one round trip inside the cavity, ϕ is the value of this phase shift for the empty cavity, p is a nonlinearity parameter that controls how quickly the phase shift changes with the light intensity inside the absorber, R is the reflection coefficient of mirrors 1 and 2 (see Fig.1). Note that the distance $E-p$ from an integer multiple of 2π determines the detuning of the cavity at low light intensities.

3 ABCD matrix analysis.

In paraxial optics it is known that any optical element may be described by a 2x2 matrix. Considering cylindrical symmetry around the optical axis and defining,for a given position z, both the perpendicular distance of any ray to the optical axis and its angle with the same axis as (y) and (θ), respectively, these (whenever the ray undergoes a transformation as it travels through an optical system) should obey Eq. (5) below, viz. using the corresponding [A,B,C,D] matrix:

$$\begin{pmatrix} y_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y_n \\ \theta_n \end{pmatrix} \quad (5)$$

In passive optical elements, such as lenses, interfaces between two media, etc., the elements A, B, C, D are constants. Nevertheless, for active" for non-linear optical elements, as it is our case, the A, B, C, D matrix elements are not necessarily constant but may be functions of various parameters.

In order to write the real valued 2D Ikeda map's Eqs. (3),(4) as a matrix system, the following values for the coefficients A, B, C and D must hold:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{E}{y_n} + Rcos(\tau_n) & -Rsin(\tau_n) \\ Rsin(\tau_n) & Rcos(\tau_n) \end{pmatrix} \quad (6)$$

In Figure 1, we sketch the diagram of our optical system, where the (a, b, c, e) matrix is the unknown map generating device, located between the plain common" mirrors M1 and M2 at a distance d/2 counted from each one, while M3 is a Phase Conjugated mirror.

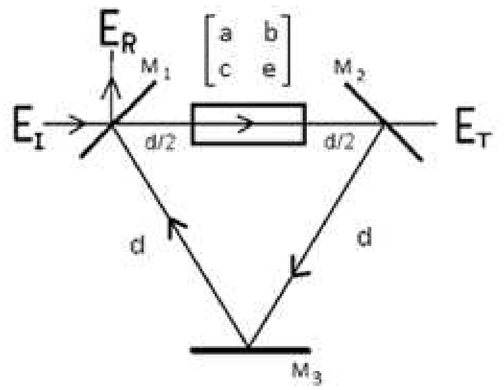

Fig. 1: Ring phase conjugated laser resonator with chaos generating element.

For this system, the total transformation matrix [A,B,C,D] for a complete round trip is written as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & e \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{3cd}{2} & b + \frac{3d}{4}(2a + 3cd + 2e) \\ -c & -\frac{3cd}{2} - e \end{pmatrix} \quad (7)$$

To reproduce the Ikeda map by a ray in the optical ring resonator, each round trip a ray described by (y_n, θ_n) has to be considered as an iteration of the desired map. Next we take the previously obtained ABCD matrix elements of the Ikeda map (6) and equate them to the total ABCD matrix of the resonator (7), this in order to generate the round trip map dynamics for (y_{n+1}, θ_{n+1}).

Note here that the results obtained by equation (7) are only valid for a very small b, $(b \approx 0)$: this due to the fact that before and after the matrix element [a,b,c,e] there is a propagation of (d - b)/2. Meanwhile, for a general case, Eq.(7) ought to be replaced by its analog:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \frac{d-b}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & e \end{pmatrix} \begin{pmatrix} 1 & \frac{d-b}{2} \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a - \frac{c}{2}(b - 3d) & \frac{1}{4}\left[b^2 c - 2b(-2 + a + 3cd + e) + 3d(2a + 3cd + 2e)\right] \\ -c & \frac{1}{2}(bc - 3cd - 2e) \end{pmatrix} \quad (8)$$

which is the total round trip transformation matrix for the general case of Ikeda Beams.

4 Ikeda Beams

The beams that are produced in an optical resonator undergoing the Ikeda map dynamics will be further called 'Ikeda beams'. In order to obtain the Ikeda beams equations (6) must be equated to the the equations found in (7), which gives rise to:

$$a + \frac{3cd}{2} = \frac{E}{y_n} + Rcos(\tau_n) \quad (9)$$

$$b + \frac{3d}{4}(2a + 3cd + 2e) = -Rsin(\tau_n) \quad (10)$$

$$-c = Rsin(\tau_n) \quad (11)$$

$$-\frac{3cd}{2} - e = Rcos(\tau_n) \quad (12)$$

The solution to this equation system for the unknowns a,b,c,e, is:

$$a = \frac{3y_n dERsin(\tau_n) + 2y_n Rcos(\tau_n) + 2E}{2y_n} \quad (13)$$

$$b = \frac{-9y_n d^2 Rsin(\tau_n) - 4y_n Rsin(\tau_n) - 6dE}{4y_n} \quad (14)$$

$$c = -Rsin(\tau_n) \quad (15)$$

$$e = -\frac{1}{2}R(2cos(\tau_n) - 3dsin(\tau_n)) \quad (16)$$

Since these solutions correspond to the [a,b,c,e] matrix, this results will be expressed in the following form:

$$\begin{pmatrix} \frac{3y_n dERsin(\tau_n) + 2y_n Rcos(\tau_n) + 2E}{2y_n} & \frac{-9y_n d^2 Rsin(\tau_n) - 4y_n Rsin(\tau_n) - 6dE}{4y_n} \\ -Rsin(\tau_n) & -\frac{1}{2}R(2cos(\tau_n) - 3dsin(\tau_n)) \end{pmatrix} \quad (17)$$

As seen, the matrix elements obtained depend on the Ikeda parameters E and R as well as on the resonator's main parameter d and on the state variable y_n . Analyzing the overall behavior of the obtained matrix, it is necessary to mention that the model may exhibit some problems caused by the factor $1/y_n$, corresponding to the upper terms of matrix (17), this is due to the fact that small values of y_n will produce very large values for these terms.

5 Ikeda Beams, General Case

As it has been previously said, the results represented by Eq.(7) are only valid for a very small b ($b \approx 0$). Thus, for the general case, the elements of matrix (6) must be equated to the ones of matrix (8), which gives the following equations

$$a - \frac{c}{2}(b-3d) = \frac{E}{y_n} + Rcos(\tau_n) \quad (18)$$

$$-c = Rsin(\tau_n) \quad (19)$$

$$\frac{1}{2}(bc - 3cd - 2e) = Rcos(\tau_n) \quad (20)$$

$$\frac{1}{4}\left[b^2c - 2b(-2+a+3cd+e) + 3d(2a+3cd+2e)\right] = -Rsin(\tau_n) \quad (21)$$

These equations generate a new solution set for the unknowns a,b,c,e:

$$\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} \frac{5E+\alpha^+ \mp \zeta}{4y_n} & \frac{csc(\tau_n)(\beta \pm \zeta)}{2Ry_n} \\ -Rsin(\tau_n) & \frac{E+\alpha^- \mp \zeta}{4y_n} \end{pmatrix} \quad (22)$$

where the following variables are defined as:

$$\zeta \equiv \sqrt{(E-2y_n)^2 + 8Ry_n sin(\tau_n)(9dE - 6dy_n + (1+27d^2)Ry_n sin(\tau_n))}$$

$$\alpha^{\pm} \equiv 2y_n(-1 \pm 2Rcos\tau_n + 9Rdsin\tau_n)$$

$$\beta \equiv -E + 2y_n - 12y_n Rdsin\tau_n$$

6 Conclusions

In this article it is shown that the introduction of a specific chaos generating device, which exhibits the chaotic behaviour of the 2-D Ikeda map, is able to generate a set of difference equations to describe the spatial dynamical behaviour of the so called I'keda Beams' within the ring phase conjugated system.

Moreover computer calculations have shown that the resulting model exhibits very complex and seemingly chaotic behaviour, as it can be observed in Figs. 2,3,4 and 5. In each of those figures every dot is the 2-D mapping for the values obtained from the final key elements of the Ikeda beams (position and angle) after each iteration (i.e. after one round trip within the resonator). All of the previously mentioned figures were obtained by a straightforward iteration of matrix (22), while using the same initial guess (y_n,θ_n)=(0.1,0.1). All figures contain plots for 10, 100 and 100,000 round trips within our resonator system identified as A), B) and C) respectively.

Furthermore the computer analysis revealed that the chaos generating element is extremely sensible to very small variations on the values corresponding to the map parameters (E, R, ϕ, p), even variations as small as $1x10^{-5}$ in those parameters can be the difference between a divergent or an stable trajectory, which we know is consistent with a system exhibiting chaos.

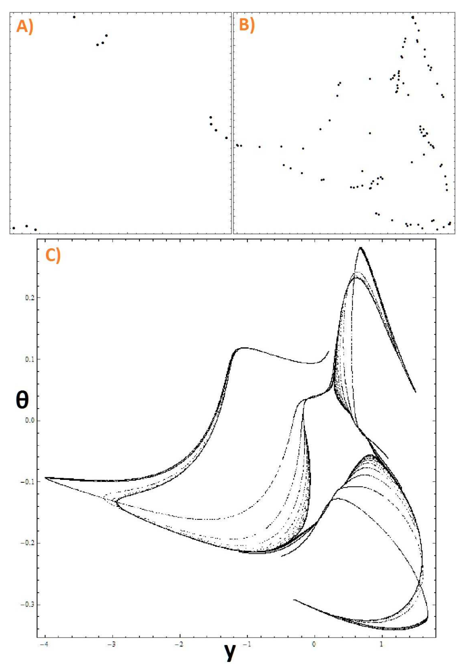

Fig. 2: Plots for iterations of matrix (22), using the following parameters d = 0.1792, E = 0.55, R = 0.2429, B = 3.25 and C = 3.05.

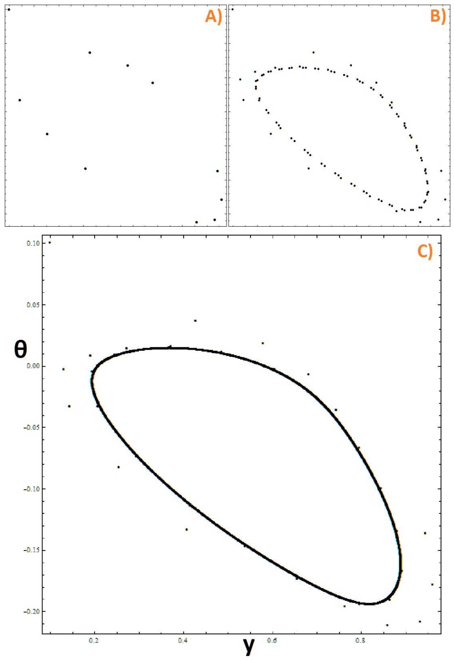

Fig. 3: Plots for iterations of matrix (22), using the following parameters d = 0.1437, E = 0.6786, R = 0.8714, B = 0.45 and C = 0.1.

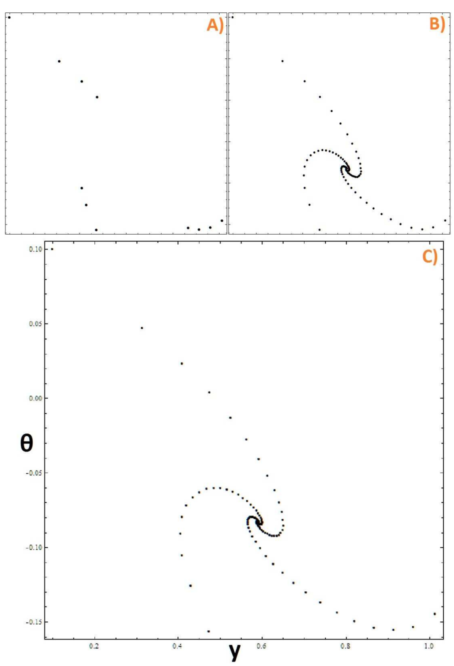

Fig. 4: Plots for iterations of matrix (22), using the following parameters d = 0.2151, E = 0.6786, R = 0.8714, B = 0.45 and C = 0.1.

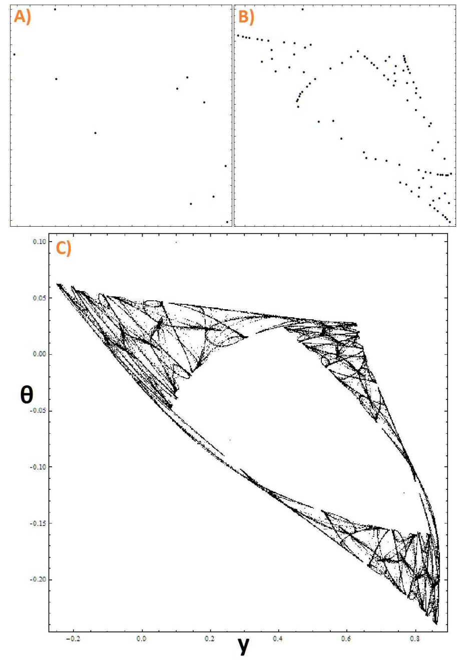

Fig. 5: Plots for iterations of matrix (22), using the following parameters d = 0.046, E = 0.609, R = 0.8714, B = 0.45 and C = 0.1.

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