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Adapted Basic Connections On the Big-Tangent Manifold

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Abstract: In this paper we give local characterisations for basic connections adapted to vertical foliation and subfoliations on the big-tangent manifold $\mathscr{T}M$ of a Riemannian space (M,g). Using some associated Vrănceanu connections we identify a triple of basic connections adapted to vertical subfoliations. Finally, we give an application of these connections in study of Lagrangians on the big-tangent manifold and also, we write in a simple form the equation of motion for scalar fields on the big-tangent manifold.

Keywords: generalized geometry, big-tangent manifold, Riemannian space, foliation.

1 Introduction and preliminary notions

1.1 Introduction

In the *generalized geometry* intitiated in [7], the tangent bundle TM of a smooth n-dimensional manifold M is replaced by the *big-tangent bundle* (or Pontryagin bundle) $TM \oplus T^*M$. On its total space the velocities and momenta are considered as independent variables. This idea was proposed and developed in [16], [17] and later was used in the study of Hamiltonian-Jacobi theory for singular Lagrangian systems [10]. On the other hand, very recently, the geometry of the total space of the big-tangent bundle, called *big-tangent manifold*, is intensively studied in [20], where as for instance are investigate several linear connections like Vrănceanu-Bott connection, connections with no multimixed torsion, projectable connections. These are linear connections on big-tangent manifold with certain properties. Another studies about the geometry of big-tangent bundle with some applications to mechanical systems can be found in [6].

On a foliated manifold, basic connections are partial connections on the transversal bundle, whose restriction along the leaves works like Bott connection, [15]. The study of basic connections was extended to manifolds which admits subfoliations (also called 2-flags), [5]. In [8], [11] we have studied some basic connections both on the tangent bundle of a Finsler space and on the cotangent bundle of a Cartan space, adapted to vertical-Liouville subfoliations defined by usual vertical foliation and the line foliation spanned by a vertical Liouville vector field. Using the framework of the geometry on the big-tangent manifold, in a recent paper [9], there is introduced the Liouville foliation on the big-tangent manifold of a Finsler space and some geometric properties in relation with some classical ones, [1], are studied.

In this paper we are interested by basic connections adapted to the vertical subfoliations on the big-tangent manifold $\mathscr{T}M$ when M is Riemannian manifold. The first section of paper presents some elementary notions about basic connections on foliated manifolds, [15], and about geometry of big-tangent manifolds, following [20]. We are interested about foliations on $\mathscr{T}M$, the vertical foliation $\mathscr{V}$, and the foliations $\mathscr{V}_1$, $\mathscr{V}_2$, by fibres of projections on T^*M, TM, respectively. In the second section we consider the big-tangent manifold of a Riemannian manifold (M,g) and we give locally conditions for connections on the normal bundles of $\mathscr{T}M$ foliated by $\mathscr{V}$, $\mathscr{V}_1$, $\mathscr{V}_2$, respectively, to be basic. In order to give examples of such connections, we determine the Levi-Civita connection of the Sasaki-type metric G defined on $\mathscr{T}M$ and the coefficients of Vrănceanu connections on $\mathscr{T}M$ foliated by $\mathscr{V}$, $\mathscr{V}_1$, $\mathscr{V}_2$, respectively (subsection 2.1.1). Next, after a briefly recall of notion of subfoliation, in subsection 2.3, for the $(n,2n)$-subfoliation $(\mathscr{V},\mathscr{V}_1)$ on $(\mathscr{T}M,G)$ a triple of basic connections adapted to this subfoliation is obtained. In section 3, following

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some ideas from [4], we give an aplication of the Vrănceanu connection on the foliated manifold $(\mathcal{T}M,\mathcal{V})$ in study of a Lagrangian and the equation of motion for scalar fiels on the big-tangent manifold.

1.2 Basic connections on a foliated manifold

A q-codimensional foliation $\mathcal{F}$ of a m-dimensional manifold M is a partition of M into $(m-q)$-dimensional submanifolds, called *leaves*. The set of vector fields tangent to leaves form an integrable subbundle F of TM, called the structural bundle of $(M,\mathcal{F})$. The transversal bundle $QF = TM/F$ is exactly the normal bundle of F in TM when M is a Riemannian manifold.

On the foliated manifold $(M,\mathcal{F})$ there is an adapted atlas whose coordinate system on the open set $V \subset M$ is $(x^i) = (x^a, x^u)$, where $a = \overline{1,q}$, $u = \overline{q+1,m}$, such that the points in the same leaf $\mathcal{L} \cap V$ have their first q coordinates equal, and are distinguished by their last $(m-q)$ coordinates. Locally, the structural bundle F is spanned by $\left\{\frac{\partial}{\partial x^u}\right\}_u$.

Also, if we consider the canonical exact sequence associated to the foliation given by an integrable subbundle F, namely

$$0 \longrightarrow F \xrightarrow{i_F} TM \xrightarrow{\pi_{QF}} QF \longrightarrow 0,$$

then we recall that a connection $\nabla : \Gamma(TM) \times \Gamma(QF) \to \Gamma(QF)$ on the normal bundle QF is said to be *basic* if

$$\nabla_X Y = \pi_{QF}[X,\widetilde{Y}] \tag{1}$$

for any $X \in \Gamma(F)$, $\widetilde{Y} \in \Gamma(TM)$ such that $\pi_{QF}(\widetilde{Y}) = Y$. Obviously, the right-hand side of (1) does not depend by choice of vector field $\tilde{Y}$, because the integrability of F.

Let g be a Riemannian metric on M and ∇^M the Levi-Civita connection on (M,g).

According to [3], the Vrănceanu connection ∇^* on $(M,g,\mathcal{F})$ is defined by

$$\begin{aligned}\nabla^*_X Y &= \pi_F \nabla^M_{\pi_F X}\pi_F Y + \pi_{QF}\nabla^M_{\pi_{QF}X}\pi_{QF}Y \\ &\quad + \pi_F\left[\pi_{QF}X, \pi_F Y\right] + \pi_{QF}\left[\pi_F X, \pi_{QF}Y\right],\end{aligned} \tag{2}$$

for any $X,Y \in \Gamma(TM)$, where π_F and π_{QF} are the projection morphisms of $\Gamma(TM)$ on $\Gamma(F)$ and $\Gamma(QF)$, respectively.

The linear connection ∇^* was defined first, using local coordinates, by Vrănceanu [21] on a non-holonomic manifold endowed with a linear connection, where by a non-holonomic manifold we mean a manifold that is endowed with two complementary distributions, at least one of which is non-integrable.

*Remark.*The restriction of the Vrănceanu connection to the normal bundle QF is an example of basic connection.

*Remark.*We notice that many of the connections that are used in the literature to study foliated manifolds can be related to the Vrănceanu connection in one way or the other. For example, Bott connection [15] is the restriction of the Vrănceanu connection to transversal distribution. Also, the adapted connection of Reinhart [14] or Vaisman connection (also called second connection) [19] are the Vrănceanu connection on that foliated manifold.

On a Riemannian foliated manifold (M,g,F), the holonomy invariance of the induced metric g_Q on the normal bundle is the condition for this foliation to be Riemannian. In this case the metric g is called bundle-like. We have the following property, [15]:

Proposition 1.1. The foliation F is Riemannian iff the restriction of the Vrănceanu connection on the normal bundle, $\nabla = \nabla^*|_{\Gamma(TM)\times\Gamma(QF)}$, is satisfying $\nabla_X g_Q = 0$ for all $X \in \Gamma(F)$.

1.3 The big-tangent manifold

Let us consider the big tangent bundle $TM \oplus T^*M$ associated to a smooth n-dimensional manifold M. The total space of the big-tangent bundle, called *big-tangent manifold*, is a $3n$-dimensional smooth manifold denoted here by $\mathcal{T}M$. Let us briefly recall some elementary notions about the geometry of big-tangent manifold $\mathcal{T}M$. For a detalied discussion about this geometry we refer [20].

The points of $\mathcal{T}M$ are triples (x,y,p), $x \in M$, $y \in T_xM$, $p \in T^*_xM$, and one has local coordinates (x^i, y^i, p_i), where $i = 1,\ldots,n = \dim M$, (x^i) are local coordinates on M, (y^i) are vector coordinates and (p_i) are covector coordinates. The change rules of these coordinates are:

$$\widetilde{x}^i = \widetilde{x}^i(x^j)\,,\, \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}y^j\,,\, \widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}p_j. \tag{3}$$

On $\mathcal{T}M$, a vector field X and a 1-form φ have the local expressions

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^i\frac{\partial}{\partial y^i} + \zeta_i\frac{\partial}{\partial p_i}\,,\; \varphi = \alpha_i dx^i + \beta_i dy^i + \gamma^i dp_i, \tag{4}$$

and the coordinate transformation (3) leads to the following transformation of vector and covector coordinates:

$$\begin{aligned}\widetilde{\xi}^i &= \frac{\partial \widetilde{x}^i}{\partial x^j}\xi^j, \widetilde{\eta}^i = \frac{\partial \widetilde{y}^i}{\partial x^j}\xi^j + \frac{\partial \widetilde{x}^i}{\partial x^j}\eta^j, \widetilde{\zeta}_i = \frac{\partial \widetilde{p}_i}{\partial x^j}\xi^j + \frac{\partial x^j}{\partial \widetilde{x}^i}\zeta_j, \\ \widetilde{\alpha}_i &= \frac{\partial x^j}{\partial \widetilde{x}^i}\alpha_j + \frac{\partial y^j}{\partial \widetilde{x}^i}\beta_j + \frac{\partial p_j}{\partial \widetilde{x}^i}\gamma^j, \widetilde{\beta}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\beta_j, \widetilde{\gamma}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}\gamma^j\end{aligned} \tag{5}$$

Also, for the big-tangent manifold $\mathcal{T}M$ we have the following projections

$$p : \mathcal{T}M \to M\,,\, p_1 : \mathcal{T}M \to TM\,,\, p_2 : \mathcal{T}M \to T^*M$$

on M and on the total spaces of tangent and cotangent bundle, respectively.

As usual, we denote by $V = V(\mathscr{T}M)$ the vertical bundle on the big-tangent manifold $\mathscr{T}M$ and it has the decomposition

$$V = V_1 \oplus V_2, \tag{6}$$

where $V_1 = p_{1*}^{-1}(V(TM))$, $V_2 = p_{2*}^{-1}(V(T^*M))$ and have the local frames $\{\frac{\partial}{\partial y^i}\}$, $\{\frac{\partial}{\partial p_i}\}$, respectively.

The subbundles V_1, V_2 are structural bundles of the vertical foliations $\mathscr{V}_1$, $\mathscr{V}_2$ of $\mathscr{T}M$ by fibers of p_2, p_1, respectively, and $\mathscr{T}M$ has a multi-foliate structure [18].

As usual, for tangent bundle and like in foliation theory, the geometry of the big-tangent manifold $\mathscr{T}M$ may be developed by considering a *horizontal bundle* H such that

$$T(\mathscr{T}M) = H \oplus V = H \oplus V_1 \oplus V_2. \tag{7}$$

The first equality of (7) produces a double grading of forms and multivectors on $\mathscr{T}M$ of bidegree of type (p,q) that means H-degree p and V-degree q. The exterior diferential admits the decomposition

$$d = d_{1,0} + d_{0,1} + d_{2,-1}, \tag{8}$$

where $d_{0,1}$ means the exterior differential along the leaves of V [19]. The second equality of (7), leads to a double grading (r,s) $(q = r+s)$ of V-degree q, V_1-degree r and V_2-degree s, respectively. This give a further decomposition of the terms of (8), for instance $d_{0,1} = d_{0,1,0} + d_{0,0,1}$.

For a chosen horizontal bundle H, a vector $X \in T_xM$ has a *horizontal lift* X^h defined by $X^h \in H_{(x,y,p)}$, $p_* X^h = X$, and $T(\mathscr{T}M)$ has local canonical bases

$$\left\{ \frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - t_i^j \frac{\partial}{\partial y^j} + t_{ij}\frac{\partial}{\partial p_j}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i} \right\}, \tag{9}$$

where t_i^j and t_{ij} are local functions of (x,y,p). The corresponding dual bases are given by

$$\{dx^i\,,\, \delta y^i = dy^i + t_j^i dx^j\,,\, \delta p_i = dp_i - t_{ij}dx^j\}. \tag{10}$$

A change of coordinates (3) implies the following transformation rules

$$\frac{\delta}{\delta \widetilde{x}^i} = \frac{\partial x^j}{\partial \widetilde{x}^i}\frac{\delta}{\delta x^j}\,,\, \delta\widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}\delta y^j\,,\, \delta\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\delta p_j, \tag{11}$$

$$\begin{aligned} \widetilde{t}_j^i &= \frac{\partial \widetilde{x}^i}{\partial x^h}\frac{\partial x^k}{\partial \widetilde{x}^j}t_k^h - \frac{\partial x^k}{\partial \widetilde{x}^j}\frac{\partial^2 \widetilde{x}^i}{\partial x^l \partial x^k}y^l, \\ \widetilde{t}_{ij} &= \frac{\partial x^h}{\partial \widetilde{x}^i}\frac{\partial x^k}{\partial \widetilde{x}^j}t_{hk} - \frac{\partial^2 x^h}{\partial \widetilde{x}^i \partial \widetilde{x}^j}p_h. \end{aligned} \tag{12}$$

Conversely, every local functions t_i^j and t_{ij} which satisfies the local change rules from (12) leads to a direct decomposition as in (7).

According to Proposition 4.1 [20], every horizontal bundle on TM has a canonical lift to a horizontal bundle on $\mathscr{T}M$ and every horizontal bundle on the cotangent bundle T^*M has a canonical lift to a horizontal bundle on $\mathscr{T}M$. More exactly, if $\{\frac{\partial}{\partial x^i} - t_i^j(x,y)\frac{\partial}{\partial y^j}\}$ is the local basis of the horizontal bundle on TM then the local horizontal basis of the canonical lift on $\mathscr{T}M$ is $\{\frac{\partial}{\partial x^i} - t_i^j(x,y)\frac{\partial}{\partial y^j} + p_h\frac{\partial t_i^h}{\partial y^j}\frac{\partial}{\partial p_j}\}$ and if $\{\frac{\partial}{\partial x^i} + t_{ij}(x,p)\frac{\partial}{\partial p_j}\}$ is the local basis of the horizontal bundle on T^*M then the local horizotal basis of the canonical lift on $\mathscr{T}M$ is $\{\frac{\partial}{\partial x^i} - p_h\frac{\partial t_{ih}}{\partial p_j}\frac{\partial}{\partial y^j} + t_{ij}\frac{\partial}{\partial p_j}\}$.

As an usual example, see [20], every linear connection Γ on M with local coefficients Γ_{jk}^i locally span a complement of the vertical distribution, that is a horizontal bundle on $\mathscr{T}M$, which has local basis

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^k\Gamma_{ik}^j\frac{\partial}{\partial y^j} + p_k\Gamma_{ij}^k\frac{\partial}{\partial p_j}. \tag{13}$$

2 Basic connections on the big-tangent manifold of a Riemannian space

In this section we consider the big-tangent manifold of a Riemannian manifold (M,g), and firstly give some local characterizations for a connection to be basic on the manifold $\mathscr{T}M$ foliated by $\mathscr{V}$, $\mathscr{V}_1$, respectively. Then, we determine the Levi-Civita connection of the Sasaki-type metric G defined on $\mathscr{T}M$ and the Vrănceanu connections on $\mathscr{T}M$ foliated by $\mathscr{V}$, $\mathscr{V}_1$, $\mathscr{V}_2$, respectively, in order to give examples of basic connections. Next, after a briefly recall of notion of subfoliation, we make a general aproach about basic connections on the $(n,2n)$-subfoliation on $(\mathscr{T}M,G)$ defined by $\mathscr{V}$ and $\mathscr{V}_1$ and we give a triple of basic connections adapted to this subfoliation.

2.1 Basic connections adapted to vertical foliations

Let us consider a Riemannian space (M,g), $\Gamma_{jk}^i(x)$ its Christoffel symbols and $\{dx^i, \delta y^i, \delta p_i\}$ the corresponding cobasis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\}$, where $\frac{\delta}{\delta x^i}$ are given by (13). Then, the formula

$$G = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j + g^{ij}(x)\delta p_i \otimes \delta p_j, \tag{14}$$

defines a metric on the big-tangent manifold $\mathscr{T}M$, which is non degenerate on V and called the *Sasaki-type metric*. Here $(g^{ij}(x))$ denotes the inverse matrix of $(g_{ij}(x))$.

Firstly, we give a local description of a basic connections associated to Riemannian manifold $(\mathscr{T}M,G)$ with respect to foliations $\mathscr{V}$, $\mathscr{V}_1$, respectively.

For this purpose, let us begin with the calculus of Lie brackets of the vector fields of adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$. We have

$$\begin{aligned}&\left[\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right]=0,\ \left[\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_j}\right]=0,\ \left[\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right]=0,\\&\left[\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right]=\Gamma^k_{ij}\frac{\partial}{\partial y^k},\ \left[\frac{\delta}{\delta x^i},\frac{\partial}{\partial p_j}\right]=-\Gamma^j_{ik}\frac{\partial}{\partial p_k},\\&\left[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right]=y^kR^l_{ikj}\frac{\partial}{\partial y^l}-p_kR^k_{ilj}\frac{\partial}{\partial p_l}\end{aligned}\tag{15}$$

where

$$R^l_{ikj}=\frac{\partial\Gamma^l_{ik}}{\partial x^j}-\frac{\partial\Gamma^l_{jk}}{\partial x^i}+\Gamma^m_{ik}\Gamma^l_{jm}-\Gamma^m_{jk}\Gamma^l_{im}.$$

Let us consider the following exact sequences associated to foliations $\mathscr{V}$ and $\mathscr{V}_1$, respectively

$$0\longrightarrow V\xrightarrow{i}T(\mathscr{T}M)\xrightarrow{\pi}H\longrightarrow 0,$$

$$0\longrightarrow V_1\xrightarrow{i_1}T(\mathscr{T}M)\xrightarrow{\pi_1}H\oplus V_2\longrightarrow 0,$$

where $i,i_1,\ \pi,\pi_1$ are the canonical inclusions and projections, respectively.

According to (1), a connection $\widetilde{\nabla}$ on H is basic with respect to the vertical foliation if

$$\widetilde{\nabla}_XZ=\pi[X,\widetilde{Z}],\tag{16}$$

for all $X\in\Gamma(V),\widetilde{Z}\in\Gamma(T(\mathscr{T}M))$ and $\pi(\widetilde{Z})=Z$.

Proposition 2.1. A connection $\widetilde{\nabla}$ on H is basic if and only if in a local adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$, we have

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=0,\quad\widetilde{\nabla}_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j}=0,\forall i,j\in\{1,2,\ldots,n\}.\tag{17}$$

Proof. Let $\widetilde{\nabla}:\Gamma(T(\mathscr{T}M))\times\Gamma(H)\to\Gamma(H)$ be a connection on H with property (17). Let $X\in\Gamma(V)$, $Z\in\Gamma(H)$, so their local form is $X=a_i\frac{\partial}{\partial y^i}+b_j\frac{\partial}{\partial p_j}$, $Z=c_k\frac{\delta}{\delta x^k}$ with a_i, b_j, c_k local differentiable functions on $\mathscr{T}M$. Then, we can compute

$$\widetilde{\nabla}_XZ=a_i\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}Z+b_j\widetilde{\nabla}_{\frac{\partial}{\partial p_j}}Z=X(c_k)\frac{\delta}{\delta x^k}.\tag{18}$$

An arbitrary vector field $\widetilde{Z}$ which projects into $Z\in H$ is by the form

$$\widetilde{Z}=Z+Z_V,$$

according to decomposition $T(\mathscr{T}M)=H\oplus V$. Since V is an integrable subbundle and, by (15), the Lie brackets $\left[\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^k}\right]$, $\left[\frac{\partial}{\partial p_j},\frac{\delta}{\delta x^k}\right]$ are vertical vector fields, we obtain

$$\begin{aligned}\pi([X,\tilde{Z}])&=\pi([X,Z])\\&=\pi\left(X(c_k)\frac{\delta}{\delta x^k}-Z(a_i)\frac{\partial}{\partial y^i}-Z(b_j)\frac{\partial}{\partial p_j}\right)\\&\quad+\pi\left(c_ka_i\left[\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^k}\right]+c_kb_j\left[\frac{\partial}{\partial p_j},\frac{\delta}{\delta x^k}\right]\right)\\&=X(c_k)\frac{\delta}{\delta x^k}.\end{aligned}$$

By the last equality and (18), we obtain (16), so $\widetilde{\nabla}$ is a basic connection on H.

Conversely, by direct calculation, in the adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$ in $\mathscr{T}M$, every basic connection $\widetilde{\nabla}$ on H is locally satisfying (17). $\square$

Also, a connection $\overline{\nabla}$ on the normal bundle $H\oplus V_2$ of foliation $\mathscr{V}_1$ is basic if

$$\overline{\nabla}_XZ=\pi_1[X,\widetilde{Z}],\tag{19}$$

for all $X\in\Gamma(V_1),\widetilde{Z}\in\Gamma(T(\mathscr{T}M))$ and $\pi_1(\widetilde{Z})=Z$.

Proposition 2.2. A connection $\overline{\nabla}$ on $H\oplus V_2$ is basic if and only if in a local adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$, we have

$$\overline{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=0,\quad\overline{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j}=0,\forall i,j\in\{1,2,\ldots,n\}.\tag{20}$$

Proof. Let $\overline{\nabla}:\Gamma(T(\mathscr{T}M))\times\Gamma(H\oplus V_2)\to\Gamma(H\oplus V_2)$ be a connection on $H\oplus V_2$ with property (20). Let $X\in\Gamma(V_1)$, $Z\in\Gamma(H\oplus V_2)$, so their local form is $X=a_i\frac{\partial}{\partial y^i}$, $Z=c_k\frac{\delta}{\delta x^k}+b_j\frac{\partial}{\partial p_j}$ with a_i, b_j, c_k local differentiable functions on $\mathscr{T}M$. Then we can compute

$$\begin{aligned}\overline{\nabla}_XZ&=a_i\overline{\nabla}_{\frac{\partial}{\partial y^i}}c_k\frac{\delta}{\delta x^k}+a_i\overline{\nabla}_{\frac{\partial}{\partial y^i}}b_j\frac{\partial}{\partial p_j}\\&=X(c_k)\frac{\delta}{\delta x^k}+X(b_j)\frac{\partial}{\partial p_j}.\end{aligned}\tag{21}$$

An arbitrary vector field $\overline{Z}$ which projects into $Z\in H\oplus V_2$ is by the form

$$\overline{Z}=Z_1+Z,$$

according to decomposition $T(\mathscr{T}M)=V_1\oplus H\oplus V_2$. Since V_1 is an integrable subbundle and, by (15), the Lie brackets $\left[\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^k}\right]\in\Gamma(V_1)$, we obtain

$$\begin{aligned}\pi_1([X,\overline{Z}])&=\pi_1([X,Z])\\&=\pi_1\left(X(c_k)\frac{\delta}{\delta x^k}-Z(a_i)\frac{\partial}{\partial y^i}+c_ka_i\left[\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^k}\right]\right)\\&=X(c_k)\frac{\delta}{\delta x^k}+X(b_j)\frac{\partial}{\partial p_j}.\end{aligned}$$

By the last equality and (21), we obtain (19), so $\overline{\nabla}$ is a basic connection on $H\oplus V_2$.

Conversely, by direct calculation, in the adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$ in $\mathscr{T}M$, every basic connection $\overline{\nabla}$ on $H\oplus V_2$ is locally satisfying (20). $\square$.

2.2 Vrănceanu connections

In order to give some examples of basic connections, in this subsection we are interested about the Vrănceanu connection on the foliated manifold $\mathscr{T}M$ with respect to vertical foliations $\mathscr{V}$, $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively.

Firstly, we give a local description of the Levi-Civita connection associated to Riemannian manifold $(\mathscr{T}M, G)$.

Let ∇ be the Levi-Civita connection on the Riemannian manifold $(\mathscr{T}M, G)$. Then we have

Lemma 2.1. Let (M, g) be a Riemannian space. Then the Levi-Civita connection ∇ on $(\mathscr{T}M, G)$ is locally expressed as follows:

$$\begin{aligned}
\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} &= \Gamma^k_{ij}\frac{\delta}{\delta x^k}+\frac{1}{2}y^l R^k_{ilj}\frac{\partial}{\partial y^k}-\frac{1}{2}p_l R^l_{ikj}\frac{\partial}{\partial p_k}\\
\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} &= -\frac{1}{2}y^h R^l_{jhm}g_{li}g^{mk}\frac{\delta}{\delta x^k}\\
&\quad +\frac{1}{2}g^{mk}\left(\frac{\partial g_{im}}{\partial x^j}-\Gamma^l_{ji}g_{lm}-\Gamma^l_{jm}g_{li}\right)\frac{\partial}{\partial y^k}\\
&= \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i}-\Gamma^k_{ji}\frac{\partial}{\partial y^k}\\
\nabla_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j} &= \frac{1}{2}p_h R^h_{jlm}g^{li}g^{mk}\frac{\delta}{\delta x^k}\\
&\quad +\frac{1}{2}g_{mk}\left(\frac{\partial g^{im}}{\partial x^j}+\Gamma^i_{jl}g^{lm}-\Gamma^m_{jl}g^{li}\right)\frac{\partial}{\partial p_k}\\
&= \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial p_i}+\Gamma^i_{jk}\frac{\partial}{\partial p_k}\\
\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} &= -\frac{1}{2}g^{mk}\left(\frac{\partial g_{ij}}{\partial x^m}-\Gamma^l_{mj}g_{li}-\Gamma^l_{mi}g_{lj}\right)\frac{\delta}{\delta x^k}\\
\nabla_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial y^j} &= 0\,,\ \nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial p_i}=0,\\
\nabla_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j} &= -\frac{1}{2}g^{mk}\left(\frac{\partial g^{ij}}{\partial x^m}+\Gamma^j_{ml}g^{li}+\Gamma^i_{ml}g^{lj}\right)\frac{\delta}{\delta x^k}.
\end{aligned}$$

Proof. Recall that the Levi-Civita connection on the Riemannian manifold $(\mathscr{T}M, G)$ is given by the Koszul formula

$$\begin{aligned}
2G(\nabla_X Y, Z) &= X(G(Y,Z))+Y(G(Z,X))-Z(G(X,Y))\\
&\quad +G([X,Y],Z)-G([Y,Z],X)+G([Z,X],Y)
\end{aligned}$$

for every $X, Y, Z \in \Gamma(\mathscr{T}M)$. Now, taking into account the relations

$$G\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=g_{ij}(x),$$

$$G\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right)=g^{ij}(x),$$

$$G\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right)=G\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial p_j}\right)=G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial p^j}\right)=0$$

by direct calculations using (15) we deduce that

$$G\left(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right),$$

$$G\left(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k}\right)=\frac{1}{2}y^h R^l_{ihj}g_{lk},$$

$$G\left(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j},\frac{\partial}{\partial p_k}\right)=-\frac{1}{2}p_h R^h_{ilj}g^{lk}$$

and then we obtain the local expression of the first relation. Similarly, we get the other expressions of the Levi-Civita connection. $\square$.

Using relation (2) and the Levi-Civita connection ∇ on the Riemannian manifold $(\mathscr{T}M, G)$ given above with respect to the adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$, we obtain the following local expression for the Vrănceanu connection ∇^* on the big-tangent manifold $\mathscr{T}M$ of a Riemannian space (M, g), endowed with the metric (14) and with vertical foliation $\mathscr{V}$:

$$\nabla^*_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^k_{ij}\frac{\delta}{\delta x^k}\,,\ \nabla^*_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=0\,,\ \nabla^*_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j}=0,$$

$$\nabla^*_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}=\Gamma^k_{ij}\frac{\partial}{\partial y^k}\,,\ \nabla^*_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=0\,,\ \nabla^*_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial y^j}=0,$$

$$\nabla^*_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial p_j}=-\Gamma^j_{ik}\frac{\partial}{\partial p_k}\,,\ \nabla^*_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j}=0\,,\ \nabla^*_{\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j}=0.$$

Similarly, we obtain the local expression of the Vrănceanu connections ∇^*_1 and ∇^*_2 on the big-tangent manifold $\mathscr{T}M$ of a Riemannian space (M, g), endowed with the metric (14) and with vertical foliations $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively:

$$\begin{aligned}
\nabla^*_{1\,\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} &= \Gamma^k_{ij}\frac{\delta}{\delta x^k}-\frac{1}{2}p_l R^l_{ikj}\frac{\partial}{\partial p_k}\,,\ \nabla^*_{1\,\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=0\\
\nabla^*_{1\,\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} &= \Gamma^k_{ij}\frac{\partial}{\partial y^k},\\
\nabla^*_{1\,\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j} &= \frac{1}{2}p_h R^h_{jlm}g^{li}g^{mk}\frac{\delta}{\delta x^k}\\
&\quad +\frac{1}{2}g_{mk}\left(\frac{\partial g^{im}}{\partial x^j}+\Gamma^i_{jl}g^{lm}-\Gamma^m_{jl}g^{li}\right)\frac{\partial}{\partial p_k}\\
&= \nabla^*_{1\,\frac{\delta}{\delta x^j}}\frac{\partial}{\partial p_i}+\Gamma^i_{jk}\frac{\partial}{\partial p_k},\\
\nabla^*_{1\,\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} &= 0\,,\ \nabla^*_{1\,\frac{\partial}{\partial p_i}}\frac{\partial}{\partial y^j}=0\,,\ \nabla^*_{1\,\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j}=0,\\
\nabla^*_{1\,\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j} &= -\frac{1}{2}g^{mk}\left(\frac{\partial g^{ij}}{\partial x^m}+\Gamma^j_{ml}g^{li}+\Gamma^i_{ml}g^{lj}\right)\frac{\delta}{\delta x^k}
\end{aligned}$$

and

$$\nabla^*_{2\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^k_{ij}\frac{\delta}{\delta x^k}+\frac{1}{2}y^lR^k_{ilj}\frac{\partial}{\partial y^k}\,,\ \nabla^*_{2\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j}=0,$$

$$\nabla^*_{2\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=-\frac{1}{2}y^hR^l_{jhm}g_{li}g^{mk}\frac{\delta}{\delta x^k}+\frac{1}{2}g^{mk}\left(\frac{\partial g_{im}}{\partial x^j}-\Gamma^l_{ji}g_{lm}-\Gamma^l_{jm}g_{li}\right)\frac{\partial}{\partial y^k}=\nabla^*_{2\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i}-\Gamma^k_{ji}\frac{\partial}{\partial y^k},$$

$$\nabla^*_{2\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=-\frac{1}{2}g^{mk}\left(\frac{\partial g_{ij}}{\partial x^m}-\Gamma^l_{mj}g_{li}-\Gamma^l_{mi}g_{lj}\right)\frac{\delta}{\delta x^k},$$

$$\nabla^*_{2\frac{\partial}{\partial p_i}}\frac{\partial}{\partial y^j}=0\,,\ \nabla^*_{2\frac{\delta}{\delta x^i}}\frac{\partial}{\partial p_j}=-\Gamma^j_{ik}\frac{\partial}{\partial p_k},$$

$$\nabla^*_{2\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j}=0\,,\ \nabla^*_{2\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j}=0.$$

Finally, we obtain

Proposition 2.3. a) The restiction of the connection ∇^* to $\Gamma(T(\mathscr{T}M))\times\Gamma(H)$ is a connection on H, denoted by $\widetilde{\nabla}^*$, which satisfies conditions (17), so it is basic with respect to vertical foliation $\mathscr{V}$.

b) The restriction of the connection ∇^*_1 to $\Gamma(T(\mathscr{T}M))\times\Gamma(H\oplus V_2)$ is a connection on $H\oplus V_2$, denoted by $\overline{\nabla}^*_1$, which satisfies conditions (19), so it is basic with respect to foliation $\mathscr{V}_1$.

Moreover, using local expression of Vrănceanu connections ∇^*, ∇^*_1, we obtain the following nonzero local coefficients of $\widetilde{\nabla}^*$, $\overline{\nabla}^*_1$:

$$\widetilde{\nabla}^*_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^i}=\Gamma^k_{ij}\frac{\delta}{\delta x^k},\quad \overline{\nabla}^*_{1\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^k_{ij}\frac{\delta}{\delta x^k}-\frac{1}{2}p_lR^l_{ikj}\frac{\partial}{\partial p_k},$$

$$\overline{\nabla}^*_{1\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j}=\frac{1}{2}p_hR^h_{jlm}g^{li}g^{mk}\frac{\delta}{\delta x^k}+\frac{1}{2}g_{mk}\left(\frac{\partial g^{im}}{\partial x^j}+\Gamma^i_{jl}g^{lm}-\Gamma^m_{jl}g^{li}\right)\frac{\partial}{\partial p_k},$$

$$\overline{\nabla}^*_{1\frac{\partial}{\partial p_i}}\frac{\partial}{\partial p_j}=-\frac{1}{2}g^{mk}\left(\frac{\partial g^{ij}}{\partial x^m}+\Gamma^j_{ml}g^{li}+\Gamma^i_{ml}g^{lj}\right)\frac{\delta}{\delta x^k}.$$

Now we can verify the condition from Proposition 1.1 to check if the foliations $\mathscr{V}$, $\mathscr{V}_1$ are Riemannian foliations.

The metric induced in the normal bundle of $\mathscr{V}$, H, is $G_H=g_{ij}(x)dx^i\otimes dx^j$ and $\widetilde{\nabla}^*_XG_H=0$ for every $X\in\Gamma(V)$ is equivalent to

$$G_H\left(\widetilde{\nabla}^*_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)+G_H\left(\frac{\delta}{\delta x^j},\widetilde{\nabla}^*_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^k}\right)=0,$$

$$G_H\left(\widetilde{\nabla}^*_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)+G_H\left(\frac{\delta}{\delta x^j},\widetilde{\nabla}^*_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^k}\right)=0,$$

which are true since $\widetilde{\nabla}^*_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=0$, $\widetilde{\nabla}^*_{\frac{\partial}{\partial p_i}}\frac{\delta}{\delta x^j}=0$, for all $i,j,k=\overline{1,n}$.

The metric induced in the normal bundle of $\mathscr{V}_1$, $H\oplus V_2$, is $G_1=g_{ij}(x)dx^i\otimes dx^j+g^{ij}(x)\delta p_i\otimes\delta p_j$ and $\overline{\nabla}^*_{1X}G_1=0$ for every $X\in\Gamma(V_1)$ is equivalent to

$$G_1\left(\overline{\nabla}^*_{1\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)+G_1\left(\frac{\delta}{\delta x^j},\overline{\nabla}^*_{1\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^k}\right)=0,$$

$$G_1\left(\overline{\nabla}^*_{1\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j},\frac{\delta}{\delta p_k}\right)+G_1\left(\frac{\delta}{\delta x^j},\overline{\nabla}^*_{1\frac{\partial}{\partial y^i}}\frac{\delta}{\delta p_k}\right)=0,$$

$$G_1\left(\overline{\nabla}^*_{1\frac{\partial}{\partial y^i}}\frac{\delta}{\delta p_j},\frac{\delta}{\delta p_k}\right)+G_H\left(\frac{\delta}{\delta p_j},\overline{\nabla}^*_{1\frac{\partial}{\partial y^i}}\frac{\delta}{\delta p_k}\right)=0,$$

which is true because $\overline{\nabla}^*_1$ is a basic connection. It follows that:

Proposition 2.4. The foliations $\mathscr{V}$ and $\mathscr{V}_1$ are Riemannian foliations and the Sasaki-type metric (14) is bundle-like with respect to both foliations on $\mathscr{T}M$.

*Remark.*By similar arguments, the foliation $\mathscr{V}_2$ is Riemannian, too.

2.3 Basic connections adapted to vertical subfoliations

In this subsection, following [5], we briefly recall the notion of a (q_1,q_2)-codimensional subfoliation on a manifold and we identify the $(n,2n)$-codimensional subfoliation $(\mathscr{V},\mathscr{V}_1)$ on the big tangent manifold $\mathscr{T}M$ where $\mathscr{V}$ is the vertical foliation and $\mathscr{V}_1$ is the foliation by fibers of projection p_2 on TM. Firstly we make a general approach about basic connections on the normal bundles related to this subfoliation and next a triple of adapted basic connections with respect to this subfoliation is given.

Definition 2.1. [5] Let M be a n-dimensional manifold and TM its tangent bundle. A (q_1,q_2)-*codimensional subfoliation* on M is a couple (F_1,F_2) of integrable subbundles F_k of TM of dimension $n-q_k$, $k=1,2$ and F_2 being at the same time a subbundle of F_1.

For a subfoliation (F_1,F_2), its normal bundle is defined as $Q(F_1,F_2)=QF_{21}\oplus QF_1$, where QF_{21} is the quotient bundle F_1/F_2 and QF_1 is the usual normal bundle of F_1. So, an exact sequence of vector bundles

$$0\longrightarrow QF_{21}\xrightarrow{i}QF_2\xrightarrow{\pi}QF_1\longrightarrow 0\qquad(22)$$

appears in a canonical way.

For a (q_1,q_2)-subfoliation (F_1,F_2) we can consider the following exact sequence of vector bundles

$$0\longrightarrow F_2\xrightarrow{i_0}F_1\xrightarrow{\pi_0}QF_{21}\longrightarrow 0\qquad(23)$$

and, according to [5], a connection ∇ on QF_{21} is said to be basic with respect to the subfoliation (F_1,F_2) if

$$\nabla_XY=\pi_0[X,\widetilde{Y}]\qquad(24)$$

for any $X\in\Gamma(F_2)$ and $\widetilde{Y}\in\Gamma(F_1)$ such that $\pi_0(\widetilde{Y})=Y$.

2.3.1 The $(n,2n)$-codimensional subfoliation $(\mathscr{V},\mathscr{V}_1)$ of $(\mathscr{T}M,G)$

Taking into account the discussion from the previous section, we have on the $3n$-dimensional big tangent manifold $\mathscr{T}M$ the $(n,2n)$-codimensional subfoliation $(\mathscr{V},\mathscr{V}_1)$. We also notice that the metric structure G on $\mathscr{T}M$ given by (14) is compatible with the subfoliated structure, that is

$$QV \cong H,\, QV_1 \cong H\oplus V_2,\, V/V_1 \cong V_2.$$

Let us consider the following exact sequences associated to the subfoliation $(\mathscr{V},\mathscr{V}_1)$

$$0 \longrightarrow V_1 \xrightarrow{i_0} V \xrightarrow{\pi_0} V_2 \longrightarrow 0,$$

where i_0, π_0 are the canonical inclusion and projection, respectively.

A triple $(\nabla^2,\overline{\nabla},\widetilde{\nabla})$ of basic connections on normal bundles V_2, $H\oplus V_2$, H, respectively, is called (according to [5]) *adapted* to the subfoliation $(\mathscr{V},\mathscr{V}_1)$ if, considering the exact sequence

$$0 \longrightarrow V_2 \xrightarrow{i'} H\oplus V_2 \xrightarrow{\pi'} H \longrightarrow 0,$$

there are the relations:

$$i'(\nabla^2_X Z_2) = \overline{\nabla}_X i'(Z_2), \quad \pi'(\overline{\nabla}_X Z) = \widetilde{\nabla}_X \pi'(Z), \tag{25}$$

for any $X\in\Gamma(V_1)$, $Z_2\in\Gamma(V_2)$, $Z\in\Gamma(H\oplus V_2)$.

By (24) a connection ∇^2 on V_2 is basic with respect to the subfoliation $(\mathscr{V},\mathscr{V}_1)$ if

$$\nabla^2_X Z_2 = \pi_0[X,\widetilde{Z}], \tag{26}$$

for all $X\in\Gamma(V_1)$, $\widetilde{Z}\in\Gamma(V)$ with $\pi_0(\widetilde{Z}) = Z_2$.

Proposition 2.5. A connection ∇^2 on V_2 is basic if and only if in a local adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$,

$$\nabla^2_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j} = 0,\, \forall i,j\in\{1,2,\ldots,n\}. \tag{27}$$

Proof. Let ∇^2 be a connection on V_2 with property (27). Let $X\in\Gamma(V_1)$, $Z_2\in\Gamma(V_2)$, so their local forms are $X = a_i\frac{\partial}{\partial y^i}$, $Z_2 = b_j\frac{\partial}{\partial p_j}$ with a_i, b_j local differentiable functions on $\mathscr{T}M$. Then we can compute

$$\begin{aligned}\nabla^2_X Z_2 &= a_i\nabla^2_{\frac{\partial}{\partial y^i}} Z_2\\ &= a_i b_j \nabla^2_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial p_j} + a_i\frac{\partial b_j}{\partial y^i}\frac{\partial}{\partial p_j}\\ &= X(b_j)\frac{\partial}{\partial p_j}.\end{aligned} \tag{28}$$

An arbitrary vertical vector field $\widetilde{Z}$ which projects into $Z_2\in V_2$ is by the form

$$\widetilde{Z} = Z_1 + Z_2,$$

according to decomposition (6), and, since V_1 is an integrable subbundle, we have

$$\begin{aligned}\pi_0([X,\widetilde{Z}]) &= \pi_0([X,Z_2])\\ &= \pi_0\left(X(b_j)\frac{\partial}{\partial p_j} - Z_2(a_i)\frac{\partial}{\partial y^i}\right)\\ &= X(b_j)\frac{\partial}{\partial p_j}.\end{aligned}$$

By the last equality and (28), we obtain (26), so ∇^2 is a basic connection on V_2.

Conversely, by direct calculation, in the adapted basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$ in $\mathscr{T}M$, every basic connection ∇^2 on V_2 is locally satisfying (27). $\square$.

Proposition 2.6. The restriction of ∇^*_2 to $\Gamma(V)\times\Gamma(V_2)$ is a connection on V_2, denoted by ∇^{*2}_2, which satisfies conditions (27), so it is a basic connection with respect to subfoliation $(\mathscr{V},\mathscr{V}_1)$.

2.3.2 A triple of basic connections adapted to subfoliation $(\mathscr{V},\mathscr{V}_1)$

In order to identify a triple of basic connections adapted to subfoliation $(\mathscr{V},\mathscr{V}_1)$, we consider the Vrănceanu connections introduced in subsection 2.1.1, and the basic connections $\widetilde{\nabla}^*$, $\overline{\nabla}^*_1$, ∇^{*2}_2 from Propositions 2.3 and 2.6.

The relations (25) are satisfied:

$$i'(\nabla^{*2}_{2\,\frac{\partial}{\partial y^i}} Z_2) = \frac{\partial Z_j}{\partial y^i}\frac{\partial}{\partial p_j} = \overline{\nabla}^*_{1\,\frac{\partial}{\partial y^i}} Z_2 = \overline{\nabla}^*_{1\,\frac{\partial}{\partial y^i}} i'(Z_2),$$

$$\pi'(\overline{\nabla}^*_{\frac{\partial}{\partial y^i}} Z) = \frac{\partial Z^h_j}{\partial y^i}\frac{\delta}{\delta x^j} = \widetilde{\nabla}^*_{\frac{\partial}{\partial y^i}} Z^h,$$

for any $Z = Z^h + Z_2 \in\Gamma(H\oplus V_2)$, locally given by $Z^h = Z^h_j\frac{\delta}{\delta x^j}$, $Z_2 = Z_i\frac{\partial}{\partial p_i}$.

Hence we have the following result:

Proposition 2.7. The triple $(\nabla^{*2}_2,\overline{\nabla}^*_1,\widetilde{\nabla}^*)$ of basic connections on normal bundles V_2, $H\oplus V_2$, H, respectively, is adapted to the subfoliation $(\mathscr{V},\mathscr{V}_1)$ of big-tangent manifold $(\mathscr{T}M,G)$.

*Remark.*If we consider the $(n,2n)$-codimensional subfoliation $(\mathscr{V},\mathscr{V}_2)$, by some analogous considerations we obtain that the restrictions of connections ∇^*_2, ∇^*_1 to $\Gamma(T(\mathscr{T}M))\times\Gamma(H\oplus V_1)$, $\Gamma(V)\times\Gamma(V_1)$, respectively, are basic connections with respect to foliation $\mathscr{V}_2$ and to subfoliation $(\mathscr{V},\mathscr{V}_2)$, respectively. These restrictions together with $\widetilde{\nabla}^*$ represent also a triple of basic connections on $(\mathscr{T}M,G)$, adapted now to subfoliation $(\mathscr{V},\mathscr{V}_2)$.

3 Lagrangians and equation of motion for scalar fields on the big tangent manifold

In [4], the equation of motion for p scalar fields Q^A, $A=\overline{1,p}$, on a foliated manifold M, are expressed using covariant derivative with respect to Vrănceanu connection on that manifold. In this section we apply that ideea for the case of big tangent manifold, $(\mathcal{T}M,\mathcal{V})$ of a Riemannian manifold (M,g), introduced in Section 1.3.

We start with a Lagrangian depending by r scalar fields $Q^A=Q^A(x,y,p)$, $A=\overline{1,r}$, on $\mathcal{T}M$:

$$\mathcal{L}=\mathcal{L}\left(g_{ij},Q^A,\frac{\delta Q^A}{\delta x^i},\frac{\partial Q^A}{\partial y^i},\frac{\partial Q^A}{\partial p_i}\right), \tag{29}$$

which is invarinat under the coordinate transformations (3).

Considering the function H locally defined by

$$H(x)=\sqrt{|det(g_{ij}(x)|},$$

from direct calculation we have the following transformation law in the intersection of two domains of local chart

$$\widetilde{H}=\left|det\left(\frac{\partial x^i}{\partial\tilde{x}^{i_1}}\right)\right|\cdot H.$$

Then

$$\mathcal{L}_0=H\cdot\mathcal{L}, \tag{30}$$

is a Lagrangian density on $\mathcal{T}M$.

The Euler-Lagrange equations for fields Q^A are

$$\frac{\partial\mathcal{L}_0}{\partial Q^A}-\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^i}\right)}\right)-\frac{\partial}{\partial y^i}\left(\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial y^i}\right)}\right)-\frac{\partial}{\partial p_i}\left(\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial p_i}\right)}\right)=0 \tag{31}$$

Taking into account relation (13), we obtain

$$\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial x^i}\right)}=\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial y^i}\right)}\Big|_{\frac{\delta Q^A}{\delta x^i}=ct.}-y^k\Gamma^i_{jk}\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}, \tag{32}$$

$$\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial p_i}\right)}=\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial p_i}\right)}\Big|_{\frac{\delta Q^A}{\delta x^i}=ct.}-p_k\Gamma^k_{ji}\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}. \tag{33}$$

Equation (31) becomes

$$\frac{\partial\mathcal{L}_0}{\partial Q^A}-\frac{\delta}{\delta x^i}\left(\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\right)-\frac{\partial}{\partial y^i}\left(\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial y^i}\right)}\right)-\frac{\partial}{\partial p_i}\left(\frac{\partial\mathcal{L}_0}{\partial\left(\frac{\partial Q^A}{\partial p_i}\right)}\right)=0, \tag{34}$$

and, from (30), it results

$$\frac{\partial\mathcal{L}}{\partial Q^A}-\frac{\delta}{\delta x^i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}\right)-\frac{\partial}{\partial y^i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial y^i}\right)}\right)-\frac{\partial}{\partial p_i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial p_i}\right)}\right)=\frac{1}{H}\frac{\delta H}{\delta x^i}\cdot\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)}. \tag{35}$$

Now we denote

$$\mathcal{L}^i_A:=\frac{\partial\mathcal{L}}{\partial\left(\frac{\delta Q^A}{\delta x^i}\right)},$$

which, from (11), are components of a horizontal vector field and

$$\mathcal{L}'^i_A:=\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial y^i}\right)},\quad \mathcal{L}''^i_A:=\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial Q^A}{\partial p_i}\right)}$$

which are components of vertical vector fields.

The derivatives of $\mathcal{L}^i_A$, $\mathcal{L}'^i_A$, $\mathcal{L}''^i_A$ with respect to Vrănceanu connection ∇^*, given locally in subsection 2.2, are

$$\mathcal{L}^i_A|_j=\frac{\delta\mathcal{L}^i_A}{\delta x^j}+\mathcal{L}^k_A\Gamma^i_{kj},\ \mathcal{L}'^i_A|_j=\frac{\partial\mathcal{L}'^i_A}{\partial y^j},\ \mathcal{L}''^i_A|_j=\frac{\partial\mathcal{L}''^i_A}{\partial p_j}.$$

Then equation (35) could be written by the form

$$\frac{\partial\mathcal{L}}{\partial Q^A}-\mathcal{L}^i_A|_i-\mathcal{L}'^i_A|_i-\mathcal{L}''^i_A|_i=\frac{1}{H}\frac{\delta H}{\delta x^i}\cdot\mathcal{L}^i_A-\mathcal{L}^k_A\Gamma^i_{ki}. \tag{36}$$

But $H=H(x)$, so we obtain by direct calculation

$$\frac{1}{H}\frac{\delta H}{\delta x^i}=\frac{1}{H}\frac{\partial H}{\partial x^i}=\frac{1}{2}g^{js}\frac{\partial g_{js}}{\partial x^i}.$$

Taking into account that Γ^k_{ij} are the Christoffel symbols on the Riemannian manifold (M,g), it follows

$$\frac{1}{H}\frac{\delta H}{\delta x^i}=\Gamma^j_{ij}.$$

Finally, the equation of motion for the scalar fields Q^A have the following nice form

$$\frac{\partial\mathcal{L}}{\partial Q^A}-\mathcal{L}^i_A|_i-\mathcal{L}'^i_A|_i-\mathcal{L}''^i_A|_i=0.$$

*Remark.*Generally, the Lagrangian (29) is also considered invariant to the action of a Lie group on the fields Q^A. In this case equations of motions also could be calculated by the means of Vrănceanu connection, but this is not the purpose of the present paper.

References

[1] Bejancu, A., Farran, H. R., *On The Vertical Bundle of a Pseudo-Finsler Manifold*. Int. J. Math. and Math. Sci. **22** (1997), No. 3, 637–642.

[2] A. Bejancu, H. R. Farran, *Foliations and Geometrical Structures*. Mathematics and Its Applications Vol. **580**, Springer, Dordrecht, 2006.

[3] A. Bejancu, H. R. Farran, *Vranceanu connection and foliations with bundle like-metrics*. Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 1, 2008, 99–113.

[4] A. Bejancu, K.L. Duggal, *Gauge theory on foliated manifolds*. Rendiconti del Seminario Matematico di Messina, Vol. I, 1991, 31–68.

[5] L.A. Cordero, X. Masa, *Characteristic classes of subfoliations*, Annales de l'Institut Fourier, 31(1981), p. 61-86.

[6] M. Gîrţu, *Geometry on the big tangent bundle*. Sci. Stud. Research, Ser. Mahthematics-Informatics **23** No. 1, (2013), 39–48.

[7] N. Hitchin, *Generalized Calabi-Yau manifolds*. Quart. J. Math., **54**, 2003, 281–308.

[8] C. Ida, A. Manea, *A vertical Liouville subfoliation on the cotangent bundle of a Cartan space and some related structures*. Int. J. Geom. Methods Mod. Phys. **11**, Issue 06, July 2014, 1450063 (2014) [21 pages] DOI: 10.1142/S0219887814500637.

[9] C. Ida, P. Popescu,*Vertical Liouville foliations on the bigtangent manifold of a Finsler space*. To appear in Filomat (2017). Preprint available to arXiv: 1402.6099v1.

[10] M. de León, D. Martín de Diego, M. Vaquero, *A Hamilton-Jacobi theory for singular Lagrangian systems in the Skinner and Rusk setting*. Int. J. Geom. Methods Mod. Phys. **09**, 1250074 (2012), 24 pages.

[11] A. Manea, C. Ida, *Adapted basic connections to a certain subfoliation on the tangent bundle of a Finsler space*. Turkish Journal of Mathematics **38**(3), 2014, 470–482.

[12] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces. Theory and Applications*. Kluwer Acad. Publ. **59**, 1994.

[13] R. Miron, D. Hrimiuc, H. Shimada, S. Sabău, *The geometry of Hamilton and Lagrange spaces*. Kluwer Acad. Publ., **118** 2001.

[14] B. L. Reinhart, *Differential Geometry of Foliations*. Berlin, Springer-Verlag, 1983.

[15] Ph. Tondeur, *Foliations on Riemannian Manifolds*. Springer-Verlag, New-York, 1988.

[16] R. Skinner, R. Rusk, *Generalized Hamiltonian dynamics I. Formulation on* $TQ \oplus T^*Q$. J. Math. Phys., **24**, 2589 (1983).

[17] R. Skinner, R. Rusk, *Generalized Hamiltonian dynamics II. Gauge transformations*. J. Math. Phys., **24**, 2595 (1983).

[18] I. Vaisman, *Almost-multifoliate Riemannian manifolds*. An. St. Univ. Iasi **16** (1970), 97-103.

[19] I. Vaisman, *Variétés riemanniene feuilletées*. Czechoslovak Math. J., **21** (1971), 46-75.

[20] I. Vaisman, *Geometry on Big-Tangent Manifolds*. Publ. Math. Debrecen, 86 (2015), no. 1-2, 213–243.

[21] G. Vrănceanu, *Sur quelque points de la theorie des espaces non holonomes*. Bul. Fac. St. Cernăuti **5** (1931) 177-205.

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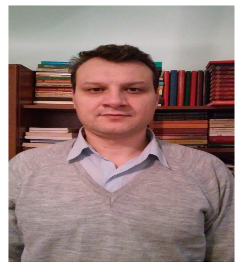

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