Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100537

The Ideal of χ^2 in a Vector Lattice of $p-$ Metric Spaces Defined by Musielak-Orlicz Functions

N. Kavitha[1] *and N. Subramanian*[2,*]

[1] Department of Mathematics, University College of Engineering,(constituent College of Anna University), Pattukkottai, India
[2] Department of Mathematics, SASTRA University, Thanjavur-613 401, India

Received: 24 Apr. 2016, Revised: 23 Jun. 2016, Accepted: 24 Jun. 2016
Published online: 1 Sep. 2016

Abstract: In this paper we introduce the concepts of ideal $\tau-$ double Cauchy and ideal $\tau-$ double analytic sequence in vector lattice defined by Musielak. Some basic properties of these concepts has been investigated.

Keywords: analytic sequence, double sequences, χ^{2FI} space, Musielak - modulus function, fuzzy $p-$ metric space,fuzzy number, solid space, symmetricity, convergence free, sequence algebra,ideal.

1 Introduction

Throughout w,Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m,n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Basarir and Solankan [3], Tripathy [4] and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_p(t) :=$$
$$\left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\},$$
$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$
$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Quite recently, in her PhD thesis, Zelter [5] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [7] and Tripathy have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. In [6] the notion of convergence of double sequences was presented by A. Pringsheim.

We need the following inequality in the sequel of the paper. For $a,b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.
Let M and Φ are mutually complementary Orlicz functions. Then, we have:

* Corresponding author e-mail: nsmaths@yahoo.com

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[8]] \quad (2)$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \quad (3)$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \quad (4)$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mn})(u) : u \geq 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty \right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{ inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1 \right\}$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1}|\Delta x_k| \text{ and}$$
$$\|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m,n \in \mathbb{N}$.

2 Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1,\ldots,x_n) = \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,
(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1,0),\ldots,\alpha d_n(x_n,0))\|_p = |\alpha|\,\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)) = (d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p)^{1/p}$ $for 1 \leq p < \infty$;
(or)
(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)) := \sup\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\}$,
for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn},0))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

The notion of ideal convergence was introduced first by Kostyrko et al.[11] as a generalization of statistical convergence which was further studied in topological spaces by Kumar et al.[12,13] and also more applications of ideals can be deals with various authors by B.Hazarika

[**?**] , B.C.Tripathy , B. Hazarika [**?**] and Shyamal Debnath et al. [32].

Definition 2.1.
A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if
(1) $\phi \in I$
(2) $A, B \in I$ imply $A \bigcup B \in I$
(3) $A \in I, B \subset A$ imply $B \in I$.
while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}\times\mathbb{N}$. A sequence $(x_{mn})_{m,n\in\mathbb{N}\times\mathbb{N}}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \left\{m,n \in \mathbb{N}\times\mathbb{N} : \|(d_1(x_1,0),\ldots,d_n(x_n,0)) - 0\|_p \geq \varepsilon\right\}$ belongs to I.

Definition 2.2.
A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if
(1) $\phi \in F$
(2) for each $A, B \in F$, we have imply $A \bigcap B \in F$
(3) each $A \in F$ and each $A \subset B$, we have $B \in F$.

Definition 2.3.
An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

Definition 2.4.
A non-trivial ideal $I \subset 2^X$ is called (i) admissible if and only if $\{\{x\} : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.
If we take $I = I_f = \{A \subseteq \mathbb{N}\times\mathbb{N} : A \textit{ is a finite subset }\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N}\times\mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{R}\times\mathbb{R}$ and the corresponding convergence coincides with the statistical convergence.
Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}\times\mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X,Y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.
Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation "$\leq$" is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}\times\mathbb{R}$ i.e. a mapping $X : R \to J(=[0,1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.5.
A sequence space E is said to be monotone if E contains the canonical pre-images of all its step spaces.
The following well-known inequality will be used throughout the article. Let $p = (p_{mn})$ be any sequence of positive real numbers with $0 \leq p_{mn} \leq sup_{mn} p_{mn} = G, D = max\{1, 2G-1\}$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq D\left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right) \text{ for all } m,n \in \mathbb{N} \text{ and } a_{mn}, b_{mn} \in \mathbb{C}.$$

Also $|a_{mn}|^{p_{mn}} \leq max\left\{1, |a|^G\right\}$ for all $a \in \mathbb{C}$.
First we procure some known results; those will help in establishing the results of this article.

Lemma 2.6.
A sequence space E is normal implies E is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [8], page 53).

Lemma 2.7.
(Kostyrko et al., [11], Lemma 5.1). If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} - A \in I$.

Definition 2.8.
Let d be a mapping from $R(I) \times R(I)$ into $R(I) \times R(I)$ and let the mappings $L, f : [0,1] \times [0,1] \to [0,1] \times [0,1]$ be symmetric, non-decreasing Musielak Orlicz in both arguments and satisfy $L \times L = 0$ and $f \times f = 1$. Denote $[d(X,Y)]_\alpha = [\lambda_\alpha(X,Y), (X,Y)]$, for $X, Y \in R(I) \times R(I)$ and $0 < \alpha < 1$.
The $(R(I) \times R(I), d, L \times L, f \times f)$ is called a fuzzy $p-$ metric space and d a fuzzy translation metric, if
(1) $d(X,Y) = \overline{0}$ if and only if $X = Y$,
(2) $d(X,Y) = d(Y,X)$ for all $X, Y \in X$,
(3) for all $X, Y, Z \in R(I) \times R(I)$, (i) $d(X,Y)(s+t) \geq L \times L(d(X,Z)(s), d(Z,Y)(t))$ whenever $s \leq \lambda_1(X,Z), t \leq \lambda_1(Z,Y)$ and $(s+t) \leq \lambda_1(X,Y)$,
(ii) $d(X,Y)(s+t) \leq f \times f(d(X,Z)(s), d(Z,Y)(t))$ whenever $s \geq \lambda_1(X,Z), t \geq \lambda_1(Z,Y)$ and $(s+t) \leq \lambda_1(X,Y)$.
A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics.

Definition 2.9.
Let $E \subset \mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and is defined by $\delta(E) = lim_{pq\to\infty} |\{m,n \in E : m \leq p, n \leq q\}|$, where the vertical bar denotes the cardinality of the respective set.

Definition 2.10.
A sequence $x = (x_{mn})$ in $\underline{a}$ topological space X is said to be $\underline{s}$tatistically convergent $\overline{0}$ if for every neighbourhood V of $\overline{0}$

$$\delta(\{m,n \in \mathbb{N} : x_{mn} \notin V\}) = 0.$$

In this case, we write $S - limx = \overline{0}$ and S denotes the set of all statistically null sequences.

Definition 2.11.
A sequence $x = (\underline{x}_{mn})$ in a topological space X is sa$\underline{i}$d to be $I-$ convergent $\overline{0}$ if for every neighbourhood V of $\overline{0}$

$$\{m,n \in \mathbb{N} : x_{mn} \notin V\} \in I.$$

In this case, we write $I - limx = \bar{0}$ and I denotes the set of all ideally null sequences.

Let X be a real vector space and $\leq$ be a partial order on this space. Then X is said to be an ordered space if it satisfies the following properties:
(i)if $x,y \in X$ and $y \leq x$, then $y+z \leq x+z$ for each $z \in X$.
(ii) if $x,y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.
If, in addition, X is a lattice with respect to the partially ordered, then X is said to be a vector lattice, if for each pair of elements $x,y \in X$ the supremum and infimum of the set x,y both exist in X. We shall write $x \vee y = sup\{x,y\}$ and $x \wedge y = inf\{x,y\}$.

For an element x of a vector lattice X, the positive part of x is defined by $x^+ = x \vee \bar{\theta}$, the negative part of x is defined by $x^- = -x \vee \bar{\theta}$, and the absolute value of x by $|x| = x \vee (-x)$, where $\bar{\theta}$ is the zero element of X.

A subset S of a vector lattice space X is said to be solid if $y \in S$ and $|y| \leq |x|$ implies $x \in S$.

A topological vector space (X,τ) is a vector space X which has a topology (linear) τ, such that the algebraic operations and addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f : X \times X \to x \times X$ defined by $f(x,y) = x+y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f : \mathbb{C} \times \mathbb{C} \to X \times X$. defined by $f(a,x) = ax$ is continuous on $\mathbb{C} \times X$.

Every linear topology τ on a vector space X has a base N for the neighbourhoods of $\bar{\theta}$ satisfying the following properties:
(1) Each $Y \in N$ is a balanced set, that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$. (2) Each $Y \in N$ is an absorbing set , that is , for every $x \in X$, there exists $a > 0$ such that $ax \in Y$. (3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A locally solid Riesz space $(X;\tau)$ is a Riesz space equipped with a locally solid topology τ.

3 χ^2- Ideal topological convergence in a vector lattice

Definition 3.1.
Let $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. A sequence (x_{mn}) of points in χ^2 is said to be $S(\tau)-$ convergent to an element $\bar{0}$ of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \notin V\right\}\right) = 0$$

(i.e.),
$lim_{uv} \frac{1}{uv}\left(\left\{m,n \leq u,v : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \notin V\right\}\right) = 0.$
In this case we write $S(\tau) - lim_{mn\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$.

4 χ^2- Ideal topological convergence in a vector lattice

Definition 4.1.
Let $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. A sequence (x_{mn}) of points in χ^2 is said to be $I(\tau)-$ convergent to an element $\bar{0}$ of χ^2 if for each $\tau-$ neighbourhood V of zero,

$$\left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \notin V\right\} \in I.$$

(i.e.),
$\left\{m,n \leq u,v : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \notin V\right\} \in \Im.$
In this case we write $I(\tau) - lim_{mn\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$.

Definition 4.2.
Let $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. A sequence (x_{mn}) of points in X if for each $\tau-$ neighbourhood V of zero, there is some $k > 0, \{m,n \in \mathbb{N} : kx_{mn} \notin V\} \in I$.

Theorem 4.3.
Let $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. Every $I(\tau)-$ convergent sequences in χ^2 has only one limit.

Proof. Suppose that $x = (x_{mn})$ is a sequence in χ^2 such that
$I(\tau) - lim_{mn\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$.
Let V be any $\tau-$ neighbourhood of zero. Also for each $\tau-$ neighbourhood V of zero there exists $Y \in N_{VecL}$ such that $Y \subseteq V$. Choose any $W \in N_{VecL}$ such that $W + W \subseteq Y$. We define the following set:

$$A_1 = \left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \in W\right\}$$

Since $I(\tau) - lim_{mn\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$ we get $A_1 \in \Im$. Now,let $A = A_1 \bigcap A_1$. Then we have $\bar{0} - \bar{0} = \bar{0} - ((m+n)!\,|x_{mn}|)^{1/m+n} + ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \in W + W \subseteq Y \subseteq V$. Hence for each $\tau-$ neighbourhood V of zero we have $\bar{0} \in V$. Since (χ^2,τ) is Hausdorff, the intersection of all $\tau-$ neighbourhoods V of zero is the singleton set $\{\bar{\theta}\}$. Thus we get the result.

Theorem 4.4.
Let $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. If a sequence (x_{mn}) in χ^2 is $I(\tau)-$ convergent, then it is $I(\tau)-$ bounded.

Proof. Suppose that (x_{mn}) is $I(\tau)$ convergent to a point $\bar{0} \in \chi^2$. Let V be an arbitrary $\tau-$ neighbourhood of zero, there exists $Y \in N_{VecL}$ such that $Y \subseteq V$. We choose $W \in N_{VecL}$ such that $W + W \subseteq Y$. Since $I(\tau) - lim_{mn\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$, the set

$$A = \left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \notin W\right\} \in I.$$

Since W is absorbing, there exists $a > 0$ such that $a \in W$. Let b be such that $|b| \leq 1$ and $b \leq a$. Since W is solid and $|b| \leq |a|$, we have $b \in W$. Also, since W is balanced $((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \in W$ implies $b\left(((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0}\right) \in W$. Then we have $b\,((m+n)!\,|x_{mn}|)^{1/m+n} = b\left(((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0}\right) + b\bar{0} \in W + W \subseteq V$, for each $m,n \in \mathbb{N} - A$. Thus $\left\{m,n \in \mathbb{N} : b\,((m+n)!\,|x_{mn}|)^{1/m+n} \notin W\right\} \in I$. Hence (x_{mn}) is $I(\tau)-$ bounded.

Theorem 4.5. $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. If a sequence $(x_{mn}), (y_{mn})$ and (z_{mn}) be three vector lattice of points in χ^2 such that
(i)$x_{mn} \leq y_{mn} \leq z_{mn}$, for all $m,n \in \mathbb{N}$,
(ii)$I(\tau) - lim_{mn\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0} = I(\tau) - lim_{mn\to\infty}((m+n)!\,|z_{mn}|)^{1/m+n} = \bar{0} = I(\tau) - lim_{mn\to\infty}((m+n)!\,|y_{mn}|)^{1/m+n}$

Proof. Let V be an arbitrary $\tau-$ neighbourhood of zero, there exists $Y \in N_{VecL}$ such that $Y \subseteq V$. We choose $W \in N_{VecL}$ such that $W + W \subseteq Y$. From given condition (ii), we have $P, Q \in \mathfrak{I}$, where $P = \left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \in W\right\}$ and $Q = \left\{m,n \in \mathbb{N} : ((m+n)!\,|z_{mn}|)^{1/m+n} - \bar{0} \in W\right\}$. Also from the given condition (i), we have $((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \leq ((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0} \leq ((m+n)!\,|z_{mn}|)^{1/m+n} - \bar{0}$
$\Rightarrow \left|((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0}\right| \leq \left|((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0}\right| + \left|((m+n)!\,|z_{mn}|)^{1/m+n} - \bar{0}\right| \in W + W \subseteq Y$. Since Y is solid, we have $((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0} \in Y \subseteq V$. Thus, $\left\{m,n \in \mathbb{N} : ((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0} \in V\right\} \in \mathfrak{I}$, for each $\tau-$ neighbourhood of V of zero. Hence $I(\tau) - lim_{mn\to\infty}((m+n)!\,|y_{mn}|)^{1/m+n} = \bar{0}$.

5 $I(\tau)-$ and $I^*(\tau)-$ convergence in vector lattice of Musielak

Definition 5.1.
Let $\left[\chi^{2\tau}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ be a vector lattice of Musielak. A sequence (x_{mn}) of points in χ^2 is said to be $I(\tau)-$ Cauchy in χ^2 if for each $\tau-$ neighbourhood V of zero, there is an integer $p, q \in \mathbb{N}$,

$$\left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - x_{pq} \notin V\right\} \in I.$$

Theorem 5.2.
If $lim_{m,n\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$, then $I(\tau) - lim_{m,n\to\infty}\left(((m+n)!\,|x_{mn}|)^{1/m+n} + \bar{0}\right) = lim_{m,n\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n}$

Proof. Let V be any $\tau-$ neighbourhood of zero. Then there exists $Y \in N_{VecL}$ such that $Y \subseteq V$. Choose $W \in N_{VecL}$ such that $W + W \subseteq Y$. Since $lim_{m,n\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n} = \bar{0}$, then there exists an integer p_0q_0 such that $m,n \geq p_0q_0$ implies that $((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \in W$. Hence

$$\left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0} \notin W\right\} \subseteq \mathbb{N} - \{p_0q_0\}.$$

Thus

$$\left\{m,n \in \mathbb{N} : \left(((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0}\right) + \bar{0} \notin V\right\} \in I.$$

This implies that $I(\tau) - lim_{m,n\to\infty}\left(((m+n)!\,|x_{mn}|)^{1/m+n} + \bar{0}\right) = lim_{m,n\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n}$.

Theorem 5.3.
$I(\tau) - lim_{m,n\to\infty}\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right) = lim_{m,n\to\infty}((m+n)!\,|x_{mn}|)^{1/m+n}$ be a vector lattice and let $x = (x_{mn})$ be a sequence in χ^2. If there is a $I(\tau)-$ convergent sequence $y = (y_{mn}) \in \chi^2$ such that $\left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} \neq ((m+n)!\,|y_{mn}|)^{1/m+n} \notin V\right\} \in I$ then x is also $I(\tau)-$ convergent

Proof. Suppose that $\left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} \neq ((m+n)!\,|y_{mn}|)^{1/m+n} \notin V\right\} \in I$ and $I(\tau) - lim_{m,n\to\infty}\left(((m+n)!\,|ymn|)^{1/m+n}\right) = \bar{0}$. Then for an arbitrary $\tau-$ neighbourhood V of zero, we have

$$\left\{m,n \in \mathbb{N} : \left(((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0}\right) \notin V\right\} \in I.$$

Now,
$\left\{m,n \in \mathbb{N} : \left(((m+n)!\,|x_{mn}|)^{1/m+n} - \bar{0}\right) \notin V\right\} \subseteq \left\{m,n \in \mathbb{N} : ((m+n)!\,|x_{mn}|)^{1/m+n} \neq ((m+n)!\,|y_{mn}|)^{1/m+n} \notin V\right\} \cup \left\{m,n \in \mathbb{N} : \left(((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0}\right) \notin V\right\}$
Therefore we have $\left\{m,n \in \mathbb{N} : \left(((m+n)!\,|y_{mn}|)^{1/m+n} - \bar{0}\right) \notin V\right\}$.

Competing interests: Authors have declared that no competing interests exist.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The authors are thankful to the editor(s) and reviewers of Applied Mathematics & Information Sciences.

References

[1] T.J.I'A.Bromwich, An introduction to the theory of infinite series, *Macmillan and Co.Ltd.* ,New York, (1965).

[2] G.H.Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.

[3] M.Basarir and O.Solancan, On some double sequence spaces, *J. Indian Acad. Math.*, **21(2)** (1999), 193-200.

[4] B.C.Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.

[5] M.Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, *Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu*, **2001**.

[6] A.Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, *Math. Ann.*, **53**, (1900), 289-321.

[7] M.Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.

[8] P.K.Kamthan and M.Gupta, Sequence spaces and series, Lecture notes, *Pure and Applied Mathematics*, **65 Marcel Dekker, In c., New York** , 1981.

[9] J.Lindenstrauss and L.Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[10] A.Wilansky, Summability through Functional Analysis, *North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam*, **Vol.85**(1984).

[11] P.Kostyrko, T.Salat and W.Wilczynski, $I-$ convergence, *Real Anal. Exchange*, **26(2)** (2000-2001), 669-686, MR 2002e:54002.

[12] V.Kumar and K.Kumar, On the ideal convergence of sequences of fuzzy numbers, *Inform. Sci.*, **178(24)** (2008), 4670-4678.

[13] V.Kumar , On I and I^*- convergence of double sequences, *Mathematical communications*, **12** (2007), 171-181.

[14] B.Hazarika, On Fuzzy Real Valued $I-$ Convergent Double Sequence Spaces, *The Journal of Nonlinear Sciences and its Applications* **(in press)**.

[15] B.Hazarika, On Fuzzy Real Valued $I-$ Convergent Double Sequence Spaces defined by Musielak-Orlicz function, *J. Intell. Fuzzy Systems*, **25(1)** (2013), 9-15, DOI: 10.3233/IFS-2012-0609.

[16] B.Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, *J. Intell. Fuzzy Systems*, **25(1)** (2013), 157-166, DOI: 10.3233/IFS-2012-0622.

[17] B.Hazarika, On $\sigma-$ uniform density and ideal convergent sequences of fuzzy real numbers, *J. Intell. Fuzzy Systems*, DOI: 10.3233/IFS-130769.

[18] B.Hazarika, Fuzzy real valued lacunary $I-$ convergent sequences, *Applied Math. Letters*, **25(3)** (2012), 466-470.

[19] B.Hazarika, Lacunary $I-$ convergent sequence of fuzzy real numbers, *The Pacific J. Sci. Techno.*, **10(2)** (2009), 203-206.

[20] B.Hazarika, On generalized difference ideal convergence in random 2-normed spaces, *Filomat*, **26(6)** (2012), 1265-1274.

[21] B.Hazarika, Some classes of ideal convergent difference sequence spaces of fuzzy numbers defined by Orlicz function, *Fasciculi Mathematici*, **52** (2014)(Accepted).

[22] B.Hazarika, $I-$ convergence and Summability in Topological Group, *J. Informa. Math. Sci.*, **4(3)** (2012), 269-283.

[23] B.Hazarika, Classes of generalized difference ideal convergent sequence of fuzzy numbers, *Annals of Fuzzy Math. and Inform.*, (in press).

[24] B.Hazarika, On ideal convergent sequences in fuzzy normed linear spaces, *Afrika Matematika*, DOI: 10.1007/s13370-013-0168-0.

[25] B.Hazarika and E.Savas, Some $I-$ convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, *Math. Comp. Modell.*, **54(11-12)** (2011), 2986-2998.

[26] B.Hazarika,K.Tamang and B.K.Singh, Zweier Ideal Convergent Sequence Spaces Defined by Orlicz Function, *The J. Math. and Computer Sci.*, (Accepted).

[27] B.Hazarika and V.Kumar, Fuzzy real valued $I-$ convergent double sequences in fuzzy normed spaces, *J. Intell. Fuzzy Systems*, (accepted).

[28] B.C.Tripathy and B.Hazarika, $I-$ convergent sequence spaces associated with multiplier sequences, *Math. Ineq. Appl.*, **11(3)** (2008), 543-548.

[29] B.C.Tripathy and B.Hazarika, Paranorm $I-$ convergent sequence spaces, *Math. Slovaca*, **59(4)** (2009), 485-494.

[30] B.C.Tripathy and B.Hazarika, Some $I-$ convergent sequence spaces defined by Orlicz functions, *Acta Math. Appl. Sinica*, **27(1)** (2011), 149-154.

[31] B.Hazarika and A.Esi, On ideal convergent sequence spaces of fuzzy real numbers associated with multiplier sequences defined by sequence of Orlicz functions, *Annals of Fuzzy Mathematics and Informatics*, (in press).

[32] Shyamal Debnath and Jayanta Debnath, Some ideal convergent sequence spaces of fuzzy real numbers, *palestine Journal of Mathematics*, **3(1)** (2014), 27-32.

N. Kavitha working as an Assistant Professor in the department of Mathematics in University College of Engineering, Pattukkottai, (A constituent college of Anna university, Chennai.), India. She had completed her Ph.D., in Anna University on 2014. She have attended 9 international conferences and one national conference. Also she have presented 2 papers in national conferences and 4 papers in international conferences. She have published 12 in reputed National and International journals.

N. Subramanian received the PhD degree in Mathematics for Alagappa University at Karaikudi, Tamil Nadu, India and also getting Doctor of Science (D.Sc) degree in Mathematics for Berhampur University, Berhampur, Odissa, India. His research interests are in the areas of summability through functional analysis of applied mathematics and pure mathematics. He has published research articles 186 in reputed international journals of mathematical and engineering sciences.