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Portfolio Optimization by Mean-Value at Risk Framework

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Abstract: The purpose of this study is to evaluate various tools used for improving performance of portfolios and assets selection using mean-value at risk models. The study is mainly based on a non-parametric efficiency analysis tool, namely Data Envelopment Analysis (DEA). Conventional DEA models assume non-negative values for inputs and outputs, but variance is the only variable in models that takes non-negative values. At the beginning variance was considered as a risk measure. However, both theories and practices indicate that variance is not a good measure of risk and has some disadvantages. This paper focuses on the evaluation process of the portfolios and replaces variance by value at risk (VaR) and tries to decrease it in a mean-value at risk framework with negative data by using mean-value at risk efficiency (MVE) model and multi objective mean-value at risk (MOMV) model. Finally, a numerical example with historical and Monte Carlo simulations is conducted to calculate value at risk and determine extreme efficiencies that can be obtained by mean-value at risk framework.

Keywords: Portfolio, Data Envelopment Analysis, Value at Risk, Efficiency, Negative data, Mean-Value at Risk Efficiency, Multi Objective Mean-Value at Risk

1 Introduction

For investors, best portfolios or assets selection and risks management are always challenging topics. Investors typically try to find portfolios or assets offering less risk and more return. Markowitz [18] works are the first type of these kinds of attempts to find such securities in a mathematical way. The model he introduced, known as Markowitz or mean-variance (MV) model, tries to decrease variance as a risk parameter in all levels of mean. This model results in an area with a frontier called efficient frontier. Morey and Morey [21] proposed mean-variance framework based on Data Envelopment Analysis, in which variances of the portfolios are used as inputs and expected return are used as outputs to DEA models. Data Envelopment Analysis has proved the efficiency for assessing the relative efficiency of Decision Making Units (DMUs) that employs multiple inputs to produce multiple outputs (Charnes et al. [8]). Briec et al. [5] tried to project points in a preferred direction on efficient frontier and evaluate points efficiencies by their distances. Demonstrated model by Briec et al. [6], which is also known as a shortage function, has some advantages. For example optimization can be done in any direction of a mean-variance space according to the investors ideal. Such as, in shortage function, efficiency of each security is defined as the distance between the asset and its projection in a pre-assumed direction. As an instance, in variance direction, efficiency is equal to the ratio between variance of projection point and variance of asset. Based on this definition if distance equals to zero, that security is on the frontier area and its efficiency equals to 1. This number, in fact, is the result of shortage function which tries to summarize value of efficiency by a number. Similar to any other model, mean-variance model has some basic assumptions. Normality is one of its important assumptions. In mean-variance models, distributions of means of securities on a particular time horizon should be normal. In contrast Mandelbrot [17] showed, not only empirical distributions of means are widely skewed, but they also have thicker tails than normal. Ariditti [1] and Kraus and Litzenberger [16] also showed that expected return in respect of third moment is positive. Ariditti [1],Kane [14], Ho and Cheung [11] showed that most investors prefer positive skewed assets or portfolios, which means that skewness is an output parameter and same as mean or expected return, should be increased. Based on Mitton and Vorkink [20] most

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investors scarify mean-variance model efficiencies for higher skewed portfolios. In this way Joro and Na [13] introduced mean-variance-skewness framework, in which skewnesses of returns considered as outputs.However, in their recommended model same as mean-variance model, optimization is done in one direction at a time. Briec et al. [6] introduced a new shortage function which obtains an efficiency measure looks to improve both mean and skewness and decreases variance at a time. Kerstens et al. [15] introduced a geometric representation of the MVS frontier related to new tools introduced in their paper. In the new models instead of estimating the whole efficient frontier, only the projection points of the assets are computed. In these models a non-linear DEA-type framework is used where the correlation structure among the units is taken into account. Nowadays, most investors think consideration of skewness and kurtosis in models are critical. Mhiri and Prigent [19] analyzed the portfolio optimization problem by introducing higher moments of return - the main financial index. However, using this approach needs variety of assumptions hold. Therefore, there is not a general willingness to incorporate higher order moments. Up to this point the assumption is that variance is a parameter that evaluates risk and it is preferred to be decreased, although, not everybody wants this. For example a venture capitalist prefers risky portfolios or assets, followed by more return than normal. In mean-variance models evaluation, such situations are considered as undesirable. But they are not really undesirable for those who are interested in risk for higher returns. There are some approaches, trying to address such ambiguities by introducing other parameters, such as semi variance. However, each approach has its own disadvantage which makes it less desirable. A new approach to manage and control risk is value at risk (VaR). This new approach focuses on the left hand side of the range of normal distribution where negative returns come with high risk. Value at risk was first proposed by Baumol [3]. The goal is to measure loss of return on left side of the portfolios return distribution by reporting a number. Based on VaR definition, it is assumed that securities have a multivariate normal distribution; however, they are also true for non-normal securities. Silvapulle and Granger [27] estimated VaR by using ordered statistics and nonparametric kernel estimation of density function. Chen and Tang [9] investigated another nonparametric estimation of VaR for dependent financial returns. Bingham et al. [4] studied VaR by using semi-parametric estimation of VaR based on normal mean-variance mixtures framework. A fully nonparametric estimation of dynamic VaR is also developed by Jeong and Kang [12] based on the adaptive volatility estimation and the nonparametric quantiles estimation. Angelidis and Benos [2] calculated VaR for Greek Stocks by employing nonparametric methods, such as historical and filtered historical simulation. Recently, the nonparametric quantile regression, along with the extreme value theory, is applied by Schaumburg [24] to predict VaR. All together Using VaR as a risk controlling parameter is the same as variance; a similar framework is applied: variance is replaced by VaR and then it is decreased in a mean-VaR space. In this study value at risk is decreased in a mean-value at risk framework with negative data. Conventional DEA models, as used by Morey and Morey [21], assume non-negative values for inputs and outputs. These models cannot be used for the case in which DMUs include both negative and positive inputs and/or outputs. Poltera et al. [23] consider a DEA model which can be applied in cases where input/output data take positive and negative values. Models which are going to be introduced in this paper are developed based on this model; although, there are also other models can be used for negative data such as Modified slacks-based measure model (MSBM), Sharpe et.al. [25], semi-oriented radial measure (SORM), Emrouznejad [10]. The paper is organized as follows. Section 2 presents a quick look at DEA models, mean-variance models of Markowitz [18], Morey and Morey [21], Joro and Na [13], and Shortage function. In section 3 Variance-covariance method, historical and Monte Carlo simulation methods, for calculating VaR, are briefly reviewed. In section 4 mean-VaR models are developed by using historical and Monte Carlo simulation methods with negative data. Section 5 represents a real global application and proposed models are applied to evaluate portfolios performance. And finally in section 6 a comparison between models is made.

2 Background

First portfolio theory for investing was published by Markowitz [18]. Markowitz approach begins with assuming that an investor has given money to invest at the present time and this money will be invested for an investors preferred time horizon. At the end of the investing period, the investor will sell all of the assets that were bought at the beginning and then either expenses or reinvests that money. Since portfolio is a collection of assets, it is better to select an optimal portfolio from a set of possible portfolios. Hence, the investor should recognize returns of portfolios' assets (or portfolios' return) and their standard deviations. This means that the investor wants to maximize expected return and minimize uncertainty (risk). Rate of return (or simply return) of the investors wealth from beginning to the end of period is calculated as follows:

Return=

$$\frac{(\text{end of period wealth}) - (\text{beginning of period wealth})}{\text{beginning of period wealth}} \tag{1}$$

or

$$\text{Return}=\log(\text{end of period wealth})-\log(\text{beginning of period wealth}). \quad (2)$$

Since Portfolio is a collection of assets, its return (r_P) can be calculated in a similar manner. Thus according to Markowitz [18], the investor should consider rates of returns associated with any of these portfolios as, what is called in statistics, a random variable. These variables can be described by mean (r_P) and standard deviation (σ_P), which are calculated as follows:

$$\bar{r}_P=\sum_{i=1}^{n}\lambda_i\bar{r}_i, \quad (3)$$

$$\sigma_P=[\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j\omega_{ij}]^{\frac{1}{2}} \quad (4)$$

where n is the number of assets in the portfolio, r_i is expected return of asset i calculated from equation 1 or 2 and r_P is expected return of portfolio. Also, λ_i is proportion of portfolios initial value invested in asset i and σ_P is standard deviation of portfolio, and ω_{ij} shows matrix of covariance of returns between assets i and j. So optimal portfolio from a set of portfolios either offering maximum expected return among a varying levels of risk or minimum risk for a varying levels of expected returns (Sharpe [26]).

Based on Markowitz [18] theory, it is required to characterize the whole efficient frontier, which for large number of assets is cumbersome. In contrast Morey and Morey [21] measured efficiency of under evaluation assets through DEA models. Data envelopment analysis (DEA) is a nonparametric method for evaluating the efficiency of systems with multiple inputs or outputs. In this section we present, not discussing in details, some basic definitions, models and concepts that will be used in other sections. Consider DMU_j $(j=1,\cdots,n)$ where each DMU consumes m inputs to produce s outputs. Also, suppose that the observed input and output vectors of DMU_j are $X_j=(x_{1j},\cdots,x_{mj})$ and $Y_j=(y_{1j},\cdots,y_{sj})$ respectively, and let $X_j\geq 0$, $X_j\neq 0$ and $Y_j\geq 0$, $Y_j\neq 0$. A basic DEA formulation in input orientation is as follows:

$$\begin{array}{lll} min\ \theta-\varepsilon(\sum_{r=1}^{s}s_r^{+}+\sum_{i=1}^{m}s_i^{-}) & & \\ s.t.\ \sum_{j=1}^{n}\lambda_j x_{ij}+s_i^{-}=\theta x_{io} & i=1,\cdots,m, & \\ \quad\ \sum_{j=1}^{n}\lambda_j y_{rj}-s_r^{+}=y_{ro} & r=1,\cdots,s, & \\ \quad\ \lambda\in\Lambda, & & \\ \quad\ s^{+},s^{-}\geq 0, & \varepsilon\geq 0 & \end{array} \quad (5)$$

where λ is a n-vector of variables λ_i, s^{+} is a s-vector of output slacks,s^{-} is a m-vector of input slacks, ε is a non-Archimedes factor, and the set Λ is defined as follows:

$$\lambda=\begin{cases}\lambda\in R_{+}^{n} \\ \qquad \text{with constant returns to scale,} \\ \lambda\in R_{+}^{n},\ 1^{T}\lambda\leq 1 \\ \qquad \text{with non-increasing returns to scale,} \\ \lambda\in R_{+}^{n},\ 1^{T}\lambda=1 \\ \qquad \text{with variable returns to scale.}\end{cases}$$

Note that subscript o refers to the under evaluation unit. A DMU is efficient if and only if $\theta=1$ and all slack variables (s^{+} and s^{-}) are equal to zero otherwise it is inefficient, (Charnes et al. [8]). In the DEA formulation (5) , the left-hand-sides of constraints define an efficient unit, while, the scalars in the right-hand sides are the inputs and outputs of the under evaluation unit and the theta is a multiplier that defines the distance from the efficient frontier. The slack variables are also used to ensure that the efficient points are fully efficient. In solving DEA models three different attitudes can be considered. DEA models can be input, output or combined oriented, where, each orientation has its own interpretation in financial fields.

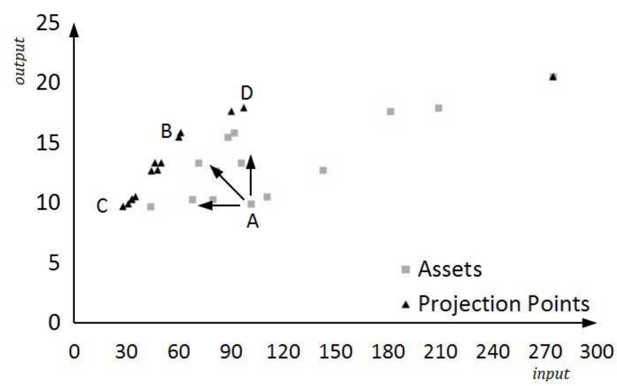

Fig. 1: Different projections (input oriented, output oriented and combined oriented).

Figure 1 illustrates different projections' orientations which are consist of input, output and combined oriented in DEA models. C is the projection point obtained by fixing level of expected return as output and minimizing variance (input oriented); B is the projection point obtaining by maximizing output (return) and minimizing input (variance) simultaneously (combined oriented), and D is the projection point obtaining by fixing variance (input) level and maximizing return (output oriented).

In recent years these models have been widely used to evaluate portfolios' efficiencies. Morey and Morey [21] used DEA model to measure efficiency of under evaluation assets only by characterizing projection points. Based on DEA model, efficiency of an asset θ is the distance between an asset and its projection. In fact, efficiency is the ratio between the variance of the projection points and the variance of the under evaluation assets. In Morey and Morey [21] framework, there are n assets, and λ_j is the weight of asset j in the projection point. r_j is the expected return of asset j. s_1 is a s-vector of output slacks and s_2 is a m-vector of input slacks. Also, ε is a non-Archimedes factor and μ_o and σ_o^2 are expected return and variance of under evaluation asset respectively.

Efficiency measure (θ) can be determined by following model,

$$\begin{array}{ll} \min\ \theta-\varepsilon(s_1+s_2) & \\ s.t.\ \ E(\sum_{j=1}^{n}\lambda_j r_j)-s_1=\mu_o & \\ \quad E[(\sum_{j=1}^{n}\lambda_j(r_j-\mu_j))^2]+s_2=\theta\sigma_o^2, & \\ \quad \sum_{j=1}^{n}\lambda_j\le 1 \qquad \forall\boldsymbol{\lambda}\ge 0. & \end{array} \tag{6}$$

Model 6 is developed based on the non-parametric efficiency analysis named Data Envelopment Analysis. Briec et al. [6] used directions in optimization. They tried to project the under evaluation assets on the efficient frontier via maximizing return and minimizing variance simultaneously in the direction of the vector $\boldsymbol{g}=(|\mu_o|,-|\delta_o^2|)$ using the following model:

$$\begin{array}{l} \max\ \beta \\ s.t.\ \ E(\sum_{j=1}^{n}\lambda_j r_j)\ge\mu_o+\beta\mu_o \\ \quad var[r(\boldsymbol{\lambda})]\le\sigma_o^2-\beta\sigma_o^2, \\ \quad \sum_{j=1}^{n}\lambda_j\le 1 \qquad \forall\boldsymbol{\lambda}\ge 0. \end{array} \tag{7}$$

where

$$var[r(\boldsymbol{\lambda})]=E[(r(\boldsymbol{\lambda})-E[r(\boldsymbol{\lambda})])^2]=\sum_{i,j=1}^{n}\lambda_i\lambda_j\omega_{ij}. \tag{8}$$

When model 7 equals zero, the under evaluation unit is on the efficient frontier. In equation 8 n is number of assets in the portfolio. λ_j is proportion of portfolios initial value invested in asset j and $\boldsymbol{\lambda}$ is a n-vector of variables λ_j. Also r_j is return of asset j and ω_{ij} is covariance of returns between asset i and asset j.

Later studies revealed that skewness is a preferred moment by investors. Based on studies of Mandelbrot [17], Ariditti [1], Kane [14] and Ho and Cheung [11], investors try to choose assets with higher rates of skewness. Therefore, Joro and Na [13] extended model 6 into mean-variance-skewness model where κ_o is the skewness of the under evaluation asset. The following model measures efficiency (θ) of the under evaluation asset.

$$\begin{array}{l} \min\theta\ -\varepsilon(s_1+s_2+s_3) \\ s.t.\ \ E[\sum_{j=1}^{n}\lambda_j r_j]-s_1=\mu_o \\ \quad E[(\sum_{j=1}^{n}\lambda_j(r_j-\mu_j))^2]+s_2=\theta\sigma_o^2, \\ \quad E[(\sum_{j=1}^{n}\lambda_j(r_j-\mu_j))^3]=\kappa_o, \\ \quad \sum_{j=1}^{n}\lambda_j\le 1, \qquad \forall\lambda\ge 0. \end{array} \tag{9}$$

Model 9 projects the asset on the efficient frontier by fixing expected return and skewness levels and minimizing variance.

In the conventional DEA models, each $DMU_j(j=1,\cdots,n)$ is specified by a pair of non-negative input and output vectors $(x_i,y_j)\in R_+^{(m+s)}$, in which inputs x_{ij} $(i=1,\cdots,m)$ are utilized to produce outputs, y_{rj} $(r=1,\cdots,s)$. These models cannot be used for the cases in which DMUs include both negative and positive inputs and/or outputs. Portela et al. [23] considered a DEA model which can be applied in cases where input/output data take both positive and negative values. Range Directional Measure (RDM) model proposed by Portela et al. [23] is as follow:

$$\begin{array}{lll} \max\ \beta & & \\ s.t.\ \ \sum_{j=1}^{n}\lambda_j x_{ij}\le x_{io}+\beta R_{io} & i=1,\cdots,m, & \\ \quad \sum_{j=1}^{n}\lambda_j y_{rj}>=y_{ro}-\beta R_{ro} & r=1,\cdots,s, & \\ \quad \sum_{j=1}^{n}\lambda_j=1, & j=1,\cdots,n. & \end{array} \tag{10}$$

where

$$R_{io}=x_{io}-\min_j\{x_{ij}:j=1,\cdots,n\}, \qquad i=1,\cdots,m, \tag{11}$$

$$R_{ro}=\max_j\{y_{rj}:j=1,\cdots,n\}-y_{ro}, \qquad r=1,\cdots,s. \tag{12}$$

Ideal point (I) within the presence of negative data, is

$$I=(max_j\{y_{rj}:r=1,\cdots,s\},\min_j\{x_{ij}:i=1,\cdots,m\}), \tag{13}$$

and the goal is to project each under evaluation assets' points to this ideal point. Other models that use negative data are modified slacks-based measure model (MSBM), Emrouznejad [10] and semi-oriented radial measure (SORM), Sharpe et al. [25].

2.1 Value at risk

Value at Risk (VaR) is defined as maximum amount of invest that one may loss in a specified time interval. Calculation of VaR can be done through different methods. In this paper variance-covariance method, historical and Monte Carlo simulations are introduced. Each method has one or more assumptions. As an instance, Variance-covariance method can be used only when assets' returns are normally distributed; therefore, before normality check should be performed. In contrast with variance-covariance method, Historical simulation is a distribution free method. In this method without searching for an exact distribution we use returns original distribution for future forecasting. In Monte Carlo method, simulated data are used to find VaR. Now one may ask about differences of these three methods. That is what will be reviewed in this paper. One of VaRs advantages is difference consideration among risk aversion investors and risk lovers followed by larger returns; moreover, it can be calculated easily. Statistically, VaR is defined as the quantile of a distribution. That is:

$$p(\Delta P_k>-VaR)=1-\alpha \tag{14}$$

Where ΔP_k defined as $P_{k+1}-P_k$ and α is defined as confidence leve. In this definition probability of losing more invest than *VaR* is equal to alpha (α). For example if a specified investment is done by 100 people for a year, alpha percent of investors may loss more money than *VaR* in one year. Based on this definition, we need a statistical distribution to estimate a confidence level for *VaR*. But the most challenging part is choosing of a distribution. So simulation is the most important step in of our calculations. In following sections we will go through different methods of *VaR* calculation.

2.1.1 Variance-covariance method

Variance-covariance method is one of the parametric methods that were suggested by Morgan [22]. In this method returns of assets should be normally distributed, and by using variance and covariance of returns, *VaR* is computed. Furthermore, time interval is usually taken as a day. Consequently, average return of each day is around zero. Consider return of a portfolio through a 200 days horizon equals to 20% then return of each day is about 0.001 percent. As mentioned earlier, this model is based on this assumption that returns of assets are assumed to follow conditional normal distribution;although, return of assets by themselves may not normally distributed because of outliers existence (distributions with fat tails). Adapting of variance-covariance method means one has accepted normal distribution assumption, then variances and covariances can easily be estimated and *VaR* can be computed. Drawbacks of this approach are,

- Wrong distribution assumption, if returns distributions are not really normally distributed;
- Input error, whenever a parameter is estimated, it followed by an error; and
- Non-stationary variables, when returns of our assets are gathered over time. So variances and covariances across assets might change.

Some works should be done to provide approaches for dealing with these weaknesses. Among these approaches, sampling and time series methods are suggested, for example. In many situations we may not know distributions and it may also be impossible to obtain them, in such cases one may use one of the other methods.

2.1.2 Historical simulation

Historical simulation is a non-parametric method in which no specific distribution is considered; In fact *VaR* is estimated by consideration of a hypothetical time series of returns and assumption that changes of future data are based on historical changes that changes in past continue in future. Also in this approach inferences are not based on normality assumption. The other difference of this method with other methods is weighting. More clearly, in this method equal weights are assigned to each day in the time series, and a potential problem occurs if data have a trend, means less return in farther past and more return in earlier past. Changes in historical distribution of returns in future are also another challenging situation. Simplicity of historical simulation method raises its weaknesses. Most of approaches estimate *VaR* based on prior data, but historical method relies much more on what happened in past. Historical data are not always reliable. Therefore, past is not all. Also as mentioned earlier data may have a trend over time. Imagine situations that data are showing a moderate but stable increase as time goes on. In historical method data points are weighted equally though, earlier data have more effect on future, so should have more weights. Furthermore, historical approach is based on what we had in the past. Since for new assets no historical data are available, values at risk cannot be estimated. To overcome these weaknesses, new methods such as weighting recent past data more and combination of time series method with historical simulation can be used.

2.1.3 Monte Carlo simulation

This method is based on stronger assumptions about distribution of returns in comparison with historical simulation method. In this method probability distribution of returns should be specified. Once distribution is specified, many samples of returns are simulated and parameters are computed based on those samples. Difficulties of Monte Carlo simulation are in two levels. First, for portfolios having many assets, many probability distributions should be specified. Second, for each asset when its distribution is established, simulation should be done many times. Bulk of computations in Monte Carlo simulation method is its main weakness point. Thats why many users prefer historical simulation.

2.1.4 Comparing approaches

Each of these approaches has advantages and disadvantages. Variance-covariance method is good approach when the distribution of returns is normal. If this assumption is not held, this method may result in misleading values. However, when our data are gathered over a short time interval, a week for example, this method can be reliable. Historical simulation approach is good since no assumption is made for probability distribution of data and this method results in reliable values assuming distribution stability of returns over time. Monte Carlo method does its best in longer time periods, where historical data is not station and normality assumption is not held.

3 Proposed models in mean-value at risk framework

In this section, bases of Mean-VaR model and foundations to compute efficient frontier is provided. Our method is based on Rang Directional Measure (RDM) model proposed by Portela et al. [23] and multi objective optimization models. Mean-Value at Risk models try to maximize proportional reduction in VaR dimension, as a risk measure, while maximizing mean in the same proportion. That proportion is efficiency of under evaluation portfolio. This section summarizes essential definition of mean-VaR models and their framework. First of all some definitions are developed. Assume a portfolio is going to be selected from n financial assets. If we show proportion of invested money in asset j with λ_j, a portfolio is a vector proportions (λ) of each n assets. While no short sales are considered sum of proportions equals 1 ($\sum_{j=1}^{n}\lambda_j = 1$). It is also obvious that all proportions are equal or greater than zero for all $j \in \{1,\cdots,n\}$ and the set of our admissible portfolios is written as:

$$\phi = \{\lambda_j \in \mathbb{R}^n; \sum_{j=1}^{n}\lambda_j = 1, \lambda_j \geq 0\} \tag{15}$$

Return of portfolio, $r(\lambda)$, is defined as:

$$r(\lambda) = \sum_{j=1}^{n}\lambda_j r_j \tag{16}$$

Expected return of this portfolio is straightforwardly computed as:

$$E(r(\lambda)) = \sum_{j=1}^{n}\lambda_j E(r_j) \tag{17}$$

Three methods to compute Value at risk of an asset was mentioned in Section 3. Frameworks remain unchanged for portfolios. From any preferred method, VaR is calculated based on returns. To compute VaR, expected returns of under evaluation portfolio over a specified time interval should be gathered.

We define $f : \phi \to \mathbb{R}^2$ as:

$$f(\boldsymbol{\lambda}) = (E[r(\boldsymbol{\lambda})], VaR[r(\boldsymbol{\lambda})] \tag{18}$$

Which represents expected return and VaR of a given portfolio λ. Consequently, expected returns and VaRs provide a two-dimensional space and a point in R^2 space which is called a $MVaR$ point. Based on defined function disposable set is:

$$f(\phi) = \{f(\boldsymbol{\lambda}); \boldsymbol{\lambda} \in \phi\}. \tag{19}$$

Here same as mean variance model, in order to get a convex set, disposal region can be extended in the following way:

$$DR = f(\phi) + (\mathbb{R}_- \times \mathbb{R}_+). \tag{20}$$

Mean-VaR models evaluate efficiency through distance of MVaR points to the efficient frontier. Same as other models two types of frontiers exist.

Definition 1 Weakly efficient frontier also known as theoretical frontier defines as:

$$\Delta^w(\phi) = \{(\mu, VaR) \in DR; \\ (-\mu', VaR') < (-\mu, VaR) \Rightarrow (\mu', VaR') \notin DR\} \tag{21}$$

This frontier is a part of the boundary of the disposal region set. Also this disposal representation set is itself an extension of the mean-VaR region in order to make it convex (including imaginary portfolios). Consequently, the theoretical frontier can contain points that are not reachable by real portfolios. Naturally, strongly efficient frontier defines as follows.

Definition 2 Strongly efficient frontier defines as:

$$\Delta^s(\phi) = \{(\mu, VaR) \in DR; \\ (-\mu', VaR') \leq (-\mu, VaR) \text{ and } (-\mu', VaR') \neq (-\mu, VaR) \\ \Rightarrow (\mu', VaR') \notin DR\} \tag{22}$$

In definitions 1 and 2 μ and VaR are mean and Value at Risk (VaR) of a point in disposal region respectively. μ' and VaR' are also mean and value at risk of an arbitrary point in Mean-VaR space. Strongly efficient frontier contains all points that are dominated in two dimensional mean-VaR space. But in weakly efficient frontier a point on the frontier may dominated in at least one of two dimensions. Based on this definition, strongly efficient frontier is contained in the weakly efficient frontier (Figure 2).

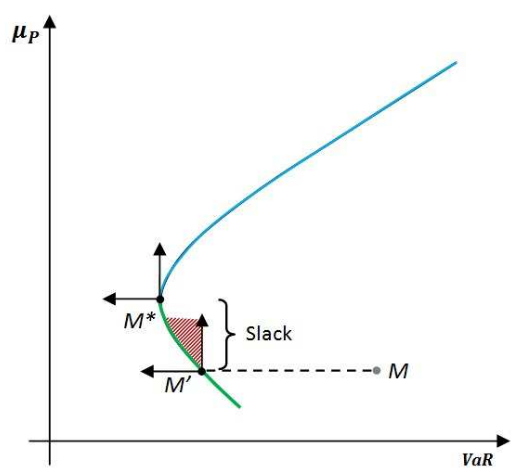

Fig. 2: Presentation of strongly and weakly efficient frontier. Green line shows weakly efficient frontier, while projection points on this part of frontier have positive slack variables. Light blue line represents strongly efficient frontier. Projection points on this part of frontier are not dominated in any dimension and all of slack variables are zero.

As can be seen in figure 2 slacks variables show minimum amount that should be added to or subtracted from a points inputs (mean) or outputs (value at risk) respectively, in order to transfers projection point to the first position where it is not dominated by any other points (point M*).

Definition 3 Based on model 10 provided by Portela et al. [23] we propose mean-VaR models can be written as mean-VaR efficiency (MVE) model and multi objective mean-VaR (MOMV) model. Let

$$\boldsymbol{g} = (R_{\mu_o}, R_{VaR_o}) \in R_+ \times R_- \ and \ R \neq 0$$

be a vector shows direction in which β is going to be maximized. MVE function defines as:

$$\xi : \mathbb{R}^2 \to (0,1],$$
$$\xi(y) = sup\{\beta; y + \beta g \in DR \| \beta \in \mathbb{R}_+\}.$$

Based on vector $\boldsymbol{g}$, definition and mentioned set of β, it is obvious that the aim is to simultaneously increase return and reduce Value at Risk of a portfolio in direction of vector $\boldsymbol{g}$. This function also cares about fundamental conditions of global optimization on non-convex sets. One should cares about directions in interpretation of models while directions affect result of MVE function. For instance proportional interpretation is suitable, if vector of direction is chosen as

$$\begin{aligned} g &= ((\max_j(\mu_j) - \mu_o), (VaR_o - \min(VaR_j))) \\ &= (R_{\mu_o}, R_{VaR_o}). \qquad j = 1, \cdots, n \end{aligned}$$

Definition 4 Let define g separately in each direction; i.e.,

$$g = (R_{\mu_o}, R_{VaR_o}) \in [0, +\infty) \times [0, +\infty). \tag{23}$$

For an under evaluation asset $y = (\mu_o, VaR_o)$ and a specified direction $g = (R_{\mu_o}, R_{VaR_o})$, based on model 10, the MVE function can be obtained through solving the following linear model:

$$\begin{aligned} &\max \ \beta \\ &s.t. \ \ E[r(\boldsymbol{\lambda})] \geq \mu_o + \beta R_{\mu_o}, \\ &\qquad VaR[r(\boldsymbol{\lambda})] \leq VaR_o + \beta R_{VaR_o}, \\ &\qquad \textstyle\sum_{j=1}^n \lambda_j = 1, \\ &\qquad \beta \geq 0, \ 0 \leq \lambda_j \leq 1 \ for \ j \in \{1, \cdots, n\}. \end{aligned} \tag{24}$$

Computation on MVE function is done based on RDM models. When this MVE function equals zero, mean-VaR point is on the weakly efficient frontier. Otherwise, $0 < \beta < 1$ indicates that mean and Value at Risk of an asset should be changed in order to result in an efficient point on the efficient frontier (amount of inefficiency). On the other hand, $1 - \beta$ is amount of efficiency. As mentioned earlier strongly efficient frontier is part of weakly efficient frontier. In such situations in order to find out projected point on which frontier is, slacks and surpluses variables are useful. Existence of slack or surplus variables in an optimum point shows that MVE function resulted in a point on the weakly efficient frontier. Means efficiency measure of under evaluation point is biased. This bias underestimates gains in return (mean) and reductions in risk (Value at Risk). This is the way that is used to distinguish between weakly efficient frontier and efficient frontier which is attainable in practice.

By using multi objective functions, the following function known as multi objective mean-VaR (MOMV) model, can be defined.

Definition 5 MOMV function in direction of vector $\boldsymbol{g}$ is defined as:

$$MF : \mathbb{R}^2 \to (0,1]$$
$$MF(y) = \sup\{\tfrac{1}{2}\textstyle\sum_i \beta_i; \mu + \beta \boldsymbol{g} \in DR\}.$$

This function tries to maximize β in directions of mean and VaR separately. Because of having more than one parameter to maximize, based on rules of optimization of multi objective functions, average of objects is tried to be maximized. Note that $\boldsymbol{\beta}$ and $\boldsymbol{g}$ are both vectors. This function evaluates arithmetic average proportional changes in each direction, which makes interpretations more complicated. Also note that MVE function might project an under evaluation asset on weakly efficient frontier while MOMV function surly projects on strongly efficient frontier.

MOMV function is computed through following model. Consider a vector $\boldsymbol{g} = (R_{\mu_o}, R_{VaR_o})$ and an under evaluation asset represented by $y = (\mu_o, VaR_o)$ in Mean-VaR space.

$$\begin{aligned} &\max \ \tfrac{1}{2}(\beta_1 + \beta_2) \\ &s.t. \ \ E[r(\boldsymbol{\lambda})] \geq \mu_o + \beta_1 R_{\mu_o}, \\ &\qquad VaR[r(\lambda)] \leq VaR_o + \beta_2 R_{VaR_o}, \\ &\qquad \textstyle\sum_{j=1}^n \lambda_j = 1, \\ &\qquad \beta_1, \beta_2 \geq 0, \ 0 \leq \lambda_j \leq 1, \ for \ j \in \{1, \cdots, n\}. \end{aligned} \tag{25}$$

If this model equals zero, in contrast with MVE model, under evaluation point is on the strongly efficient frontier. On the other hand if β_i doesnt equals zero, each β_i shows proportional changes in mean and value at risk respectfully. As a result because projection in each direction is independent of other directions, projected points are for sure on the efficient frontier. Because of this flexibility, MOMV function always results in projection points having zero slacks or surpluses. As a consequence, by this model, the weakly and strongly efficient frontiers always coincide. Also, as can be seen, using MOMV model leads to clustered projection points. This clustering occurs while MOMV model is a more flexible model than MVE model in determination of optimal directions. It is well-known that the multi objective models (like MOMV model) always result in larger or equal optimal values than single objective models (like MVE model). Therefore, MOMV model efficiencies are always less than or equal to the MVE model efficiencies. Note that in special cases optimization can be done in one direction.

For example vector $\boldsymbol{g}$ in direction of mean R_o can be set to zero. So optimization is done by minimizing risk. Same way can be used to maximize return.

4 Application in Iranian stock companies

In this section a comparison study is conducted to compare methods introduced in previous sections. To do this a sample of 20 corporations from Tehran stock is randomly selected. Each of these corporations can be considered as a portfolio. Returns of these assets over 62 days have been gathered [1]. Also missing data over holidays estimated through spline interpolation method. Efficiency of each asset is going to be evaluated and methods of computing efficiencies compared. Table 1 reports expected return and also value at risk of each asset (columns 3-9). In this table value at risks are calculated by using historical and Monte Carlo simulation methods. As mentioned previously, in order to calculate VaR by using Monte Carlo method, distribution of assets must be known. Here because of mathematical difficulties to obtain distributions, sampling methods are being used instead. By consideration of a specific margin of error, number of required samples to present the whole population for each asset is defined and sample sizes are reported in table 1 column 2. Now by using boot strapping method, we have repeated sampling scheme for many times, for example 1000, and value at risk of each sample is calculated. Now average of these 1000 VaRs is unbiased estimate of population VaR.

$$\bar{VaR} = \frac{1}{1000}\sum_{j=1}^{1000} VaR_j \tag{26}$$

Where VaR_j is Value at Risk of sample j and $\bar{VaR}$ is unbiased estimate of populations Value at Risk.

In table 2 directions, which are used in MVE and MOMV models, are provided. Note that, DEA model with negative data should be used, since expected returns might be negative. Therefore, directions are calculated by using equations 11 and 12.

Now based on Values at Risk in table 1 and directions in table 2, and using model 24, efficiency of each asset is calculated. Results are presented in table 3. As mentioned earlier, β shows amount of inefficiency. So, an asset is efficient unless β equals zero. Based on data in table 3, asset 10 in all levels of historical and Monet Carlo simulation is efficient. However, asset 1 in higher levels of Historical simulation and all levels of Monte Carlo simulation is efficient. For asset 1 it can be interpreted, as the confidence level of risk increases, it gets efficient. This shows that asset 1 is suitable for investors, who desire to invest with higher confidence on amount of risk that they may face.

[1] Time period is from 21 April to 21 June 2014.

Same data is used and efficiency of assets is calculated by using MOMV model (25). Results are provided in table 4. Table 4 reports values of inefficiency (β). Interpretations are same as before. Based on MOMV model asset 10 is the efficient asset. Also same as MVE model in higher levels of confidence, asset 1 gets efficient. Also by comparing results of table 3 and table 4, it can be concluded that results of MOMV model are generally greater than results of MVE model. It is a general characteristic of multi objective models. However, general conclusion and results obtained from MVE and MOMV models do not change. As the last result, table 5 presents average of β for both MVE and MOMV model for different methods of obtaining Value at Risk. Table 5 shows that average of β from MOMV model, in all confidence levels, is higher than average of β from MVE model. Also, it can be inferred as the confidence level increases value of β grows.

5 Geometric representations

This section goes through visualization of Mean-Value at Risk region in order to show efficient frontier and position of under evaluation assets. In first step 10000 portfolios made from mentioned assets, are shown in figure 3 in three different confidence levels. In this figure portfolios are made using values at risk calculated through historical simulations. Note that since portfolios are made based on normally distributed weights,

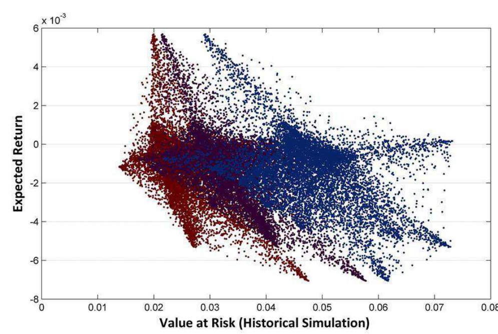

Fig. 3: This figure shows disposable region made by 10000 portfolios from assets of table 1. In this figure value at risks are calculated from historical simulation.

Figure 3 illustrates as the risk's confidence level, increases the whole feasible region moves rightward. Therefore, as the confidence level increases, investors get surer about the amount of risk that may face on a predefined level of return. In fact, in higher levels of α, risk of an under evaluation asset is calculated more preciously. Same conclusions can be made for portfolios made by values at risk calculated through Monte Carlo

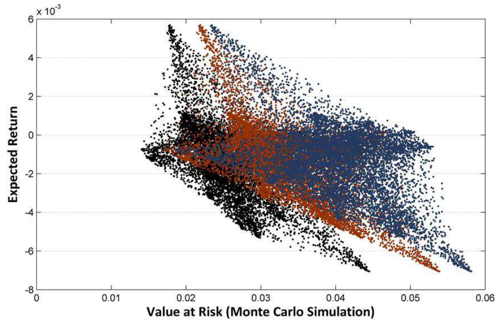

Fig. 4: This figure shows disposable region made by 10000 portfolios from assets of table 1. In this figure value at risks are calculated from Monte Carlo simulation.

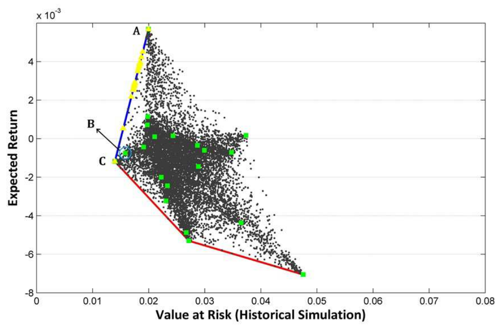

Fig. 5: Under evaluation assets (light green dots) and their projection points (yellow dots) on efficient frontier for 90% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection point and under evaluation point of asset 10 coincides. Therefore, asset 10 is completely efficient. Also asset 16 (point C) is efficient at 90% confidence risk level. Efficiencies are obtained by solving MVE model.

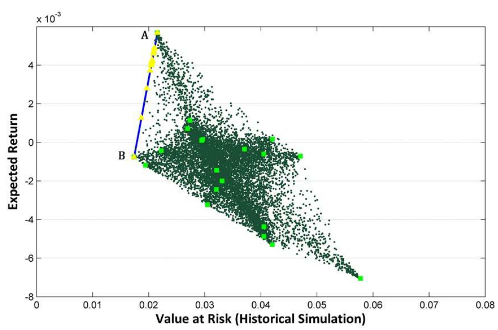

Fig. 6: Under evaluation assets (light green dots) and their projection points (yellow dots) on efficient frontier for 95% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVE model.

simulation (Figure 4). They are mostly in the middle of disposable area. However, by using weights which are uniformly distributed, portfolios will disperse uniformly.

Now each asset can be considered as a portfolio and its performance can be evaluated. Figures 5-10 show assets position in Mean-VaR region among all 10000 portfolios. In these figures projection of each asset is also shown. These projection points are obtained via MVE model. In all of these figures it can be seen that asset 10 is on the efficient frontier and its projection and corresponding point are equivalent (point A in figures 5-10). This is also true for asset 1. As the risks confidence level increases, this asset gets efficient. On the other hand, asset 16 (point C in figure 5), which is efficient at 90% confidence risk level of historical method, in higher levels of confidence becomes inefficient. Also, in all of figures it can be seen that all of projection points are on the blue straight line. This line represents strongly efficient frontier. In figures 5 red line shows weakly efficient frontier.

Same conclusions can be made for assets evaluated by values at risk computed from Monet Carlo method. In Figures 11-16, projection points which are obtained from MVMO model are shown. In these figures corresponding points of under evaluation assets are projected mainly on two points. Point A represents asset 10 and point B represents asset 1.

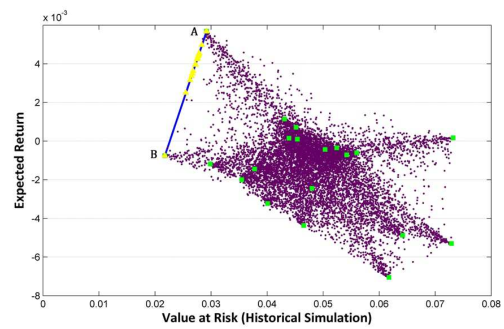

Fig. 7: Under evaluation assets (light green dots) and their projection points (yellow dots) on efficient frontier for 99% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVE model.

Same as before, conclusions for Monte Carlo simulation are like as historical simulation. Figures 14-16 are made according to values at risk computed by Monte Carlo simulation. In fact, since they are efficient and β_is are also defined separately in each direction, these two points are acting like an index for other points. Therefore, points are projected in a clustered way.

Table 1: Original data of under evaluation assets

Asset number	Number of sample	Mean	Original Data – Value at risk – Historical sim. 90%	95%	99%	Monte Carlo sim. 90%	95%	99%
1	17	-0.00075	0.0158	0.0174	0.0217	0.0141	0.0172	0.0186
2	20	-0.00035	0.0287	0.0371	0.0524	0.0299	0.0375	0.0426
3	22	-0.00487	0.0267	0.0405	0.0642	0.0276	0.0408	0.0509
4	23	-0.00071	0.0348	0.0471	0.0542	0.0344	0.0439	0.0492
5	11	-0.00245	0.0234	0.0320	0.0480	0.0249	0.0306	0.0311
6	54	0.00072	0.0198	0.0269	0.0452	0.0194	0.0275	0.0448
7	60	-0.00323	0.0231	0.0305	0.0401	0.0231	0.0304	0.0400
8	49	0.00015	0.0243	0.0296	0.0439	0.0249	0.0295	0.0429
9	15	0.00017	0.0373	0.0421	0.0732	0.0357	0.0468	0.0505
10	16	0.00568	0.0200	0.0216	0.0292	0.0178	0.0219	0.0235
11	16	-0.00530	0.0272	0.0420	0.0729	0.0297	0.0436	0.0492
12	38	-0.00060	0.0299	0.0405	0.0560	0.0316	0.0408	0.0530
13	27	-0.00043	0.0191	0.0223	0.0503	0.0188	0.0258	0.0382
14	21	-0.00705	0.0476	0.0578	0.0617	0.0445	0.0539	0.0582
15	15	-0.00437	0.0365	0.0406	0.0465	0.0333	0.0388	0.0406
16	22	-0.00118	0.0140	0.0194	0.0299	0.0149	0.0191	0.0231
17	26	-0.00200	0.0223	0.0331	0.0355	0.0231	0.0302	0.0338
18	19	0.00010	0.0211	0.0294	0.0454	0.0216	0.0293	0.0339
19	34	0.00114	0.0198	0.0274	0.0431	0.0199	0.0261	0.0374
20	18	-0.00144	0.0289	0.0321	0.0378	0.0274	0.0315	0.0333

Table 2: Directions used to project points on the efficient frontier

Asset number	Directions – Max Mean-Mean	VaR-Min VaR – Historical sim. 90%	95%	99%	Monte Carlo sim. 90%	95%	99%
1	0.0064	0.0019	0	0	0	0	0
2	0.0060	0.0147	0.0197	0.0307	0.0158	0.0203	0.024
3	0.0106	0.0127	0.0231	0.0425	0.0135	0.0236	0.0324
4	0.0064	0.0208	0.0297	0.0325	0.0202	0.0267	0.0307
5	0.0081	0.0094	0.0146	0.0263	0.0107	0.0134	0.0126
6	0.0050	0.0058	0.0095	0.0234	0.0053	0.0103	0.0262
7	0.0089	0.0091	0.0131	0.0183	0.0090	0.0132	0.0214
8	0.0055	0.0104	0.0122	0.0222	0.0107	0.0123	0.0243
9	0.0055	0.0234	0.0247	0.0514	0.0216	0.0296	0.032
10	0	0.0060	0.0042	0.0074	0.0037	0.0047	0.0049
11	0.0110	0.0132	0.0246	0.0512	0.0156	0.0264	0.0306
12	0.0063	0.0160	0.0231	0.0342	0.0174	0.0236	0.0345
13	0.0061	0.0051	0.0049	0.0286	0.0047	0.0086	0.0196
14	0.0127	0.0336	0.0404	0.0400	0.0304	0.0367	0.0396
15	0.0101	0.0225	0.0232	0.0248	0.0191	0.0216	0.0221
16	0.0069	0.0000	0.0020	0.0081	0.0007	0.0019	0.0045
17	0.0077	0.0083	0.0157	0.0137	0.0089	0.013	0.0152
18	0.0056	0.0071	0.0120	0.0237	0.0075	0.0121	0.0154
19	0.0045	0.0059	0.0100	0.0214	0.0058	0.0088	0.0189
20	0.0071	0.0149	0.0147	0.016	0.0132	0.0143	0.0148

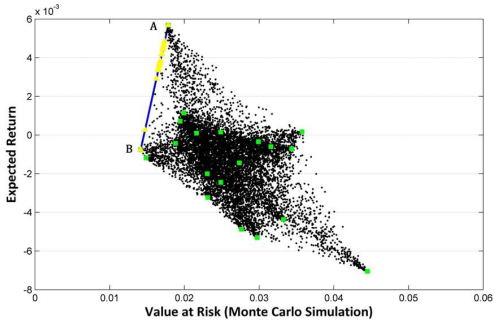

Fig. 8: Under evaluation assets (light green dots) and their projection points (yellow dots) on efficient frontier for 90% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, in contrast with historical simulation in 90% confidence level, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVE model.

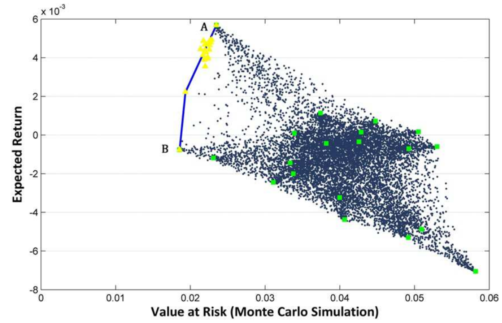

Fig. 10: Under evaluation assets (light green dots) and their projection points (yellow dots) on efficient frontier for 99% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVE model.

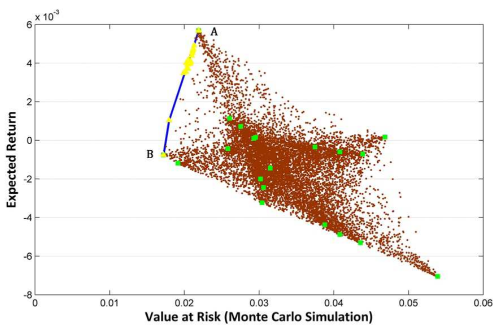

Fig. 9: Under evaluation assets (light green dots) and their projection points (yellow dots) on efficient frontier for 95% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVE model.

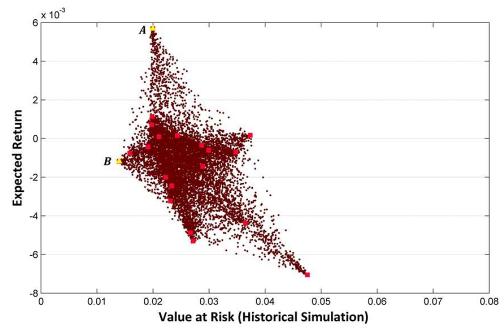

Fig. 11: Under evaluation assets (light red dots) and their projection points (yellow dots) on efficient frontier for 90% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVMO model.

In conclusion, efficiency measures calculated via MVE model in comparison with MOMV model are smaller in different confidence levels. Based on MOMV model, efficient company on a 90% confidence level is asset number 10 for both historical and Monte Carlo simulation methods. Asset 10 is RayanSaipa Co. and asset 1 is Mellat Bank. Also, on all confidence levels all of efficiencies calculated by Monte Carlo method are smaller than efficiencies calculated by historical method. These results show that Monte Carlo simulation method is much more accurate than historical method.

6 Conclusions

This paper introduced a measure for portfolio performance evaluation using mean-value at risk efficiency (MVE) model and multi objective mean-value at risk (MOMV) model. Morey and Morey [21], Joro and Na [13] and Kerstence et al. [15] proposed models for evaluating portfolios efficiency in which DEA model was employed. In these models a non-linear DEA-type framework was used in which, the correlation structure among the units was taken into account. In these models variance was considered as a risk measure. However, both theories and practices indicate that variance is not a good risk measure and has disadvantages. In this paper we introduced two models for portfolio optimization problem, where, most literature only consider the

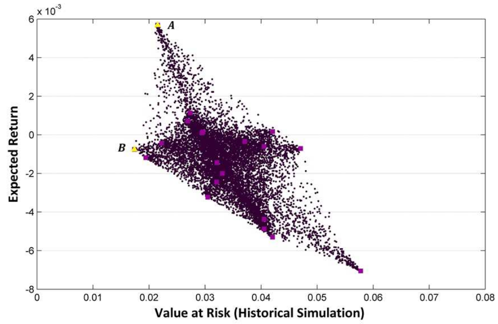

Fig. 12: Under evaluation assets (purple dots) and their projection points (yellow dots) on efficient frontier for 95% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVMO model.

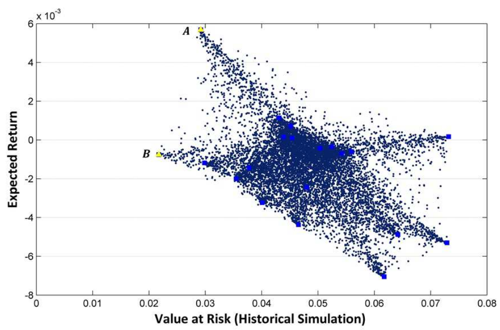

Fig. 13: Under evaluation assets (light blue dots) and their projection points (yellow dots) on efficient frontier for 99% confidence level. Point A represents asset 10 and point B stands for asset 1. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVMO model.

computational formulas to measure risk by *VaR*. In these models mean was considered as output and *VaR* was considered as input. Since mean might take negative values, conventional DEA method is not appropriate to solve these models. So MVE and MOMV models are developed based on Range Directional Measure (RDM) model in order to take negative values as outputs or inputs. Methods to calculate VaR are also briefly reviewed and advantages and disadvantages of each method are also mentioned. Finally, MVE and MOMV models are applied to data from 20 Companies from Tehran Stock. In addition, a numerical example based on historical and Monte Carlo simulation for calculating value at risk with different confidence levels are presented to demonstrate mentioned models. The detailed results

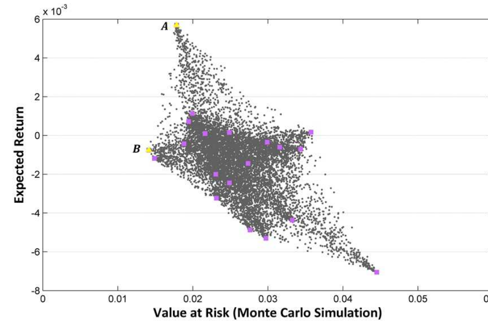

Fig. 14: Under evaluation assets (pinkish purple dots) and their projection points (yellow dots) on efficient frontier for 90% confidence level. Point A represents asset 10 and point B stands for asset 1. Using MOMV function, because of having flexibility in determination of optimization directions, leads to a higher amount of clustered projection points. As can be seen, all of points are projected on points A or B. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVMO model.

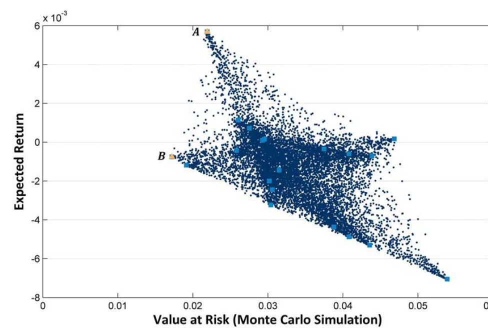

Fig. 15: Under evaluation assets (light blue dots) and their projection points (orange dots) on efficient frontier for 95% confidence level. Point A represents asset 10 and point B stands for asset 1. Using MOMV function, because of having flexibility in determination of optimization directions, leads to a higher amount of clustered projection points. As can be seen all of points are projected on points A or B. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVMO model.

are presented in tables 2 to 5. Results show that Monte Carlo simulation method is much more accurate than historical method. We can also use MVE and MOVM models to study other mean-risk portfolio optimization problems, where the risk is obtained by other measures, such as CVaR, Semi-variance, and so on. These topics can be considered for future studies.

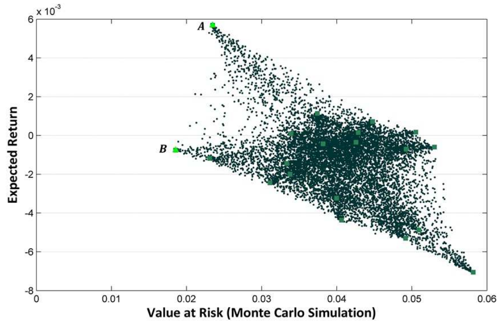

Fig. 16: Under evaluation assets (green dots) and their projection points (light green dots) on efficient frontier for 99% confidence level. Point A represents asset 10 and point B stands for asset 1. Using MOMV function, because of having flexibility in determination of optimization directions, leads to a higher amount of clustered projection points. As can be seen all of points are projected on points A or B. Figure shows that projection points and under evaluation points of assets 10 and 1 coincide. Therefore, assets 1 and 10 are completely efficient. Efficiencies are obtained by solving MVMO model.

Table 3: Efficiencies obtained by solving MVE model

Asset number	Efficiencies-MVE function					
	Historical sim.			Monte Carlo sim.		
	90%	95%	99%	90%	95%	99%
1	0.2	0	0	0	0	0
2	0.7	0.82	0.8	0.81	0.81	0.83
3	0.73	0.86	0.86	0.81	0.85	0.88
4	0.77	0.88	0.81	0.85	0.85	0.86
5	0.64	0.79	0.79	0.76	0.76	0.74
6	0.41	0.67	0.74	0.55	0.66	0.84
7	0.65	0.78	0.74	0.74	0.76	0.83
8	0.61	0.74	0.74	0.73	0.71	0.83
9	0.79	0.85	0.87	0.85	0.86	0.86
10	0	0	0	0	0	0
11	0.74	0.87	0.88	0.83	0.86	0.87
12	0.72	0.85	0.82	0.82	0.83	0.87
13	0.43	0.53	0.79	0.55	0.64	0.8
14	0.87	0.91	0.86	0.9	0.9	0.9
15	0.81	0.86	0.8	0.85	0.84	0.83
16	0	0.36	0.54	0.21	0.32	0.5
17	0.6	0.8	0.67	0.72	0.75	0.77
18	0.5	0.73	0.75	0.65	0.71	0.75
19	0.39	0.68	0.72	0.56	0.61	0.78
20	0.72	0.78	0.69	0.79	0.76	0.76

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Table 4: Efficiencies obtained by solving MOMV model

Asset number	Efficiencies-MOMV function					
	Historical sim.			Monte Carlo sim.		
	90%	95%	99%	90%	95%	99%
1	0.47	0	0	0	0	0
2	0.8	0.89	0.88	0.88	0.88	0.9
3	0.76	0.91	0.91	0.86	0.9	0.92
4	0.86	0.93	0.89	0.91	0.91	0.92
5	0.68	0.86	0.86	0.83	0.82	0.8
6	0.48	0.78	0.84	0.65	0.77	0.91
7	0.67	0.84	0.8	0.79	0.82	0.89
8	0.71	0.83	0.83	0.83	0.81	0.9
9	0.87	0.92	0.93	0.91	0.92	0.92
10	0	0	0	0	0	0
11	0.77	0.92	0.93	0.88	0.91	0.92
12	0.81	0.91	0.89	0.89	0.9	0.93
13	0.44	0.57	0.87	0.6	0.73	0.87
14	0.91	0.95	0.91	0.94	0.94	0.94
15	0.87	0.91	0.85	0.9	0.89	0.89
16	0	0.53	0.54	0.53	0.53	0.53
17	0.64	0.87	0.73	0.79	0.82	0.84
18	0.58	0.83	0.84	0.75	0.81	0.84
19	0.49	0.79	0.83	0.68	0.73	0.87
20	0.8	0.86	0.77	0.86	0.84	0.83

Table 5: Camparison of Singel and Multi objective outputs

Model	**Historical Simulation**			**Monte Carlo Simulation**		
	%90	%95	%99	%90	%95	%99
Single Objective	0.56	0.69	0.69	0.65	0.67	0.73
Multi Objective	0.63	0.76	0.76	0.72	0.75	0.78

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