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The Consistent Value: Potentializability and Equivalent Relations on Fuzzy Games

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Abstract: Based on several equivalent relations, this paper is aimed to axiomatize the family of all solutions that admit a potential in the framework of fuzzy transferable-utility (TU) games. Further, several axiomatic results of the consistent value are also proposed by applying these equivalent relations.

Keywords: Fuzzy TU games, potential, the consistent value

1 Introduction

The potential approach is a useful notion in various spheres. For example, a vector field H is said to be “conservative” if there exists a differentiable mapping h such that H is the gradient of h. The mapping h is said to be the potential function for H. Many vector fields, including electric force fields and gravitational fields, are conservative. The notion “conservative” is generalized from the classic physical result regarding the preservation of energy. This result asserts that the entirety of the kinetic energy and the potential energy of a mote moving over a conservative vector field is constant. In the framework of traditional transferable-utility (TU) games, Hart and Mas-Colell [11] initially introduced the potential approach to axiomatize the Shapley value [25]. Subsequently, Ortmann [22,23] and Calvo and Santos [10] demonstrated several equivalent relations to axiomatize the family of all traditional TU solutions that admit a potential.

The theory of fuzzy TU games commenced with the investigation of Aubin [1,2] where the opinions of a fuzzy TU game and the fuzzy core are introduced. In the framework of fuzzy TU games, many solution concepts have been applied wildly. For example, Aubin [1,2], Butnariu [7], Hwang [12], Hwang and Liao [13] and Tijs et al. [27] investigated the extended cores of fuzzy TU games; Branzei et al. [4], Butnariu and Klement [8], Butnariu and Kroupa [9], Hwang and Liao [14], Li and Zhang [15,16] and Tsurumi et al. [28] investigated the extended Shapley values [25] of fuzzy TU games; Branzei et al. [3,5], Muto et al. [20] and Tijs et al. [27] investigated the extended compromise values, the extended stable sets, the extended Weber sets of fuzzy TU games; and fuzzy and multiobjective games were also analyzed by Nishizaki and Sakawa [21]. Related results also can be found in Branzei et al. [6], Molina and Tejada [18], Sakawa and Nishizaki [24] and so on. By both considering the players and their activity levels, Hwang and Liao [14] proposed the *consistent value* which is a fuzzy extension of the Shapley value [25]. Hwang and Liao [14] also showed that there exists a unique potential and the resulting payoff vector coincides with the consistent value.

In the framework of fuzzy TU games, we build on the results proposed by Calvo and Santos [10], Ortmann [22,23] and Hwang and Liao [14].

1. In Section 3, we axiomatize the family of all fuzzy solutions that admit a potential, and show that any solution that admits a potential turns out to be the consistent value of a specific game. After applying the property of *independence of individual expansions*, we provide several equivalent relations among the potentializability of a solution, the properties of *balanced contributions* and *full-action path independence*.
2. In Section 3, we axiomatize the consistent value by means of these equivalent relations.

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2 Preliminaries

Let U be the universe of agents. For $i \in U$ and $b_i \in [0,1]$, we set $B_i = [0,b_i]$ to be the activity level space of agent i, where level 0 denotes no participation. Let $B^N = \prod_{i\in N} B_i$ be the product set of the activity level spaces for agents in N. For all $T \subseteq N$, a agent-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $\theta^T \in B^N$, which is the vector with $\theta^T_i = 1$ if $i \in T$, and $\theta^T_i = 0$ if $i \in N \setminus T$. In $\mathbb{R}^N$, we denote 0_N to be the zero vector.

A **fuzzy TU game**[1] is a triple (N,b,v), where $N \neq \emptyset$ is a finite set of agents, $b = (b_i)_{i\in N}$ is the vector that presents the highest activity level for each agent, and $v : B^N \to \mathbb{R}$ is a characteristic mapping with $v(0_N) = 0$ which distributes to each activity level vector $\alpha = (\alpha_i)_{i\in N} \in B^N$ the value that the agents can gain when each agent i participates at activity level α_i. A game (N,b,v) will sometimes be denoted by its characteristic mapping v if no confusion can occur. Denote the class of all fuzzy TU games by Γ. Given $(N,b,v) \in \Gamma$ and $\alpha \in B^N$, we write (N,α,v) for the **fuzzy TU subgame** defined by restricting v to $\{\beta \in B^N \mid \beta_i \leq \alpha_i \text{ for all } i \in N\}$.

Given $(N,b,v) \in \Gamma$, let $K^{N,b} = \{(i,k_i) \mid i \in N, k_i \in B^+_i\}$, where $B^+_i = (0,b_i]$. A **solution** on Γ is a map ψ distributing to each $(N,b,v) \in \Gamma$ an element

$$\psi(N,b,v) = \big(\psi_{i,k_i}(N,b,v)\big)_{(i,k_i)\in K^{N,b}} \in \mathbb{R}^{K^{N,b}}.$$

Here $\psi_{i,k_i}(N,b,v)$ is the value of the agent i when he takes activity level k_i to participate game v. For convenience, we define $\psi_{i,0}(N,b,v) = 0$ for each $i \in N$.

Let $N \subseteq U$, $S \subseteq N$, $i \in N$ and $\alpha \in \mathbb{R}^N$. We write α_S to be the restriction of α at S. For convenience we introduce the notation α_{-i} to represent $\alpha_{N\setminus\{i\}}$ and let $\beta = (\alpha_{-i},k_i) \in \mathbb{R}^N$ be defined by $\beta_{-i} = \alpha_{-i}$ and $\beta_i = k_i$. Furthermore, let $j \in N \setminus \{i\}$, α_{-ij} to represent $\alpha_{N\setminus\{i,j\}}$ and (α_{-ij},k_i,k_j) to represent $((\alpha_{-i},k_i)_{-j},k_j)$.

Hwang and Liao [14] provided a generalization of the Shapley value of fuzzy TU games as follows.

Definition 1. *The* **consistent value** *of fuzzy TU games, γ, is the function on Γ which associates with each $(N,b,v) \in \Gamma$, each agent $i \in N$ and each $k_i \in B^+_i$ the value*[2]

$$\begin{aligned}&\gamma_{i,k_i}(N,b,v)\\ =&\sum_{\substack{S\subseteq N\\ i\in S}} \tfrac{(|S|-1)!(|N|-|S|)!}{|N|!}\cdot\Big[v\big((b_{-i},k_i)_S,0_{N\setminus S}\big)-v\big((b_{-i},0)_S,0_{N\setminus S}\big)\Big].\end{aligned}$$

[1] A fuzzy TU game, originally introduced by Aubin [1,2], is a pair (N,v^*), where v^* is a mapping such that $v^* : [0,1]^N \longrightarrow \mathbb{R}$ and $v^*(0_N) = 0$. In fact, $(N,v^*) = (N,\theta^N,v)$.

[2] Without loss of generality, we can assume that $b_i > 0$ for all $i \in N$.

3 The equivalence theorem and axiomatizations

In this section, we provide several equivalent relations to axiomatize the family of all fuzzy solutions that admit a potential.

Given a function $P : \Gamma \longrightarrow \mathbb{R}$ which associates a number $P(N,b,v) \in \mathbb{R}$ to each $(N,b,v) \in \Gamma$. For each $(i,k_i) \in K^{N,b}$, Hwang and Liao [14] defined that

$$D^{i,k_i}P(N,b,v) = P\big(N,(b_{-i},k_i),v\big) - P\big(N,(b_{-i},0),v\big).$$

Definition 2.(Hwang and Liao [14]) *A solution ψ on Γ admits a* **potential** *if there exists a function $P : \Gamma \to \mathbb{R}$ satisfies for all $(N,b,v) \in \Gamma$ and for all $(i,k_i) \in K^{N,b}$, $\psi_{i,k_i}(N,b,v) = D^{i,k_i}P(N,b,v)$.*

A function $P : \Gamma \longrightarrow \mathbb{R}$ is said to be *0-normalized* if $P(N,0_N,v) = 0$ for each $N \subseteq U$. And P is said to be *efficient* if for all $(N,b,v) \in \Gamma$,

$$\sum_{i\in N} D^{i,b_i}P(N,b,v) = v(b).$$

The existence of a potential and a 0-normalized potential are equivalent, since the function

$$P^0(N,b,v) = P(N,b,v) - P(N,0_N,v)$$

is a 0-normalized potential if P is a potential. Furthermore, a solution ψ on Γ admits one 0-normalized potential at most.

Remark. Hwang and Liao [14] showed that a solution ψ on Γ admits a uniquely 0-normalized and efficient potential P if and only if ψ is the consistent value γ on Γ. For all $(N,b,v) \in \Gamma$ and for all $(i,k_i) \in K^{N,b}$

$$\gamma_{i,k_i}(N,b,v) = D^{i,k_i}P(N,b,v).$$

To state the equivalence theorem, some more definitions will be needed.

Definition 3. *Let ψ be a solution on Γ.*

- **Efficiency (EFF):** *For all $(N,b,v) \in \Gamma$, $\sum_{i\in N} \psi_{i,b_i}(N,b,v) = v(b)$.*
- **Balanced contributions (BC):** *For all $(N,b,v) \in \Gamma$ and for all $(i,k_i),(j,k_j) \in K^{N,b}, i \neq j$,*

$$\begin{aligned}&\psi_{i,k_i}\big(N,(b_{-j},k_j),v\big) - \psi_{i,k_i}\big(N,(b_{-j},0),v\big)\\ =&\psi_{j,k_j}\big(N,(b_{-i},k_i),v\big) - \psi_{j,k_j}\big(N,(b_{-i},0),v\big).\end{aligned}$$

- **Independence of individual expansions (IIE)** *if for all $(N,b,v) \in \Gamma$ and for all $(i,k_i) \in K^{N,b}$, $k_i \neq b_i$,*

$$\psi_{i,k_i}\big(N,(b_{-i},k_i),v\big) = \psi_{i,k_i}\big(N,(b_{-i},k_i+1),v\big) = \cdots = \psi_{i,k_i}(N,b,v).$$

Inspired by Myerson [19], Hwang and Liao [14] extended the property of balanced contributions to fuzzy games. For any two agents i,j and their activity levels k_i,k_j, the payoff for the activity level k_i of agent i will occur difference when agent j takes the activity level k_j and agent j vanishes from the game. Vice versa, the payoff for the activity level k_j of agent j will occur difference when agent i takes the activity level k_i and agent i vanishes from the game. What BC asserts is that this two differences will be coincident. IIE asserts that whenever a agent gets available higher activity level the payoff for all original activity levels is not varied under condition that other agents are immobile.

Some considerable weakenings of the previous properties are as follows. **Weak efficiency (WEFF)** simply asserts that for all $(N,b,v)\in\Gamma$ with $|N|=1$, ψ satisfies EFF. **Upper balanced contributions (UBC)** only requires that BC holds if $k_i=b_i$ and $k_j=b_j$. **Weak independence of individual expansions (WIIE)** simply asserts that for all $(N,b,v)\in\Gamma$ with $|N|=1$, ψ satisfies IIE.

*Remark.*Hwang and Liao [14] axiomatized the consistent value γ by means of EFF, BC and IIE.

In the framework of traditional TU games, Calvo and Santos [10] showed that any traditional TU solution that admits a potential turns out to be the Shapley value of an auxiliary game. A fuzzy extension of an auxiliary game is defined as follows.

Definition 4.*Given a solution ψ on Γ and a game $(N,b,v)\in\Gamma$, the* **auxiliary fuzzy game** (N,b,v_ψ) *is defined as follows. For all $\alpha\in B^N$,*

$$v_\psi(\alpha)=\sum_{i\in N}\psi_{i,\alpha_i}(N,\alpha,v).$$

Note that if ψ satisfies efficiency then $v=v_\psi$.

In the framework of TU games, Ortmann [22] proposed the property of path independence to axiomatize the family of all traditional TU solutions that admit a potential. In order to propose an analogous result in fuzzy games, some more definitions and notations are needed.

An **full-action order** for $(N,b,v)\in\Gamma$ is a bijection $\sigma: N\to N$. The amount of all full-action orders for (N,b,v) is $|N|!$. Let σ,σ' be two full-action orders for (N,b,v), we say that σ' is a **transposition** of σ if there exist $i,j\in N$ with $i\neq j$ and $\sigma(j)=\sigma(i)+1$, such that $\sigma'(i)=\sigma(j)$, $\sigma'(j)=\sigma(i)$ and $\sigma'(p)=\sigma(p)$ for all $p\in N\setminus\{i,j\}$. It is easy to see that each full-action order can be transformed to another full-action order by means of transpositions.

Let σ be an full-action order. The activity level vector that is present after the t-th agent according to σ, denoted by $s^{\sigma,t}$, is defined by

$$s_i^{\sigma,t}=\begin{cases} b_i \text{ if } \sigma(i)\le t\,;\\ 0 \ \text{ otherwise,}\end{cases}$$

for all $i\in N$.

Definition 5.*Let ψ be a solution on Γ. We say that ψ satisfies* **full-action path independence (FAPI)** *if*

$$\sum_{i\in N}\psi_{i,b_i}(N,s^{\sigma,\sigma(i)},v)=\sum_{i\in N}\psi_{i,b_i}(N,s^{\sigma',\sigma'(i)},v),$$

for all $(N,b,v)\in\Gamma$ and for all orders σ,σ'.

Subsequently, we show that the property UBC coincides with the property FAPI.

Lemma 1.*A solution ψ on Γ satisfies FAPI if and only if ψ satisfies UBC.*

*Proof.*Let $(N,b,v)\in\Gamma$. It is trivial if $|N|=1$. Assume that $|N|\geq 2$. First we show that if ψ satisfies FAPI, then it satisfies UBC. Let $i,j\in N$. Let σ_i and σ_j be two full-action orders with $\sigma_i(i)=\sigma_j(j)=|N|$, $\sigma_i(j)=\sigma_j(i)=|N|-1$ and $\sigma_i(p)=\sigma_j(p)$ for all $p\notin\{i,j\}$. Since ψ satisfies FAPI,

$$\begin{aligned}0=&\sum_{p\in N}\psi_{p,b_p}(N,s^{\sigma_i,\sigma_i(p)},v)-\sum_{p\in N}\psi_{p,b_p}(N,s^{\sigma_j,\sigma_j(p)},v)\\ =&\,\psi_{j,b_j}(N,(b_{-i},0),v)+\psi_{i,b_i}(N,b,v)-\psi_{i,b_i}(N,(b_{-j},0),v)\\ &-\psi_{j,b_j}(N,b,v).\end{aligned}$$

Therefore,

$$\begin{aligned}&\psi_{i,b_i}(N,b,v)-\psi_{j,b_j}(N,b,v)\\ =&\,\psi_{i,b_i}(N,(b_{-j},0),v)-\psi_{j,b_j}(N,(b_{-i},0),v).\end{aligned}$$

So, ψ satisfies UBC.

Conversely, suppose that ψ satisfies UBC. Let $(N,b,v)\in\Gamma$ and σ,σ' be two full-action orders for (N,b,v). Since each full-action order can be transformed to another full-action order by applying transpositions, we can assume that σ' is a transposition of σ. Let $i,j\in N$ with $i\neq j$ and $\sigma(j)=\sigma(i)+1$, such that $\sigma'(i)=\sigma(j)$, $\sigma'(j)=\sigma(i)$ and $\sigma'(p)=\sigma(p)$ for all $p\in N\setminus\{i,j\}$. Since σ' is a transposition of σ, for all $t\in N\setminus\{i,j\}$,

$$\psi_{t,b_t}(N,s^{\sigma,\sigma(t)},v)=\psi_{t,b_t}(N,s^{\sigma',\sigma'(t)},v). \tag{1}$$

Since σ' is a transposition of σ, by equation (1),

$$\begin{aligned}&\sum_{p\in N}\psi_{p,b_p}(N,s^{\sigma',\sigma'(p)},v)-\sum_{p\in N}\psi_{p,b_p}(N,s^{\sigma,\sigma(p)},v)\\ =&\,\psi_{j,b_j}(N,(b_{-i},0),v)+\psi_{i,b_i}(N,b,v)-\psi_{i,b_i}(N,(b_{-j},0),v)\\ &-\psi_{j,b_j}(N,b,v).\end{aligned} \tag{2}$$

Since ψ satisfies UBC,

$$\begin{aligned}&\psi_{j,b_j}(N,(b_{-i},0),v)+\psi_{i,b_i}(N,b,v)\\ =&\,\psi_{i,b_i}(N,(b_{-j},0),v)+\psi_{j,b_j}(N,b,v).\end{aligned} \tag{3}$$

By equations (2) and (3),

$$\sum_{p\in N}\psi_{p,b_p}(N,s^{\sigma,\sigma(p)},v)=\sum_{p\in N}\psi_{p,b_p}(N,s^{\sigma',\sigma'(p)},v).$$

Next, we present the main result in this section.

Theorem 1. *Let ψ be a solution on Γ. The following are equivalent :*

1. *ψ admits a potential*
2. *ψ satisfies BC and WIIE*
3. *ψ satisfies UBC and IIE*
4. *ψ satisfies FAPI and IIE*
5. *For all $(N,b,v) \in \Gamma$, $\psi(N,b,v) = \gamma(N,b,v_\psi)$.*

Proof. Please see the Appendix.

Finally, we axiomatize the consistent value by means of Lemma 1 and Theorem 1.

Theorem 2.

1. *The consistent value γ is the only solution satisfying EFF, IIE, and UBC.*
2. *The consistent value γ is the only solution satisfying EFF, WIIE, and BC.*

Proof. The results are Theorems 2 in Hwang and Liao [14]. In fact, this theorem also can be proved by Remarks 1, 2 and Theorem 1 in this paper.

Different from the axiomatic results proposed by Hwang and Liao [14], we offer an alternative axiomatization of the consistent value based on the equivalence between UBC and FAPI.

Theorem 3. *A solution ψ on Γ satisfies EFF, IIE and FAPI if and only if $\psi = \gamma$*

Proof. It follows from Theorems 1, 2 and Lemma 1.

4 Conclusions

In this paper, we build on the results proposed by Hwang and Liao [14] on fuzzy TU games. Several differences between Hwang and Liao's [14] work and ours are as follows.

– In the framework of fuzzy TU games, We provide the full-action path independence (FAPI) property. This property and related results do not appear in Hwang and Liao [14].
– Different from the results proposed by Hwang and Liao [14], several equivalent relations are proposed to axiomatize the family of all fuzzy solutions that admit a potential.
– Different from the technique of the proofs in Hwang and Liao [14], we axiomatize the consistent value by means of the equivalent relations proposed in this paper. Further, we propose alternative axiomatization of the consistent value.

In addition to providing axiomatic results of the consistent value, these mentioned above raise one question in the framework of fuzzy TU games.

– Whether related results of the Shapley value on standard TU games could be described in the framework of fuzzy TU games.

To our knowledge, these issues are still open questions.

Appendix

The proof of Theorem 1. Let ψ be a solution on Γ. By Lemma 1, we have that $3 \Leftrightarrow 4$.
To verify $1 \Rightarrow 2$, suppose ψ admits a potential P. Let $(N,b,v) \in \Gamma$ and $(i,k_i),(j,k_j) \in K^{N,b}$, $i \neq j$,

$$\begin{aligned}
&\psi_{i,k_i}\big(N,(b_{-j},k_j),v\big) - \psi_{i,k_i}\big(N,(b_{-j},0),v\big)\\
=&\big[P\big(N,(b_{-ij},k_i,k_j),v\big) - P\big(N,(b_{-ij},0,k_j),v\big)\big]\\
&-\big[P\big(N,(b_{-ij},k_i,0),v\big) - P\big(N,(b_{-ij},0,0),v\big)\big]\\
=&\big[P\big(N,(b_{-ij},k_i,k_j),v\big) - P\big(N,(b_{-ij},k_i,0),v\big)\big]\\
&-\big[P\big(N,(b_{-ij},0,k_j),v\big) - P\big(N,(b_{-ij},0,0),v\big)\big]\\
=&\psi_{j,k_j}\big(N,(b_{-i},k_i),v\big) - \psi_{j,k_j}\big(N,(b_{-i},0),v\big).
\end{aligned}$$

Hence, ψ satisfies BC. To see that ψ satisfies WIIE, we show that it satisfies IIE. Let $(N,b,v) \in \Gamma$ and $(i,k_i) \in K^{N,b}, k_i \neq b_i$. For $k_i \leq l \leq b_i$

$$\begin{aligned}
&\psi_{i,k_i}\big(N,(b_{-i},l),v\big)\\
=&P\big(N,(b_{-i},k_i),v\big) - P\big(N,(b_{-i},0),v\big)\\
=&\psi_{i,k_i}(N,b,v).
\end{aligned}$$

That is, ψ satisfies IIE.
To verify $2 \Rightarrow 3$, suppose ψ satisfies BC and WIIE. Clearly, ψ satisfies UBC. It remains to show that ψ satisfies IIE. Let $(N,b,v) \in \Gamma$. The proof proceeds by induction on $|N|$. It is true for $|N| = 1$ by WIIE. Assume that ψ satisfies IIE for $|N| \leq t-1$, where $t \geq 2$.
The case $|N| = t$: For $(i,k_i) \in K^{N,b}$ with $k_i \neq b_i$, let $p \in N$ and $p \neq i$. For all $t = 0,1,2,\cdots,b_i - k_i$, consider the game $\big(N,(b_{-ip},k_i+t,b_p),v\big)$, by BC of ψ,

$$\begin{aligned}
&\psi_{i,k_i}\big(N,(b_{-ip},k_i+t,b_p),v\big) - \psi_{i,k_i}\big(N,(b_{-ip},k_i+t,0),v\big)\\
=&\psi_{p,b_p}\big(N,(b_{-ip},k_i,b_p),v\big) - \psi_{p,b_p}\big(N,(b_{-ip},0,b_p),v\big).
\end{aligned}$$

Hence, for all $t = 0,1,2,\cdots,b_i - k_i$,

$$\begin{aligned}
&\psi_{i,k_i}\big(N,(b_{-ip},k_i+t,b_p),v\big)\\
=&\psi_{i,k_i}\big(N,(b_{-ip},k_i+t,0),v\big) + \psi_{p,b_p}\big(N,(b_{-ip},k_i,b_p),v\big)\\
&-\psi_{p,b_p}\big(N,(b_{-ip},0,b_p),v\big).
\end{aligned}$$

Since $|S(b_{-ip},k_i+t,0)| < |N|$, by induction hypotheses, for all $t = 0,1,2,\cdots,b_i - k_i$,

$$\psi_{i,k_i}\big(N,(b_{-ip},k_i,0),v\big) = \psi_{i,k_i}\big(N,(b_{-ip},k_i+t,0),v\big).$$

So, for all $t = 0,1,2,\cdots,b_i - k_i$,

$$\begin{aligned}
&\psi_{i,k_i}\big(N,(b_{-ip},k_i,b_p),v\big)\\
=&\psi_{i,k_i}\big(N,(b_{-ip},k_i,0),v\big) + \psi_{p,b_p}\big(N,(b_{-ip},k_i,b_p),v\big)\\
&-\psi_{p,b_p}\big(N,(b_{-ip},0,b_p),v\big)\\
=&\psi_{i,k_i}\big(N,(b_{-ip},k_i+t,0),v\big) + \psi_{p,b_p}\big(N,(b_{-ip},k_i,b_p),v\big)\\
&-\psi_{p,b_p}\big(N,(b_{-ip},0,b_p),v\big)\\
=&\psi_{i,k_i}\big(N,(b_{-ip},k_i+t,b_p),v\big).
\end{aligned}$$

That is, ψ satisfies IIE.
To verify $3 \Rightarrow 5$, suppose that ψ satisfies UBC and IIE. Let $(N,b,v) \in \Gamma$. The proof proceeds by induction on

$|N|$. Assume that $|N| = 1$ and $N = \{i\}$. By EFF of γ and definition of v_ψ,

$$\gamma_{i,b_i}(N,b,v_\psi) = v_\psi(b) = \psi_{i,b_i}(N,b,v).$$

Let $k_i \in B_i^+, k_i \neq b_i$. By IIE and EFF of γ and definition of v_ψ,

$$\begin{aligned} &\gamma_{i,k_i}(N,b,v_\psi) \\ &= \gamma_{i,k_i}(N,(b_{-i},k_i),v_\psi) \\ &= v_\psi(b_{-i},k_i) \\ &= \psi_{i,k_i}(N,(b_{-i},k_i),v) \\ &= \psi_{i,k_i}(N,b,v). \end{aligned}$$

Assume that $\psi(N,b,v) = \gamma(N,b,v)$ if $|N| \leq l-1$, where $l \geq 2$.
The case $|N| = l$: By UBC of both ψ and γ, and induction hypotheses, for $i,j \in N$,

$$\begin{aligned} &\psi_{i,b_i}(N,b,v) - \psi_{j,b_j}(N,b,v) \\ &= \psi_{i,b_i}\big(N,(b_{-j},0),v\big) - \psi_{j,b_j}\big(N,(b_{-i},0),v\big) \\ &\quad \text{(by UBC of } \psi) \\ &= \gamma_{i,b_i}\big(N,(b_{-j},0),v_\psi\big) - \gamma_{j,b_j}\big(N,(b_{-i},0),v_\psi\big) \\ &\quad \text{(by induction hypotheses)} \\ &= \gamma_{i,b_i}(N,b,v_\psi) - \gamma_{j,b_j}(N,b,v_\psi). \\ &\quad \text{(by UBC of } \gamma) \end{aligned}$$

So, for all $i,j \in N$,

$$\psi_{i,b_i}(N,b,v) - \gamma_{i,b_i}(N,b,v_\psi) = \psi_{j,b_j}(N,b,v) - \gamma_{j,b_j}(N,b,v_\psi).$$

Let $c = \psi_{i,b_i}(N,b,v) - \gamma_{i,b_i}(N,b,v_\psi)$ for all $i \in N$. By definition of v_ψ and EFF of γ,

$$\begin{aligned} &|N| \cdot c \\ &= \sum_{i \in N} \psi_{i,b_i}(N,b,v) - \sum_{i \in N} \gamma_{i,b_i}(N,b,v_\psi) \\ &= v_\psi(b) - v_\psi(b) \\ &= 0. \end{aligned}$$

Therefore, $c = 0$. Hence $\psi_{i,b_i}(N,b,v) = \gamma_{i,b_i}(N,b,v)$ for all $i \in N$. It remains to show that $\psi_{i,k_i}(N,b,v) = \gamma_{i,k_i}(N,b,v_\psi)$ for all $i \in N$ and $k_i \in B_i^+, k_i \neq b_i$. By induction hypotheses and UBC of both ψ and γ, for all $i,j \in N$ and $k_i \in B_i^+, k_i \neq b_i$,

$$\begin{aligned} &\psi_{i,k_i}\big(N,(b_{-i},k_i),v\big) - \psi_{j,b_j}\big(N,(b_{-i},k_i),v\big) \\ &= \psi_{i,k_i}\big(N,(b_{-ij},k_i,0),v\big) - \psi_{j,b_j}\big(N,(b_{-i},0),v\big) \\ &\quad \text{(by UBC of } \psi) \\ &= \gamma_{i,k_i}\big(N,(b_{-ij},k_i,0),v_\psi\big) - \gamma_{j,b_j}\big(N,(b_{-i},0),v_\psi\big) \\ &\quad \text{(by induction hypotheses)} \\ &= \gamma_{i,k_i}\big(N,(b_{-i},k_i),v_\psi\big) - \gamma_{j,b_j}\big(N,(b_{-i},k_i),v_\psi\big). \\ &\quad \text{(by UBC of } \gamma) \end{aligned}$$

So, for all $i,j \in N$ and $k_i \in B_i^+, k_i \neq b_i$,

$$\begin{aligned} &\psi_{i,k_i}\big(N,(b_{-i},k_i),v\big) - \gamma_{i,k_i}\big(N,(b_{-i},k_i),v_\psi\big) \\ &= \psi_{j,b_j}\big(N,(b_{-i},k_i),v\big) - \gamma_{j,b_j}\big(N,(b_{-i},k_i),v_\psi\big). \end{aligned}$$

Let $d = \psi_{i,k_i}\big(N,(b_{-i},k_i),v_\psi\big) - \gamma_{i,k_i}\big(N,(b_{-i},k_i),v_\psi\big)$ for all $i \in N$ and $k_i \in B_i^+, k_i \neq b_i$. By definition of v_ψ and EFF of γ, $d = 0$. Hence, by IIE of both ψ and γ, for all $i \in N$ and $k_i \in B_i^+, k_i \neq b_i$,

$$\begin{aligned} &\psi_{i,k_i}(N,b,v) \\ &= \psi_{i,k_i}\big(N,(b_{-i},k_i),v\big) \\ &= \gamma_{i,k_i}\big(N,(b_{-i},k_i),v_\psi\big) \\ &= \gamma_{i,k_i}(N,b,v_\psi). \end{aligned}$$

To verify $5 \Rightarrow 1$, suppose that $\psi(N,b,v) = \gamma(N,b,v_\psi)$ for all $(N,b,v) \in \Gamma$. Since the consistent value γ admits a potential P_γ, we define a function of ψ as $P_\psi(N,b,v) = P_\gamma(N,b,v_\psi)$ for all $(N,b,v) \in \Gamma$. Then for all $(i,k_i) \in K^{N,b}$,

$$\begin{aligned} &P_\psi\big(N,(b_{-i},k_i),v\big) - P_\psi\big(N,(b_{-i},0),v\big) \\ &= P_\gamma\big(N,(b_{-i},k_i),v_\psi\big) - P_\gamma\big(N,(b_{-i},0),v_\psi\big) \\ &= \gamma_{i,k_i}(N,b,v_\psi) \\ &= \psi_{i,k_i}(N,b,v). \end{aligned}$$

Hence, ψ admits the potential P_ψ by Definition 2.

References

[1] Aubin JP Coeur et valeur des jeux flous á paiements latéraux. Comptes Rendus de l'Académie des Sciences 279:891-894 (1974)
[2] Aubin JP Cooperative fuzzy games. Mathematics of Operations Research 6:1-13 (1981)
[3] Branzei R, Dimitrov D, Tijs S Convex fuzzy games and participation monotonic allocation schemes. Fuzzy Sets and Systems 139:267-281 (2003)
[4] Branzei R, Dimitrov D, Tijs S Egalitarianism in convex fuzzy games. Mathematical Social Sciences 47:313-325 (2004)
[5] Branzei R, Dimitrov D, Tijs S Hypercubes and compromise values for cooperative fuzzy games. Eupopean Journal of Operational Research 155:733-740 (2004)
[6] Branzei R, Dimitrov D, Tijs S Models in cooperative game theory. Springer, Verlag (2005)
[7] Butnariu D Fuzzy games: a description of the concept. Fuzzy Sets and Systems 1:181-192 (1978)
[8] Butnariu D, Klement T Triangular norm-based measures and games with fuzzy coalitions. Kluwer Academic Publishers, Dordrecht. (1993)
[9] Butnariu D, Kroupa T Shapley mappings and the cumulative value for n-person games with fuzzy coalitions. Eupopean Journal of Operational Research 186:288-299 (2008)
[10] Calvo E, Santos JC Potential in cooperative TU-games. Mathematical Social Sciences 34:175-190 (1997)
[11] Hart S, Mas-Colell A Potential, value and consistency. Econometrica 57:589-614 (1989)
[12] Hwang YA Fuzzy games: a characterization of the core. Fuzzy Sets and Systems 158:2480-2493 (2007)
[13] Hwang YA, Liao YH Max-consistency, complement-consistency, and the core of fuzzy games. Fuzzy Sets and Systems 159:152-163 (2007)
[14] Hwang YA, Liao YH The consistent value of fuzzy games. Fuzzy Sets and Systems 160:644-656 (2009)
[15] Li S, Zhang Q The measure of interaction among players ingames with fuzzy coalitions. Fuzzy Sets and Systems 159:119-137 (2008)

[16] Li S, Zhang Q A simplified expression of the Shapley function for fuzzy game. Eupopean Journal of Operational Research 196:234-245 (2009)
[17] Maschler M, Owen G The consistent Shapley value for hyperplane games. International Journal of Game Theory 18:389-407 (1989)
[18] Molina E, Tejada J The equalizer and the lexicographical solutions for cooperative fuzzy games: characterizations and properties. Fuzzy Sets and Systems 125:369-387 (2002)
[19] Myerson R Conference structures and fair allocation rules. International Journal of Game Theory 9,169-182 (1980)
[20] Muto S, Ishihara S, Fukuda S, Tijs S and Branzei R Generalized cores and stable sets for fuzzy games. International Game Theory Review 8:95-109 (2006)
[21] Nishizaki I, Sakawa M Fuzzy and multiobjective games for conflict resolution. Physica-Verlag, Heidelberg (2001)
[22] Ortmann KM Preservation of differences, potential, conservity. Working paper 236, Inst. of Math. Ec., University of Bielefeld (1995a)
[23] Ortmann KM Conservation of energy in nonatomic games. Working paper 237, Inst. of Math. Ec., University of Bielefeld (1995b)
[24] Sakawa M, Nishizaki I A lexicographical concept in an *n*-person cooperative fuzzy games. Fuzzy Sets and Systems 61:265-275 (1994)
[25] Shapley LS A value for *n*-person game, In: Kuhn, H.W., Tucker, A.W.(eds.). Contributions to the Theory of Games II, Princeton:307-317 (1953)
[26] Sharkey WW, Telser LG Supportable cost functions for the multiproduct firm. Journal of Economic Theory 18:23-37 (1978)
[27] Tijs S, Branzei R, Ishihara S, Muto S On cores and stable sets for fuzzy games. Fuzzy Sets and Systems 146:285-296 (2004)
[28] Tsurumi M, Tanino T, Inuiguchi M A Shapley function on a class of cooperative fuzzy games. Eupopean Journal of Operational Research 129:596-618 (2001)

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