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Stability Analysis of Implicit-Explicit Class for Solving ODEs and DDEs

Iman H. Ibrahim and Fatma M. Yousry*

Faculty of Women for Arts, Science, and Education, Ain Shams University, Cairo, Egypt.

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Abstract: Implicit-Explicit schemes have been widely used, it reduces the computational work for solving differential equations which have both stiff and non-stiff parts . In this paper an implicit-explicit linear multistep method for solving ordinary and delay differential equations is introduced. In both cases we are going to study the stability of the method using two approaches and the stability regions will be ploted. Numerical tests are introduced.

Keywords: Implicit-Explicit method , Stiff system , Stability regions, Delay differential equations, Splitting method.

1 Introduction

In many applications large systems of ordinary differential equations (ODEs) have both stiff and non-stiff parts. Methods for stiff problems are usually implicit and at each time step a non-linear set of algebraic equations need to be solved. This can be very expensive. It is possible to split the differential equation into a fast part and a slow part. The fast part contains only linear terms and the slow part includes all the other terms. Therefore we need to develop methods where only the linear terms are implicit and all other terms are explicit, [9,14].

Many authors solve these systems by integrating the stiff part implicitly and the non-stiff part explicitly, see [1,15,13]. Some authors develop methods as implicit-explicit (IMEX) linear multistep methods intended for such applications, see [2,4,10,11].

The delay differential equations (DDEs) can be found have the same property. The importance of DDEs is evidenced by many different areas in which they describe physical systems, such as electrostatic charge problems, automatic controls, machine tools, biological system, and a number of theory problems, see [1]. It has been shown that the linear multistep methods used for ODEs can be used to generate the solution of DDEs when the step size is finite . The aim of this paper is to entroduce implicit-explicit class for solving ODEs and DDEs and study its stability.

2 Implicit-Explicit linear multistep method

Implicit-Explicit (IMEX) Schemes are more efficient techniques for solving the time dependent equation in the form:

$$\acute{y}(t) = F(t, y(t)) + G(t, y(t)) \qquad , \quad t \geqslant 0 \tag{2.1}$$

where F and G represent the non-stiff and the stiff parts of the system respectively.

For the numerical treatment of (2.1) we consider the IMEX linear multistep formula

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h \sum_{j=0}^{k-1} \beta_j\, F(t_{n+j}, y_{n+j}) + h \sum_{j=0}^{k} \gamma_j\, G(t_{n+j}, y_{n+j}) \quad , \gamma_0 \neq 0 \tag{2.2}$$

In this paper we construct an IMEX technique based on the k-step, k-order formulas depends on a free parameter, [3] which takes the form:

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h\, \beta_k \left(f_{n+k} - \beta^* f_{n+k-1} \right) \ , \tag{2.3}$$

with respect to $\acute{y}(t) = f(t, y(t)))$, $y(t_0) = y_0$ where the derivative function is continuous and satisfying Lipschitz condition, by the suitable choice of the free parameter the obtained class has good stability properties.

Now, the discussion is focused on the IMEX Scheme (2.2) based on the form (2.3) with extrapolation $f_{n+1} \approx 2 f_n - f_{n-1}$ for the explicit part as follows :

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h\, \beta_k \{(2-\beta^*)\, f_{n+k-1} - f_{n+k-2}\} + h\, \beta_k \{g_{n+k} - \beta^* g_{n+k-1}\} \tag{2.4}$$

* Corresponding author e-mail: imanhfz.asa@gmail.com

where $f_{n+j} = f(t_{n+j}, y_{n+j})$, $g_{n+j} = g(t_{n+j}, y_{n+j})$ and h is the time step size. Due to the extrapolation process the order of the scheme (2.4) will be reduced to two.

3 Stability analysis of the method for ordinary differential equations

In this section the stability analysis of (2.4) when it is applied to (2.1) is investigated , consider the scalar test equation

$$\acute{y}(t) = \lambda\, y(t) + \mu\, y(t) \quad , \tag{3.1}$$

where λ and μ are complex constants representing the eigen values of the explicit and implicit operators respectively.

The characteristic equation takes the form:

$$\sum_{j=0}^{k} \alpha_j\, \xi^{n+j} = \lambda_1\, \beta_k \left\{ (2-\beta^*)\, \xi^{n+k-1} - \xi^{n+k-2} \right\} + \mu_1\, \beta_k \left\{ \xi^{n+k} - \beta^* \xi^{n+k-1} \right\} \tag{3.2}$$

where $\lambda_1 = \lambda\, h$ and $\mu_1 = \mu\, h$,this equation takes the form

$$A(\xi) - \lambda_1 B(\xi) - \mu_1\, C(\xi) = 0 \tag{3.3}$$

where A, B and C are the following polynomials

$$A(\xi) = \sum_{j=0}^{k} \alpha_j\, \xi^{n+j} , \tag{3.4}$$

$$B(\xi) = \beta_k \left\{ (2-\beta^*)\, \xi^{n+k-1} - \xi^{n+k-2} \right\} ,$$

$$C(\xi) = \beta_k \left\{ \xi^{n+k} - \beta^* \xi^{n+k-1} \right\} .$$

For stability, ξ must satisfy the condition $|\xi| \leqslant 1$, with strict inequality for multiple roots, see [5,6]. Here, the stability analysis is explained and the stability regions are ploted by using two different approaches. These two approaches provide an understanding of the phenomena from different points of view, in the following Figures the shaded parts are the stability regions.

3.1 First approach

3.1.1 Stability of explicit methods

To study the stability of explicit methods, put $\mu_1 = 0$ in (3.3), so,

$$A(\xi) - \lambda_1\, B(\xi) = 0 .$$

The boundary of the stability regions of the explicit method is given by :

$$\lambda_1 = \frac{A(e^{i\theta})}{B(e^{i\theta})} \quad , \quad \theta \in [-\pi, \pi] \tag{3.5}$$

Table 1: The range of λ_1 for various steps

k	Range of λ_1
1	$-0.462 < \lambda_1 < 0$
2	$-1.538 < \lambda_1 < 0$
3	$-2.769 < \lambda_1 < 0$
4	$-2.333 < \lambda_1 < 0$
5	$-0.922 < \lambda_1 < 0$
6	$-0.533 < \lambda_1 < 0$
7	$-0.166 < \lambda_1 < 0$

the bounds of λ_1 in the real axis for k-step up to 7 are given in Table 1:

The stability regions for explicit methods (2.4) of k-step up to 7 are given in Figures 1&2.

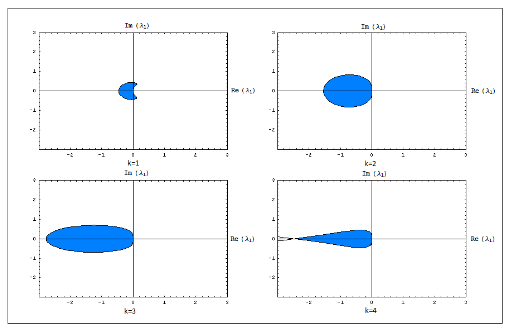

Fig. 1: The stability regions of method (2.4) when $\mu_1 = 0$, for k up to 4

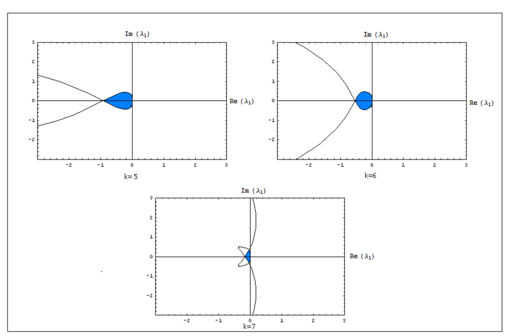

Fig. 2: The stability regions of method (2.4) when $\mu_1 = 0$, for k=5,6,7

Table 2: The angle α of the A(α)-stable method (2.4) for k=4, 5 & 6

k	α
4	81°
5	61°
6	32°

3.1.2 Stability of Implicit-Explicit methods

To study the stability of implicit-explicit methods with respect to the implicit eigenvalues we define

$$\varphi_\lambda(\xi) = \frac{A(\xi) - \lambda_1 B(\xi)}{C(\xi)}.$$

$\lambda_1^* = -0.1$ is choosen from the stability region of the explicit method for each k as a common value. This choice gives a large stability region. For step k= 4 as an example,

$$\varphi_\lambda(\xi) = \frac{\alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \alpha_4\xi^4 - \lambda_1\left(\beta_4(2-\beta^*)\xi^3 - \xi^2\right)}{\beta_4\left(\xi^4 - \beta^*\xi^3\right)}$$

where $\alpha_0 = \frac{(3+\beta^*)}{25-3\beta^*}$, $\alpha_1 = \frac{-(16+6\beta^*)}{25-3\beta^*}$, $\alpha_2 = \frac{(36+18\beta^*)}{25-3\beta^*}$, $\alpha_3 = \frac{-(48+10\beta^*)}{25-3\beta^*}$, $\beta_4 = \frac{12}{25-3\beta^*}$ and the boundary of the stability region for the implicit-explicit method is given by

$$\varphi_{\lambda^*}(e^{i\theta}) = \frac{\alpha_0 + \alpha_1 e^{i\theta} + \alpha_2 e^{2i\theta} + \alpha_3 e^{3i\theta} + \alpha_4 e^{4i\theta} - \lambda_1^*\left(\beta_4(2-\beta^*)e^{3i\theta} - e^{2i\theta}\right)}{\beta_4\left(e^{4i\theta} - \beta^* e^{3i\theta}\right)}$$

For $\beta^* = 0.4$ the stability regions for IMEX methods with steps up to 7 are given in Figures 3 & 4.

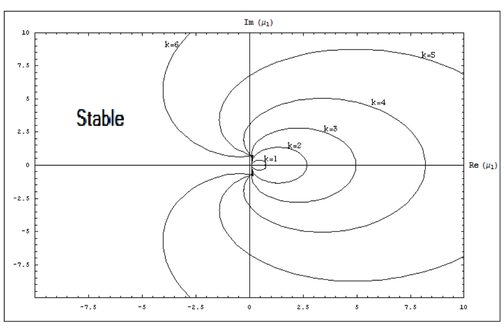

Fig. 3: The stability regions of method (2.4) when $\mu_1 = 0$, for k up to 6

The method (2.4) of steps 1,2,3 are $A-stable$ and that of steps 4,5,6 are $A(\alpha)-stable$ and the angles α are tabulated in Table 2:

In this approach, the stability of (2.4) is determined by the location of the roots of the characteristic equation (3.3), for a root ξ, stability region requires that $|\,\xi\,| \leqslant 1$ with strict inequality for multiple roots .

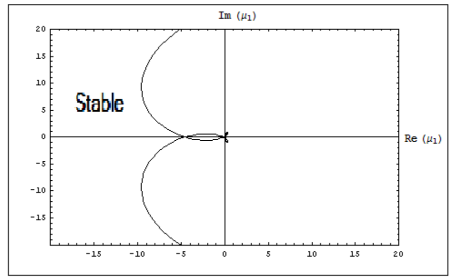

Fig. 4: The stability regions of IMEX method (2.4) when $\mu_1 = 0$, k=7

3.2 Second approach

Here the stability of the method (2.4) is studied in λ_1, μ_1 plane to obtain the values of λ_1, μ_1 that make the roots of (3.3) less than one.

For $k = 3$, in this case the characteristic equation of (2.4) takes the form:

$$\alpha_0 + \xi(\alpha_1 + \beta_3\lambda_1) + \xi^2$$
$$(\alpha_2 - 2\beta_3\lambda_1 + \beta^*\beta_3\lambda_1 + \beta^*\beta_3\mu_1) + \xi^3(\alpha_3 - \beta_3\mu_1) = 0$$

To study the stability region in $\lambda_1 - \mu_1$ plane, the following Theorem is needed.

Theorem 1.[8]*The condition for which all the roots of the equation*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

lie inside the unit circle, are that the principal minor determinants of the Hermitian matrix (A_{rs}) are positive definite , where

$$A_{rs} = \sum_{l=0}^{\min(r,s)} \{\bar{a}_{n+l-r}\, a_{n+l-s} - a_{r-l}\,\bar{a}_{s-l}\} \qquad , r,s = 0(1)n-1$$

and $\bar{a}_i$ is the conjugate element of a_i .

For direct applications, the Hermitian matrix (A$_{rs}$) for $n = 3$ represented explicitly by :

$$(A_{33}) = \begin{pmatrix} a_3\bar{a}_3 - a_0\bar{a}_0 & a_2\bar{a}_3 - a_0\bar{a}_1 & a_1\bar{a}_3 - a_0\bar{a}_2 \\ a_3\bar{a}_2 - a_1\bar{a}_0 & a_3\bar{a}_3 + a_2\bar{a}_2 - a_1\bar{a}_1 - a_0\bar{a}_0 & a_2\bar{a}_3 - a_0\bar{a}_1 \\ a_3\bar{a}_1 - a_2\bar{a}_0 & a_3\bar{a}_2 - a_1\bar{a}_0 & a_3\bar{a}_3 - a_0\bar{a}_0 \end{pmatrix},$$

is positive definite,where
$a_0 = \alpha_0$
$a_1 = (\alpha_1 + \beta_3\lambda_1)$
$a_2 = (\alpha_2 - 2\beta_3\lambda_1 + \beta^*\beta_3\lambda_1 + \beta^*\beta_3\mu_1)$
$a_3 = (\alpha_3 - \beta_3\mu_1)$.

If the coefficients are real in the characteristic equation then $\bar{a}_i = a_i$, so

$$D_1 = a_3^2 - a_0^2$$

$$D_2 = \begin{vmatrix} a_3^2 - a_0^2 & a_3a_2 - a_1a_0 \\ a_3a_2 - a_1a_0 & a_3^2 + a_2^2 - a_1^2 - a_0^2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_3^2 - a_0^2 & a_3a_2 - a_1a_0 & a_3a_1 - a_2a_0 \\ a_3a_2 - a_1a_0 & a_3^2 + a_2^2 - a_1^2 - a_0^2 & a_3a_2 - a_1a_0 \\ a_3a_1 - a_2a_0 & a_3a_2 - a_1a_0 & a_3^2 - a_0^2 \end{vmatrix}$$

Thus, to find the region of stability we choose the common domain in $\lambda_1 - \mu_1$ plane satisfied by the three conditions : $D_1 > 0$, $D_2 > 0$, $D_3 > 0$.

The stability regions of (2.4) for steps k up to 7 are plotted in Figures 5 & 6.

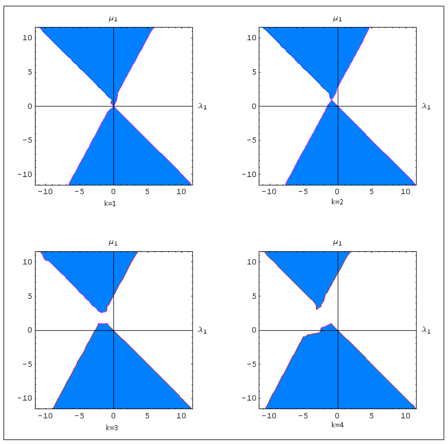

Fig. 5: The numerical stability regions of formula (2.4) applied on the equation (3.1) for k up to 4

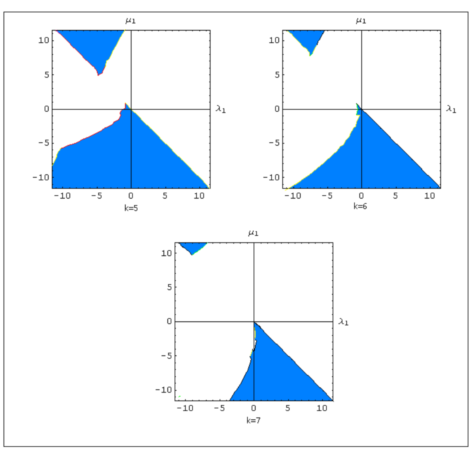

Fig. 6: The numerical stability regions of formula (2.4) applied on the equation (3.1) for k=5,6,7

4 Stability analysis of the method for delay differential equation

Now, an IMEX linear multistep methods for the numerical solution of delay differential equation which are composed of stiff and non-stiff parts is introduced.

The similar form of (2.1) is given by:

$$\acute{y}(t) = f^1(t, y(t)) + f^2(t, y(t-\tau)) \ , \tau, t \geqslant 0 \ , \qquad (4.1)$$

where f^1 and f^2 represent the stiff and non-stiff parts, respectively .

For the numerical solution of (4.1), we consider the IMEX linear multistep methods

$$\textstyle\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h \sum_{j=0}^{k} \, \beta_j \, f^1(t_{n+j}, y_{n+j}) + h \sum_{j=0}^{k-1} \, \gamma_j \, f^2(t_{n+j}, y_{n+j}), \qquad (4.2)$$

IMEX scheme (4.2) is constructed by applying a given known implicit linear multistep methods to the whole problem (4.1) and then replace the implicit term which occurs in the case f^2 by a suitable extrapolation formula to preserve the implicitness of the whole formula.

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = $$
$$h\,\beta_k \{(2-\beta^*) \ f_{n+k-1}^2 - f_{n+k-2}^2\} + h\,\beta_k \, \{ f_{n+k}^1 - \beta^* \, f_{n+k-1}^1 \} \qquad (4.3)$$

Consider the scalar test problem

$$\acute{y}(t) = \mu\, y(t) \ + \lambda\, y(t-\tau), \tau \geqslant 0, \qquad (4.4)$$

where μ and λ represent the complex eigen values of the stiff and non-stiff parts respectively and τ is the delay constant, $\tau = \nu h$, where $\nu \in I^+$.

The characteristic equation is represented by

$$A(\xi) - \lambda_1 B(\xi) - \mu_1 C(\xi) = 0 \qquad (4.5)$$

where

$$A(\xi) = \sum_{j=0}^{k} \alpha_j \, \xi^{n+j} \qquad (4.6)$$
$$B(\xi) = \beta_k \left\{ (2-\beta^*)\, \xi^{n+k-1-\nu} - \xi^{n+k-2-\nu}) \right\}$$
$$C(\xi) = \beta_k \left\{ \xi^{n+k} - \beta^* \xi^{n+k-1} \right\} \qquad ,$$
$$\mu_1 = \mu\, h \ , \lambda_1 = \lambda\, h \ , \tau \geqslant 0,$$

the stability analysis of (4.3) is discussed by the location of the roots of the characteristic equation for different values of ν we can choose, $\nu = 2$ and $\nu = 5$ as examples.

We study the stability method when it is applied to DDEs by the former two approaches .

Table 3: The range of λ_1 for various steps

k	$Range\ of\ \ \lambda_1\ ,\ \nu=2$	$Range\ of\ \ \lambda_1\ ,\ \nu=5$
1	$-0.462<\lambda_1<0$	$-0.293<\lambda_1<0$
2	$-0.433<\lambda_1<0$	$-0.218<\lambda_1<0$
3	$-0.485<\lambda_1<0$	$-0.267<\lambda_1<0$
4	$-0.445<\lambda_1<0$	$-0.266<\lambda_1<0$
5	$-0.444<\lambda_1<0$	$-0.268<\lambda_1<0$
6	$-0.454<\lambda_1<0$	$-0.267<\lambda_1<0$
7	$-0.458<\lambda_1<0$	$-0.263<\lambda_1<0$

4.1 First approach

4.1.1 Stability of explicit methods

To study the stability of explicit methods put $\mu_1=0$ in (4.5) so, obtain

$$A(\xi)-\lambda_1\,B(\xi)=0,$$

The intersection of λ_1 with real axis given by

$$\lambda_1=\frac{A(\xi)}{B(\xi)}$$

In the case of $\nu=2\,\&\,5$ the bounds of λ_1 in the real axis for steps k up to 7 are given in Table 3

The boundary of the stability regions of the explicit method is given by

$$\lambda_1=\frac{A(e^{i\theta})}{B(e^{i\theta})}\qquad,\quad \theta\in[-\pi,\pi] \tag{4.7}$$

The stability regions for explicit methods for steps k up to 7 (with $\nu=2\,\&\,5$) are given in Figures 7 - 10.

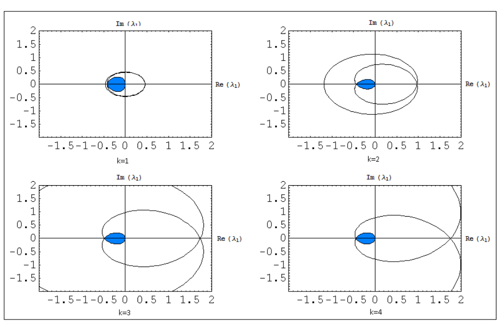

Fig. 7: The stability regions of method (4.3) when $\mu_1=0$ for k up to 4 with $\nu=2$

4.1.2 Stability of Implicit-Explicit methods

To study the stability of implicit-explicit methods , let

$$\varphi_\lambda(\xi)=\frac{A(\xi)-\lambda_1\,B(\xi)}{C(\xi)}$$

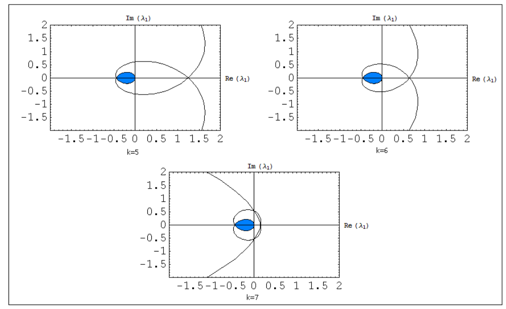

Fig. 8: The stability regions of method (4.3) when $\mu_1=0$ for k=5,6,7 with $\nu=2$

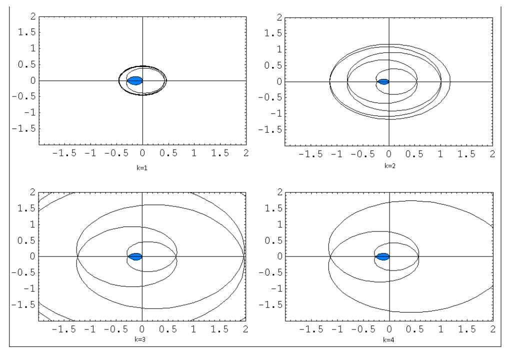

Fig. 9: The stability regions of method (4.3) when $\mu_1=0$ for up to 4 with $\nu=5$

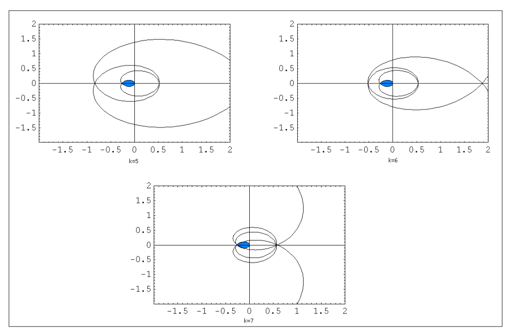

Fig. 10: The stability regions of method (4.3) when $\mu_1=0$ for k=5,6,7 with $\nu=5$

Also we choose $\lambda_1^*=-0.1$ from the stability region of the explicit method as a common value. For step k= 4

$$\varphi_\lambda(\xi)=\frac{\alpha_0+\alpha_1\xi+\alpha_2\xi^2+\alpha_3\xi^3+\alpha_4\xi^4-\lambda_1\left(\beta_4\,(2-\beta^*)\,\xi^{3-\nu}-\xi^{2-\nu}\right)}{\beta_4\,(\xi^4-\beta^*\xi^3)}$$

The boundary of the stability region of IMEX method is given by

$$\varphi_{\lambda^*}(e^{i\theta})=\frac{\alpha_0+\alpha_1e^{i\theta}+\alpha_2e^{2i\theta}+\alpha_3e^{3i\theta}+\alpha_4e^{4i\theta}-\lambda_1^*\left(\beta_4(2-\beta^*)e^{(3-\nu)i\theta}-e^{(2-\nu)i\theta}\right)}{\beta_4\left(e^{4i\theta}-\beta^*e^{3i\theta}\right)}$$

For $\beta^* = 0.4$ the stability regions of Implicit-Explicit methods (4.3) for steps k up to 7 (with $v = 2\,\&\,5$) are given in Figures 11-14.

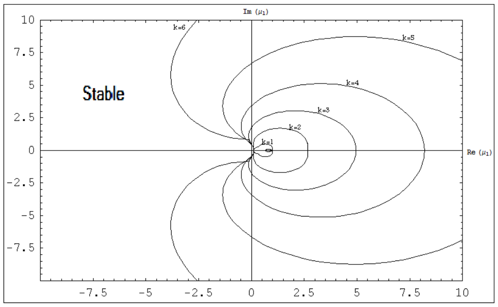

Fig. 11: The stability regions of IMEX method (4.3) when $\mu_1 = 0$ for k up to 6 with $v = 2$

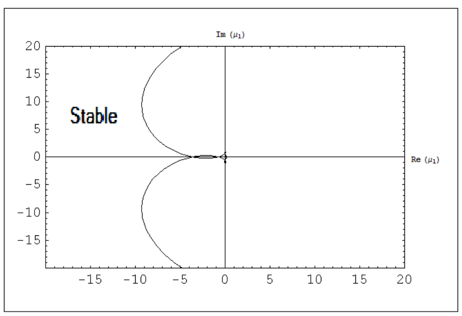

Fig. 12: The stability regions of IMEX method (4.3) for k=7 with $v = 2$

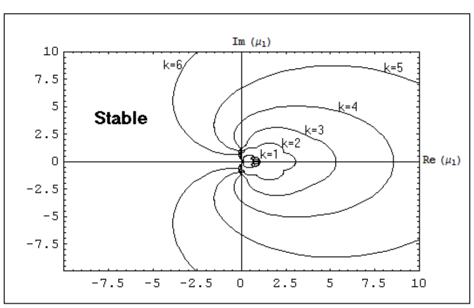

Fig. 13: The stability regions of IMEX method (4.3) for k up to 6 with $v = 5$

In the case of $v = 2$ the method of steps 1 and 2 of (4.3) is $A-stable$ and for steps from 3 to 6 is $A(\alpha)-stable$, for

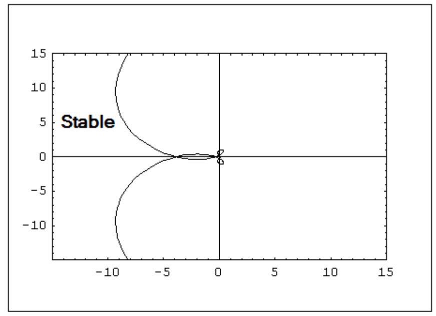

Fig. 14: The stability regions of IMEX method (4.3) for k=7 with $v = 5$

Table 4: The angle of the $A(\alpha)$-stable method(4.3) for k=3, 4, 5 & 6 for different values of υ.

k	α for $v = 2$	α for $v = 5$
2	90°	81°
3	81°	74°
4	71°	74°
5	62°	57°
6	34°	30°

$v = 5$ the method of step 1 of (3.3) is $A-stable$ and for steps from 2 to 6 is $A(\alpha)-stable$, α is tabulated in Table 4:

4.2 Second approach

To study the stability region of (4.3), consider the characteristic equation of (4.3) which takes the form:

$$\alpha_{0+}\alpha_1\xi+\alpha_2\xi^2+\ldots+\alpha_{k-1}\xi^{k-1}+\alpha_k\xi^k-\beta_k\lambda_1((2-\beta^*)\xi^{k-1-v}-\xi^{k-2-v})$$
$$-\mu_1\beta_k(\xi^k-\beta^*\xi^{k-1})=0\,.$$

The method with k=3 & $v = 2$ is discussed in details, its characteristic equation is:

$$\beta_3\lambda_1+\xi(\alpha_0-2\beta_3\lambda_1+\beta^*\beta_3\lambda_1)+\alpha_1\xi^2+\xi^3(\alpha_2+\beta^*\beta_3\mu_1)+\xi^4(\alpha_3-\beta_3\mu_1)=0\,.$$

Applying theorem 1 to study the stability region in $\lambda_1 - \mu_1$ plane.

$a_0 = \beta_3\lambda_1$,
$a_1 = (\alpha_0 - 2\beta_3\lambda_1 + \beta^*\beta_3\lambda_1)$,
$a_2 = \alpha_1$,
$a_3 = (\alpha_2 + \beta^*\beta_3\mu_1)$,
$a_4 = (\alpha_3 - \beta_3\mu_1)$,

if the coefficients are real in the characteristic equation then $\overline{a_i} = a_i$, so

$$D_1 = a_4^2 - a_0^2$$

$$D_2 = \begin{vmatrix} a_4^2 - a_0^2 & a_4a_3 - a_1a_0 \\ a_4a_3 - a_1a_0 & a_4^2 + a_3^2 - a_1^2 - a_0^2 \end{vmatrix}$$

$$D_3=\begin{vmatrix} a_4^2-a_0^2 & a_4a_3-a_1a_0 & a_4a_2-a_2a_0 \\ a_4a_3-a_1a_0 & a_4^2+a_3^2-a_1^2-a_0^2 & a_4a_3+a_3a_2-a_2a_1-a_1a_0 \\ a_4a_2-a_2a_0 & a_4a_3+a_3a_2-a_2a_1-a_1a_0 & a_4^2+a_3^2-a_1^2-a_0^2 \end{vmatrix}$$

$$D_4=\begin{vmatrix} a_4^2-a_0^2 & a_4a_3-a_1a_0 & a_4a_2-a_2a_0 & a_4a_1-a_3a_0 \\ a_4a_3-a_1a_0 & a_4^2+a_3^2-a_1^2-a_0^2 & a_4a_3+a_3a_2-a_2a_1-a_1a_0 & a_4a_2-a_2a_0 \\ a_4a_2-a_2a_0 & a_4a_3+a_3a_2-a_2a_1-a_1a_0 & a_4^2+a_3^2-a_1^2-a_0^2 & a_4a_3-a_1a_0 \\ a_4a_1-a_3a_0 & a_4a_2-a_2a_0 & a_4a_3-a_1a_0 & a_4^2-a_0^2 \end{vmatrix}$$

Thus, to find the stability region choose the common domain in $\lambda_1-\mu_1$ plane satisfied by the four conditions $:D_1>0$, $D_2>0$, $D_3>0$, $D_4>0$.

The numerical stability regions are plotted in Figures 15 - 18.

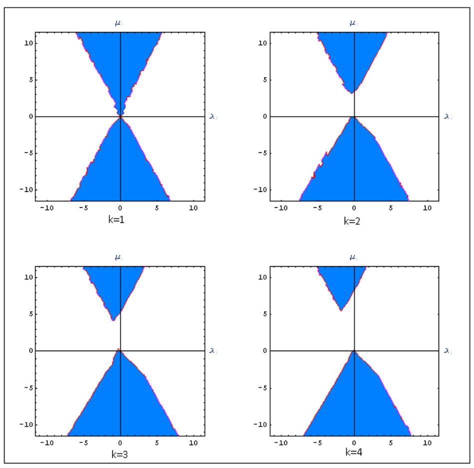

Fig. 15: The numerical stability regions of formula (4.3) for k up to 4 with $v=2$

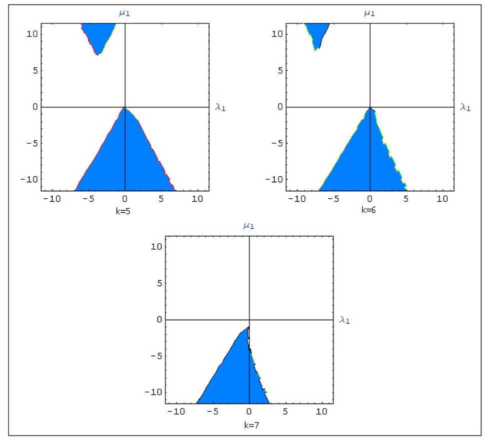

Fig. 16: The numerical stability regions of formula (4.3) for k=5,6,7 with $v=2$

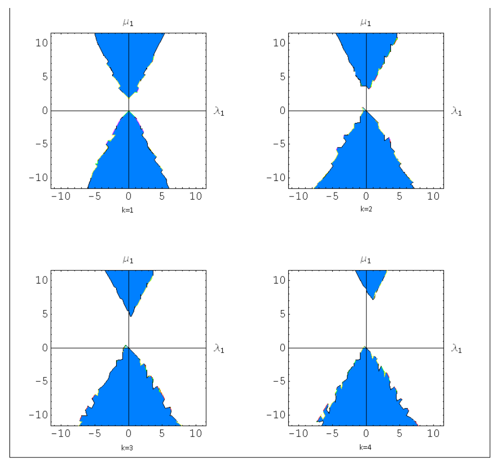

Fig. 17: The numerical stability regions of formula (4.3) for k up to 4 with $v=5$

5 Numerical tests

Test 1[12]

Consider the differential equations

$y_1'(t)=f_1+g_1=(y_2(t)+2\sin t)+(-2\,y_1(t))$

$y_2'(t)=f_2+g_2=(998\,y_1(t))+(-999y_2(t)-999(\sin t-\cos t))$

with initial conditions $y_1(0)=2$, $y_2(0)=3$, its exact solutions are

$y_1(t)=\exp(-t)(2+\exp(-t)\sin t)$, $y_2(t)=\exp(-t)(2+\exp(-t)\cos t)$.

Test 2

Consider the differential equation

$x'(t)=f+g=-\sin t+\lambda(\,x(t)-\cos t)$

$\lambda=-10^4$ with the initial condition $x(0)=1$, the exact solution is

$x(t)=\exp(\lambda\,t)(x(0)-1)+\cos t.$

Test 3 [7]

Consider the stiff delay differential equation

$y\prime(t)=f^1+f^2=p\,y(t)+(-\exp(p-1)\;y(t-1))$

$y(t)=\exp((p-1)t)\qquad t\prec 0,$

with exact solution $y(t)=\exp((p-1)t)$, the results are given for $t\in[0,2]$, $p=-100$, $p=-24$.

We solve these tests by formula (2.4) of step four, $\beta^*=0.4$ with different values of h at different values of t, the percentage of error of $y(t)$ (% Er$(y(t))$) of tests 1 , 2 and 3 are given in Tables 5 -7 respectively .

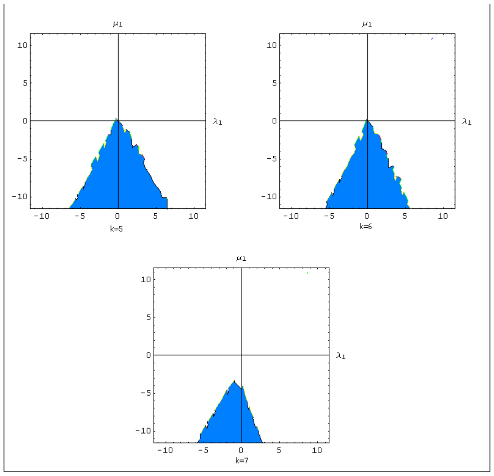

Fig. 18: The numerical stability regions of formula (4.3) for k=5,6,7 with $\nu = 5$

Table 5: The percentage errors of y(t) for Test 1

t	h	$\%Er(y_1(t))$	$\%Er(y_2(t))$
5	0.001	8.50372E-03	5.85746E-02
10	0.001	2.63299E-01	8.48673
5	0.0001	8.51057E-05	5.70159E-05
10	0.0001	2.68187E-03	1.49796E-02

Table 6: The percentage errors of y(t) for Test 2

t	h	$\%er(x(t))$
10	0.0001	2.09786E-06
20	0.0001	2.50075E-06

Table 7: The percentage errors of y(t) for Test 3

t	h	%Er$(y(t)), p = -100$	%Er$(y(t)), p = -24$
1	0.001	7.00549	2.03674E-01
2	0.001	26.3506	7.57421E-01
1	0.0001	3.27045E-02	1.813151E-03
2	0.0001	1.21607E-01	6.74318E-03

6 Conclusion

In many applications it is convenient to use splitting methods to take advantage of the special structure of the differential operator that is decomposed into a sum of two or more parts. In this paper we discussed the stability of some splitting methods which are based on a special form of the extension of BDF formula .

IMEX multistep method discussed in this paper for solving ODEs and DDEs have certain advantages such that reduce computational costs per step while preserving the stability properties of the implicit algorithm.

With certain choices of the free parameter in the splitting method, the stability regions can be improved. We note that the stability regions of the formula (4.3) reduced with the increasing of ν for the DDEs. The influence of explicitness and implicitness in the stability of the global method were analyzed and stability regions were plotted for ODEs and DDEs.

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Iman H. Ibrahim earned her BSc degree in mathematics from the Faculty of women, Ain Shams university, Cairo, Egypt. She obtained her MSc degree in mathematical analysis and PhD degree in numerical analysis from Faculty of women, Ain Shams university. Her research interest is in numerical analysis.

Fatma M. Yousry earned her B.Sc. degree in mathematics from Faculty of women, Ain Shams university, Cairo, Egypt. She obtained her MSc and PhD degrees in numerical analysis from Faculty of women Ain Shams university. Her research interest is in numerical analysis.