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Unifying Some Implicit Common Fixed Point Theorems for Hybrid Pairs of Mappings in G-Metric Spaces through Altering Distance Function

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Abstract: In this paper, the notion of sub-compatibility for hybrid pair of mappings in the framework of G-metric spaces, is introduced. The role of an appropriate implicit function concerning altering distance function is also highlighted which envelops a host of contraction conditions, in one go. Employing this implicit relation some common fixed point theorems are proved for two hybrid pairs of single and multivalued mappings in the structure of G-metric spaces. While proving our results, we utilize the idea of compatibility for hybrid mappings due to Kneko et al. [1] together with subsequentially continuity due to Bouhadjera et al. [2] (also alternately reciprocal continuity due to Singh et al. [3] together with sub-compatibility) as patterned in Imdad et al. [4]. In view of remarks given in E. Karapinar et al. [5], our fixed point results can not be reduced to the results which are observed in Jleli et al. [6], in the setting of hybrid pairs of mappings. This leads that our results are not the consequences of any fixed point results on metric spaces from the existing literature. Some illustrative examples associated with their pictorial justifications are also presented which substantiate the genuineness of the hypotheses and the degree of utility of our results.

Keywords: G-metric spaces, multi-valued mappings, subcompatible mappings, subsequential continuity and reciprocal continuity, compatible mappings, occasionally weakly compatible mappings.

1 Introduction and Preliminaries

During the last few decades, the celebrated Banach contraction principle, also known as the Banach fixed point theorem [7], has become one of the core topics of applied mathematical analysis. As a consequence, a number of generalizations, extensions, and improvement of the praiseworthy Banach contraction principle in various direction have been explored and reported by various authors. The characterization of the renowned Banach fixed point theorem in the setting of multi-valued maps is one of the most outstanding ideas of research in fixed point theory. The remarkable examples in this trend were given by Nadler [8], Mizoguchi and Takahashi [9], and Berinde and Berinde [10]

In 1992, Dhage [11] introduced the concept of D-metric spaces. Afterwards notable results established in this space. The paper [12] by Kim et al. is one of them. In 2004, Mustafa and Sims ([13], [14]) shown that most of the results concerning Dhage's D-metric spaces are invalid and thereafter they introduced a new generalized metric space structure and called it G-metric space. In this type of spaces a nonnegative real number is assigned to every triplet of elements. The authors also portrayed some fixed point theorems [15], [16] in perspective of G-metric spaces. Tagging on these initial papers, several researchers established many fixed point results on the setting of G-metric spaces ([17]- [22]). Recently, Abbas et al. [23] proved remarkable theorems in the framework of G-metric spaces.

Definition 1 [14] Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following properties:

(G-1) $G(x,y,z) = 0$ if $x = y = z$;
(G-2) $0 < G(x,x,y)$, for all $x,y \in X$ with $x \neq y$;

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(G-3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$ with $y \neq z$;

(G-4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$, symmetry in all three variables;

(G-5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$.

The function G is called a generalized or a G-metric on X and the pair (X,G) is called a G-metric space.

Definition 2 [14] Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that the sequence $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to+\infty} G(x,x_n,x_m) = 0,$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathcal{N}$ such that

$$G(x,x_n,x_m) < \varepsilon,$$

for all $m,n > N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim\limits_{n,m\to+\infty} x_n = x$.

Proposition 3 [14] *Let (X,G) be a G-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}$ is G-convergent to x;
(2) $G(x_n,x_n,x) \to 0$ as $n \to +\infty$;
(3) $G(x_n,x,x) \to 0$ as $n \to +\infty$;
(4) $G(x_n,x_m,x) \to 0$ as $n,m \to +\infty$.

Definition 4 [14] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $N \in \mathcal{N}$ such that

$$G(x_n,x_m,x_l) < \varepsilon,$$

for all $n,m,l \geq N$, that is $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 5 [14] *Let (X,G) be a G-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}$ is G-Cauchy;
(2) For every $\varepsilon > 0$, there is $N \in \mathcal{N}$ such that $G(x_n,x_n,x_m) < \varepsilon$ for all $n,m \geq N$.

Definition 6 [14] A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Recently, Kaewcharoen et al. [24] established the notion of hybrid pair of mappings in G-metric spaces, as follows.

Let X be a G-metric space. We denote $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H_G(\cdot,\cdot,\cdot)$ be the Hausdorff G-distance on $CB(X)$ i.e.

$$H_G(A,B,C) = \max\Big\{\sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,A,C), \sup_{x\in C} G(x,A,B)\Big\},$$

where

$$G(x,B,C) = d_G(x,B) + d_G(B,C) + d_G(x,C),$$

$$d_G(B,C) = \inf\{d_G(a,b), a \in B, b \in C\},$$
$$d_G(x,B) = \inf\{d_G(x,y), y \in B\}$$

and

$$d_G(x,y) = G(x,y,y) + G(y,x,x),$$

for all $x,y \in X$.
Recall that $G(x,y,C) = \inf\{G(x,y,z) : z \in C\}$.

Let $T : X \to CB(X)$ be a multi-valued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Remark 7 [24] Let X be a G-metric space, $x \in X$ and $B \subseteq X$. Then for each $y \in B$, we have

$$\begin{aligned} G(x,B,B) &= d_G(x,B) + d_G(B,B) + d_G(x,B) \\ &\leq 2d_G(x,y) \\ &= 2(G(x,x,y) + G(x,y,y)) \\ &\leq 2(G(x,y,y) + G(x,y,y) + G(x,y,y)) \\ &\leq 6G(x,y,y). \end{aligned}$$

Similarly $G(x,y,A) \leq G(x,y,z)$, $\forall x,y \in X$,$\forall z \in A$.

The following terminology is also standard.

Definition 8 Let (X,G) be a G-metric space with $f,g : X \to X$ and $T,S : X \to CB(X)$.

(1) $x \in X$ is a fixed point of f (resp. T) if $x = fx$ (resp. $x \in Tx$). The set of all fixed points of f (resp., T) is denoted by $F(f)$ (resp. $F(T)$).
(2) $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of $f$and T is denoted by $C(f,T)$.
(3) $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and $T$is denoted by $F(f,T)$.
(4) $x \in X$ is a common fixed point of f, g, S and T if $fx = x = gx \in Tx \cap Sx$.

Kaneko et al. [1] extended the notion of compatible maps to the setting of single and multi-valued maps. Later on, Jungck et al. [25] weakened the aforesaid concept by introducing the concept of weak compatibility for hybrid pair of mappings.

Definition 9 [25] $F : X \to CB(X)$ and $f : X \to X$ are weakly compatible if they commute at their coincidence points; i.e., $\{x \in X : fx \in Fx\} \subset \{x \in X : fFx = Ffx\}$.

Further in a paper, Al-Thagafi et al. [26] coined the concept of occasionally weakly compatible mappings which is weaker than weakly compatible mappings. In 2007, Abbas et al. [27] extended the definition of occasionally weakly compatible maps to the setting of multivalued mappings.

Definition 10 [27] $f : X \to X$ and $F : X \to CB(X)$ are said to be occasionally weakly compatible maps (shortly owc) if and only if there exists some point x in X such that $fx \in Fx$ and $fFx \subseteq Ffx$; i.e., $\{x \in X : fx \in Fx\} \cap \{x \in X : fFx \subset Ffx\} \neq \phi$.

Remark 11 In a paper [28], Doric et al. asserted that, the occasionally weak compatibility does not produce new common fixed point results, when involved mappings have a unique point of coincidence and therefore it reduces to weak compatibility in the case of single-valued mappings. However, this conclusion does not hold good in the case of hybrid pairs of mappings ([28] Example 2.5). Hence the occasionally weakly compatible property still produces new results for hybrid pairs of mappings.

Pant [29] introduced the concept of reciprocally continuous maps for pairs of single-valued maps further, Singh et al. [3] extended the idea of reciprocal continuity to the setting of single and multi-valued maps as follows.

Definition 12 [3] $F : X \to CB(X)$ and $f : X \to X$ are reciprocally continuous on X (resp. at $t \in X$) if and only if $fFx \in CB(X)$ for each $x \in X$ (resp. $fFt \in CB(X)$) and $\lim_{n\to\infty} fFx_n = fA$, $\lim_{n\to\infty} Ffx_n = Ft$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Fx_n = A \in CB(X), \qquad \lim_{n\to\infty} fx_n = t \in A.$$

In 2009, Bouhadjera et al. [30] introduced the notion of subsequential continuity for single-valued mappings and afterward to the setting of single and multi-valued mappings in [2], which is the weaker concept of reciprocally continuity.

Definition 13 [2] Mappings $f : X \to X$ and $F : X \to CB(X)$ are subsequentially continuous on X (resp. at $t \in X$) if and only if $fFx \in CB(X)$ for each $x \in X$ (resp. $fFt \in CB(X)$) and there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n \text{ and}$$
$$\lim_{n\to\infty} fFx_n = fA, \lim_{n\to\infty} Ffx_n = Ft.$$

Following example exhibits the above definition.

Example 14 Let $X = [0,\infty)$ with $G : X \times X \times X \to R^+$ be the G-metric space defined by

$$G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}, \qquad \forall x,y,z \in X.$$

Define $f : X \to X$ and $F : X \to CB(X)$ by

$$fx = \begin{cases} 1-x, & if\ x < 1; \\ x, & if\ x \geq 1 \end{cases}$$

and

$$Fx = \begin{cases} [1,1+x], & if\ x \leq 1; \\ [0,1], & if\ x > 1. \end{cases}$$

First of all, notice that $fFx \in CB(X)$ for all $x \in X$.
Consider the sequence $\{x_n\} = \{\frac{1}{2n}\}$ for $n = 1,2,\cdots$, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty}(1-x_n) = 1 = t \in \{1\} = A = \lim_{n\to\infty} Fx_n.$$

Also we have

$$\lim_{n\to\infty} Ffx_n = \lim_{n\to\infty} F(1-x_n) = [1,2] = F(t) = F(1)$$

and

$$\lim_{n\to\infty} fFx_n = \lim_{n\to\infty} f[1,1+x_n] = 1 = fA = f(1).$$

Therefore f and F are subsequentially continuous.
From the same example we will show that f and F are neither continuous nor reciprocally continuous. It is clear that f and F are discontinuous at $t = 1$. Now, we consider the sequence $\{x_n\} = \{1+\frac{1}{n}\}$ for $n = 1,2,\cdots$, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = 1 = t \in [0,1] = A = \lim_{n\to\infty} Fx_n.$$

Further, we have
$\lim_{n\to\infty} Ffx_n = \lim_{n\to\infty} Fx_n = [0,1] \neq F(t) = F(1) = [1,2]$.
Hence f and F are not reciprocally continuous.
Bouhadjera et al. [30] also developed the concept of subcompatible mappings for single-valued mappings, acknowledging this concept we define notion of subcompatibility for hybrid mappings (single-valued and multi-valued mappings) in the framework of G-metric spaces.

Definition 15 *Maps $F : X \to CB(X)$ and $f : X \to X$ are said to be sub compatible if and only if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Fx_n = A \in CB(X),$$
$$\lim_{n\to\infty} fx_n = t \in A, \text{ and } \lim_{n\to\infty} H_G(Ffx_n, fFx_n, fFx_n) = 0.$$

Next example validates the aforesaid definition and also distinguishes the concept of subcompatibility to compatibility.

Example 16 Let $X = [0,\infty)$ with $G : X \times X \times X \to R^+$ be the G-metric space defined by

$$G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}, \qquad \forall x,y,z \in X.$$

Define $f : X \to X$ and $F : X \to CB(X)$ by

$$fx = \begin{cases} x, & if\ x < 1; \\ 2, & if\ x \geq 1 \end{cases}$$

and

$$Fx = \begin{cases} \{1\}, & if\ x < 1; \\ [2,1+x], & if\ x \geq 1. \end{cases}$$

Consider the sequence $\{x_n\} = \{1+\frac{1}{n}\}$ for $n = 1,2,\cdots$, we have

$$\lim_{n\to\infty} fx_n = 2 = t \in \{2\} = A = \lim_{n\to\infty} Fx_n.$$

Also we have

$$\lim_{n\to\infty} F f x_n = \lim_{n\to\infty} F(2) = [2,3],$$
$$\lim_{n\to\infty} fF x_n = \lim_{n\to\infty} f[2,1+x_n] = \{2\}$$

and
$$\lim_{n\to\infty} H_G(F f x_n, fF x_n, fF x_n) = 0.$$

Therefore f and F are subcompatible mappings.

Now, we consider the sequence $\{x_n\} = \{1-\frac{1}{n}\}$ for $n = 1,2,\cdots$.
In this case, we have

$$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} x_n = 1 \in \{1\} = \lim_{n\to\infty} F x_n,$$

but
$$\lim_{n\to\infty} H_G(F f x_n, fF x_n, fF x_n) \neq 0.$$

It implies that f and F are not compatible.

Further, Imdad et al. [4] enhanced the results of Bouhadjera et al. [30] and showed that the results in [30] can easily recovered by replacing subcompatibility with compatibility or subsequential continuity with reciprocally continuity.

In this paper, owning the above idea of Imdad et al. [4], an endeavour has been made to find the common fixed point for two hybrid pairs of mappings using the notion of sub compatibility and reciprocal continuity (Alternatively subsequential continuity and compatibility) involving implicit relation and altering distance functions.

Definition 17 [31] An altering distance function is a mapping $\psi : [0,\infty) \to [0,\infty)$ such that
(ψ_1) $\psi(t)$ is increasing and continuous,
(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

It is clear that the mappings $\psi(t) = t$ is an altering distance function.

2 An Implicit Relation

Definition 18 Let Φ be a family of all continuous functions $\phi : (R^+)^6 \to R$ satisfying the conditions:

(Φ_1) ϕ is non-decreasing in its first variable and non-increasing in its second, fifth and sixth variable.
(Φ_2) $\phi(t,t,0,0,t,t) \leq 0 \Rightarrow t = 0$.

Example 19 We give some examples of the members of Φ .

(1) $\phi(t_1,t_2,\cdots,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6)$, where $a,b,c \geq 0$, $a+2c < 1$.
(2) $\phi(t_1,t_2,\cdots,t_6) = t_1 - k\max\{t_2,t_3,\cdots,t_6\}$, $k \in (0,1)$.
(3) $\phi(t_1,t_2,\cdots,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}$, $k \in (0,1)$.
(4) $\phi(t_1,t_2,\cdots,t_6) = t_1 - a\max\left\{\frac{t_2+t_3}{2},\frac{t_4+t_5}{2}\right\} - bt_6$, where $a,b \geq 0$, $a+b < 1$.
(5) $\phi(t_1,t_2,\cdots,t_6) = t_1 - at_2 - b(t_3+t_4) - c\sqrt{t_5t_6}$, where $a,b,c \geq 0$, $a+c < 1$.
(6) $\phi(t_1,t_2,\cdots,t_6) = t_1 - at_2 - b\frac{t_3^2+t_4^2}{1+t_3+t_4} - c\max\{t_5,t_6\}$, where $a,b,c \geq 0$, $a+c < 1$.
(7) $\phi(t_1,t_2,\cdots,t_6) = t_1 - at_2 - b\max\{t_3,t_5\} - c\max\{t_4,t_6\}$, where $a,b,c \geq 0$, $a+b+c < 1$.
(8) $\phi(t_1,t_2,\cdots,t_6) = t_1^2 - a\frac{t_2^2.t_3^2+t_4^2.t_5^2}{1+t_3} - bt_6^2$, where $a,b \geq 0$, $b < 1$.
(9) $\phi(t_1,t_2,\cdots,t_6) = t_1 - k\max\{t_2+t_3, t_3+t_4, t_4+t_5, t_3+t_6\}$, $k \in (0,1)$.

Certainly, apart from aforesaid examples, there are many other examples that satisfy the condition of Φ defined in Definition 18.

3 Main Results

In this section, our main theorem runs as follows.

Theorem 20. *Let (X,G) be a G-metric space, f and g be self-mappings of X, and S and T be mappings from X into $CB(X)$ satisfy the following conditions:*

(1) For all $x,y \in X$, $\phi \in \Phi$,

$$\phi(\psi(H_G(Tx,Sy,Sy)),\psi(G(fx,gy,Sy)), \psi(G(fx,Tx,Tx)),\psi(G(gy,Sy,Sy)), \psi(G(fx,Sy,Sy)),\psi(G(Tx,gy,Sy))) \leq 0. \tag{1}$$

(2) The pairs (f,T) is reciprocally continuous and subcompatible.
(3) The pairs (g,S) is occasionally weakly compatible.

Then f,g,S and T have a unique common fixed point.

Proof. Suppose that pair (f,T) is reciprocally continuous and subcompatible pair of mappings, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f x_n = t \in A = \lim_{n\to\infty} T x_n, \tag{2}$$

where $t \in X$ and $A \in CB(X)$, and satisfying

$$\lim_{n\to\infty} H_G(T f x_n, fT x_n, fT x_n) = 0$$

and
$$\lim_{n\to\infty} fT x_n = fA, \lim_{n\to\infty} T f x_n = Tt.$$

And also, we have
$$fA \subseteq Tt. \tag{3}$$

Since the other pair (g,S) is occasionally weakly compatible, there exists $u \in X$ such that $gu \in Su$ and

$$gSu \subseteq Sgu. \tag{4}$$

First of all, we claim that $gu = t$, for this, utilizing inequality (1) with $x = x_n$ and $y = u$, one gets

$$\phi(\psi(H_G(Tx_n,Su,Su)),\psi(G(fx_n,gu,Su)),\psi(G(fx_n,Tx_n,Tx_n)),\psi(G(gu,Su,Su)),\psi(G(fx_n,Su,Su)),\psi(G(Tx_n,gu,Su)))\le 0.$$

Which on making $n\to\infty$ reduces to

$$\phi(\psi(H_G(A,Su,Su)),\psi(G(t,gu,Su)),\psi(G(t,A,A)),\psi(G(gu,Su,Su)),\psi(G(t,Su,Su)),\psi(G(A,gu,Su)))\le 0.$$

Now, for $t\in A$ and $gu\in Su$ and in view of definition of H, we obtain

$$G(t,gu,gu)\le H_G(A,Su,Su)$$

and using Remark 7 and by (Φ_1), one has

$$\phi(\psi(G(t,gu,gu)),\psi(G(t,gu,gu)),0,0,\psi(G(t,gu,gu)),\psi(G(t,gu,gu)))\le 0.$$

So that

$$\psi(G(t,gu,gu))=0.$$

Which implies that

$$G(t,gu,gu))=0,$$

Which amounts to say that $gu=t$ and so $t\in Su$. Employing occasionally weak compatibility of (g,S), one gets

$$gt\in St.$$

Next to show that $gt=t$. Using inequality (1) with $x=x_n$ and $y=t$, we acquire

$$\phi(\psi(H_G(Tx_n,St,St)),\psi(G(fx_n,gt,St)),\psi(G(fx_n,Tx_n,Tx_n)),\psi(G(gt,St,St)),\psi(G(fx_n,St,St)),\psi(G(Tx_n,gt,St)))\le 0.$$

Which on letting $n\to\infty$, gives rise

$$\phi(\psi(H_G(A,St,St)),\psi(G(t,gt,St)),\psi(G(t,A,A)),\psi(G(gt,St,St)),\psi(G(t,St,St)),\psi(G(A,gt,St)))\le 0.$$

In view of definition of H and Remark 7, we have

$$\phi(\psi(G(t,gt,gt)),\psi(G(t,gt,gt)),0,0,\psi(G(t,gt,gt)),\psi(G(t,gt,gt)))\le 0.$$

So that

$$\psi(G(t,gt,gt))=0.$$

Therefore, we have

$$G(t,gt,gt))=0,$$

it implies that $gt=t$ and also $gt=t\in St$. which leads that t is a common fixed point of g and S.

Next, we show that t is also a common fixed point of mappings f and T.
Utilizing inequality(1) with $x=t$, $y=t$, one acquires

$$\phi(\psi(H_G(Tt,St,St)),\psi(G(ft,gt,St)),\psi(G(ft,Tt,Tt)),\psi(G(gt,St,St)),\psi(G(ft,St,St)),\psi(G(Tt,gt,St)))\le 0.$$

Since $fA\subseteq Tt$, $t\in A$ and $ft\in Tt$. Then by the definition of H and Remark 7, we have

$$\phi(\psi(G(ft,t,t)),\psi(G(ft,t,t)),0,0,\psi(G(ft,t,t)),\psi(G(ft,t,t)))\le 0.$$

It implies that

$$\psi(G(ft,t,t))=0.$$

So that

$$G(ft,t,t)=0 \text{ and } ft=t.$$

which amounts to say that t is also a common fixed point of f and T. Consequently, t is a common fixed point of f,g,S and T.

For the uniqueness, suppose that w is another common fixed point of f,g,S and T. Then by inequality (1) with $x=t$ and $y=w$,

$$\phi(\psi(H_G(Tt,Sw,Sw)),\psi(G(ft,gw,Sw)),\psi(G(ft,Tt,Tt)),\psi(G(gw,Sw,Sw)),\psi(G(ft,Sw,Sw)),\psi(G(Tt,gw,Sw)))\le 0.$$

Hence, we have

$$\phi(\psi(G(t,w,w)),\psi(G(t,w,w)),\psi(G(t,t,t)),\psi(G(w,w,w)),\psi(G(t,w,w)),\psi(G(t,w,w)))\le 0,$$

and so,

$$\phi(\psi(G(t,w,w)),\psi(G(t,w,w)),0,0,\psi(G(t,w,w)),\psi(G(t,w,w)))\le 0.$$

It implies that $\psi(G(t,w,w)=0$, and so $G(t,w,w)=0$. Therefore, we have $t=w$. This completes the proof.

Next result is obtained for a pair of subsequentially continuous and compatible mappings.

Theorem 21. *Let $f,g:X\to X$ and $S,T:X\to CB(X)$ be the mappings in G-metric space (X,G) satisfying the following conditions:*

(1) For all $x,y\in X$, $\phi\in\Phi$,

$$\phi(\psi(H_G(Tx,Sy,Sy)),\psi(G(fx,gy,Sy)),\psi(G(fx,Tx,Tx)),\psi(G(gy,Sy,Sy)),\psi(G(fx,Sy,Sy)),\psi(G(Tx,gy,Sy)))\le 0.$$

(2) The pairs (f,T) is subsequentially continuous and compatible.

(3) The pairs (g,S) is occasionally weakly compatible.

Then f,g,S and T have a unique common fixed point in X.

Proof. Suppose that the pair (f,T) is subsequentially continuous and compatible as well. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$$

and

$$\lim_{n\to\infty} Tfx_n = Tt,\ \lim_{n\to\infty} fTx_n = fA$$

for some $t \in X$. And by using compatibility of pairs (f,T), we have

$$\lim_{n\to\infty} H_G(Tfx_n, fTx_n, fTx_n) = 0,$$

that is, $fA \subseteq Tt$. The rest of proof follows on the similar lines as in Theorem 3.1.

Setting $S = T$ in Theorem 3.1 and Theorem 3.2, resulting the following corollary.

Corollary 22 *Let f and g be self-mappings of a G-metric space (X,G) and T be a mapping from X into $CB(X)$ satisfying the following conditions:*

(1) For all $x,y \in X$, $\phi \in \Phi$,

$$\begin{aligned}\phi(&\psi(H_G(Tx,Ty,Ty)), \psi(G(fx,gy,Ty)),\\ &\psi(G(fx,Tx,Tx)), \psi(G(gy,Ty,Ty)),\\ &\psi(G(fx,Ty,Ty)), \psi(G(Tx,gy,Ty))) \leq 0.\end{aligned}$$

(2) The pairs (f,T) is reciprocally continuous and subcompatible (Alternately subsequentiallly continuous and compatible).

(3) The pairs (g,T) is occasionally weakly compatible.

Then f,g and T have a unique common fixed point in X.

In view of enrichment by Imdad et al. [4] in the setting of single-valued mappings, next theorem is presented for hybrid pair of mappings.

Theorem 23. *Let $f,g : X \to X$ and $S,T : X \to CB(X)$ be the mappings in G-metric space (X,G). If pairs (f,T) and (g,S) are compatible and subsequentially continuous (Alternately subcompatible and reciprocally continuous), then*

(1) f and T have a coincidence point.

(2) g and S have a coincidence point.

Furthermore, suppose that for all $x,y \in X$ and $\phi \in \Phi$,

$$\begin{aligned}\phi(&\psi(H_G(Tx,Sy,Sy)), \psi(G(fx,gy,Sy)),\\ &\psi(G(fx,Tx,Tx)), \psi(G(gy,Sy,Sy)),\\ &\psi(G(fx,Sy,Sy)), \psi(G(Tx,gy,Sy))) \leq 0.\end{aligned} \tag{5}$$

Then f,g,S and T have a unique common fixed point in X.

Proof. Case I. Suppose that the pair (f,T) (also (g,S)) is subsequentially continuous and compatible. Then there exists a sequence $\{x_n\}$ in X such that for some $t \in X$,

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$$

and

$$\lim_{n\to\infty} Tfx_n = Tt,\ \lim_{n\to\infty} fTx_n = fA.$$

By using compatibility of pairs (f,T), we have

$$\lim_{n\to\infty} H_G(Tfx_n, fTx_n, fTx_n) = 0,$$

and so, $H_G(Tt, fA, fA) = 0$. It implies that $fA \subseteq Tt$. If $t \in A$, then $ft \in Tt$. This means that t is a coincidence point of (f,T).

Whereas with respect to pair (g,S), there exists a sequence $\{y_n\}$ in X such that for some $z \in X$,

$$\lim_{n\to\infty} gy_n = z \in B = \lim_{n\to\infty} Sy_n$$

and

$$\lim_{n\to\infty} Sgy_n = Sz,\ \lim_{n\to\infty} gSy_n = gB.$$

Also compatibility of pair (g,S) yeilds

$$\lim_{n\to\infty} H_G(Sgy_n, gSy_n, gSy_n) = 0,$$

it implies that $H_G(Sz,gB,gB) = 0$. Hence we have $gB \subseteq Sz$ and also $gz \in Sz$. Which leads that z is a coincidence point of (g,S).

Now we prove that both the coincidence point z and t of pairs (g,S) and (f,T) respectively, are equal i.e. $z = t$.

By using inequality (5), one yields

$$\begin{aligned}\phi(&\psi(H_G(Tx_n,Sy_n,Sy_n)), \psi(G(fx_n,gy_n,Sy_n)),\\ &\psi(G(fx_n,Tx_n,Tx_n)), \psi(G(gy_n,Sy_n,Sy_n)),\\ &\psi(G(fx_n,Sy_n,Sy_n)), \psi(G(Tx_n,gy_n,Sy_n))) \leq 0.\end{aligned}$$

Letting $n \to \infty$ and by the definition of H and ϕ, we have

$$\begin{aligned}\phi(\psi(G(t,z,z)), \psi(G(t,z,z)), \psi(G(t,t,t)), \psi(G(z,z,z)),\\ \psi(G(t,z,z)), \psi(G(t,z,z))) \leq 0.\end{aligned}$$

Hence, we have
$\phi(\psi(G(t,z,z)), \psi(G(t,z,z)), 0, 0, \psi(G(t,z,z)), \psi(G(t,z,z))) \leq 0,$
and so, $\psi(G(t,z,z)) = 0$. It implies that

$$G(t,z,z) = 0 \text{ and } t = z.$$

Now we claim that $ft = t$. On the contrary, suppose that $ft \neq t$, then by inequality (5) with $x = t$, $y = y_n$, we obtain

$$\begin{aligned}\phi(&\psi(H_G(Tt,Sy_n,Sy_n)), \psi(G(ft,gy_n,Sy_n)),\\ &\psi(G(ft,Tt,Tt)), \psi(G(gy_n,Sy_n,Sy_n)),\\ &\psi(G(ft,Sy_n,Sy_n)), \psi(G(Tt,gy_n,Sy_n))) \leq 0.\end{aligned}$$

In view of ϕ, H and Remark 7, one gets

$$\begin{aligned}\phi(\psi(H_G(ft,z,z)), \psi(G(ft,z,z)), 0, 0, \psi(G(ft,z,z)),\\ \psi(G(ft,z,z))) \leq 0.\end{aligned}$$

So that

$$\psi(G(ft,z,z)) = 0.$$

It implies that $G(ft,z,z)=0$ or $G(ft,t,t)=0$. This yields that $ft=t\in Tt$.
Again suppose that $gt\neq t$, then by using inequality (5), we obtain

$$\begin{aligned}\phi(&\psi(H_G(Tt,St,St)),\psi(G(ft,gt,St)),\\ &\psi(G(ft,Tt,Tt)),\psi(G(gt,St,St)),\\ &\psi(G(ft,St,St)),\psi(G(Tt,gt,St)))\leq 0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\phi(\psi(G(t,gt,gt)),\psi(G(t,gt,gt)),0,0,\psi(G(t,gt,gt)),\\ \psi(G(t,gt,gt)))\leq 0.\end{aligned}$$

This implies that

$$\psi(G(t,gt,gt))=0 \text{ and so } G(t,gt,gt)=0.$$

Which leads that $gt=t\in St$. Therefore t is a common fixed point of f,g,S and T.

Uniqueness is an easy consequence of inequality (5).

Case II. Suppose that pair (f,T) (also (g,S) is subcompatible and reciprocally continuous. Then there exists a sequence $\{x_n\}$ in X such that for some $t\in X$,

$$\lim_{n\to\infty} fx_n=t\in A=\lim_{n\to\infty} Tx_n$$

and

$$\lim_{n\to\infty} H_G(fTx_n,Tfx_n,Tfx_n)=0.$$

Hence, we have

$$H_G(fA,Tt,Tt)=0.$$

This implies that $fA\subseteq Tt$ and $ft\in Tt$. This mean that t is a coincidence point of (f,T).

In respect to pair (g,S), there exists a sequence $\{y_n\}$ in X such that for some $z\in X$

$$\lim_{n\to\infty} gy_n=z\in B=\lim_{n\to\infty} Sy_n$$

and

$$\lim_{n\to\infty} H_G(gSy_n,Sgy_n,Sgy_n)=0.$$

Hence, we have

$$H_G(gB,Sz,Sz)=0.$$

This implies that $gB\subseteq Sz$ and $gz\in Sz$. This means that z is a coincidence point of (g,S). The rest of the proof can be completed on the similar lines of Case I.

Now we furnish an illustrative example to highlight the validity of Theorem 3.2 and Theorem 3.3 (Case I).

Example 24 Consider $X=[0,\infty)$ equipped with the $G-$metric defined by

$$G(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\},$$

and define $f,g:X\to X$ and $T,S:X\to CB(X)$ as follows:

$$fx=gx=\begin{cases}1, & if\ x<3,\\ x, & if\ 3\leq x\end{cases}$$

and

$$Tx=Sx=\begin{cases}[0,\frac{x}{4}] & ,if\ x<3,\\ \{3\} & ,if\ 3\leq x.\end{cases}$$

Now consider the sequence $\{x_n\}=\{3+\frac{1}{n}\}$ in X. Then, we have

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} f(3+\frac{1}{n})=\lim_{n\to\infty}(3+\frac{1}{n})=3=:t$$

and

$$\lim_{n\to\infty} Tx_n=\lim_{n\to\infty} T(3+\frac{1}{n})=\lim_{n\to\infty}\{3\}=\{3\}=:A.$$

Therefore,

$$\lim_{n\to\infty} fx_n=3=t\in\{3\}=A=\lim_{n\to\infty} Tx_n. \tag{6}$$

Also, we have

$$\lim_{n\to\infty} Tfx_n=\lim_{n\to\infty} T(3+\frac{1}{n})=\{3\}=T\{3\}=T(t)$$

and

$$\lim_{n\to\infty} fTx_n=\lim_{n\to\infty} f(\{3\})=\{3\}=f\{3\}=f(A).$$

Then f and T (also g and S) are subsequentially continuous mappings. Again with (6), f and T satisfy

$$\lim_{n\to\infty} H_G(Tfx_n,fTx_n,fTx_n)=0.$$

Hence f and T (also g and S) are compatible mappings.

Here pair (g,S) is occasionally weakly compatible for coincidence point $x=3$, and also we have $gSx=\{3\}\subseteq\{3\}=Sgx$.

Now, in order to check the contractive condition (3.1), we define the altering distance function

$$\psi(t)=t \tag{7}$$

while $\phi\in\Phi$ is given by

$$\phi(t_1,t_2,\cdots,t_6)=t_1-k\max\{t_2+t_3,t_3+t_4,t_4+t_5,t_3+t_6\},\quad \text{where}\quad k\in(0,1). \tag{8}$$

Invoking

(7) and (8) to our contractive condition. Now, we have to verify

$$\begin{aligned}H_G(Tx,Sy,Sy)-k\max\{&G(fx,gy,Sy)+G(fx,Tx,Tx),\\ &G(fx,Tx,Tx)+G(gy,Sy,Sy),\\ &G(gy,Sy,Sy)+G(fx,Sy,Sy),\\ &G(fx,Tx,Tx)+G(Tx,gy,Sy)\}\leq 0\end{aligned}$$

or

$$H_G(Tx,Sy,Sy) \le k\max\{G(fx,gy,Sy)+G(fx,Tx,Tx), G(fx,Tx,Tx)+G(gy,Sy,Sy), G(gy,Sy,Sy)+G(fx,Sy,Sy), G(fx,Tx,Tx)+G(Tx,gy,Sy)\}, \quad (9)$$

where $k \in (0,1)$. In order to verify (9), it is sufficient to show that

$$H_G(Tx,Sy,Sy) \le k\ [G(fx,Tx,Tx)+G(Tx,gy,Sy)], \quad (10)$$

where $k \in (0,1)$. Without loss of generality, we assume that $0 \le x \le y$. Also, we have

$$d_G(x,y) = 2|x-y|, \ \ \forall x,y \in X.$$

Consider the following possible cases:

Case I. When $0 \le x \le y < 3$. Then, we have

$$\begin{aligned} H_G(Tx,Sy,Sy) &= H_G\Big(\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big) \\ &= \max\Big[\sup_{a\in[0,\frac{x}{4}]} G\Big(a,\Big[0,\frac{y}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big), \\ &\quad \sup_{b\in[0,\frac{y}{4}]} G\Big(b,\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big)\Big]. \end{aligned}$$

Since $x \le y$ so $[0,\frac{x}{4}] \subseteq [0,\frac{y}{4}]$, this implies that

$$d_G\Big(\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big) = 0 = \underset{a\in[0,\frac{x}{4}]}{d_G}\Big(a,\Big[0,\frac{y}{4}\Big]\Big).$$

Then, we have

$$G(a,[0,\frac{y}{4}],[0,\frac{y}{4}]) = 0.$$

Now

$$\begin{aligned} G\Big(b,\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big) &= d_G\Big(b,\Big[0,\frac{x}{4}\Big]\Big)+ \\ &\quad d_G\Big(b,\Big[0,\frac{y}{4}\Big]\Big)+d_G\Big(\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big) \\ &= \begin{cases} 0, & if\ b \le \frac{x}{4}, \\ 2b-\frac{x}{2}, & if\ b > \frac{x}{4}. \end{cases} \end{aligned}$$

This implies that

$$\sup_{b\in[0,\frac{y}{4}]} G\Big(b,\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big) = \frac{y}{2}-\frac{x}{2}.$$

Finally, we have $H_G(Tx,Sy,Sy) = \frac{y}{2}-\frac{x}{2}$.

Now taking

$$\begin{aligned} G(fx,Tx,Tx) &= G\Big(1,\Big[0,\frac{x}{4}\Big],\Big[0,\frac{x}{4}\Big]\Big) \\ &= 2d_G\Big(1,\Big[0,\frac{x}{4}\Big]\Big) \\ &= 4\Big(1-\frac{x}{4}\Big) \\ &= 4-x \end{aligned}$$

and

$$\begin{aligned} G(Tx,gy,Sy) &= G\Big(\Big[0,\frac{x}{4}\Big],1,\Big[0,\frac{y}{4}\Big]\Big) \\ &= d_G\Big(1,\Big[0,\frac{x}{4}\Big]\Big)+d_G\Big(1,\Big[0,\frac{y}{4}\Big]\Big) \\ &\quad + d_G\Big(\Big[0,\frac{x}{4}\Big],\Big[0,\frac{y}{4}\Big]\Big) \\ &= 2\Big(1-\frac{x}{4}\Big)+2\Big(1-\frac{y}{4}\Big) \\ &= 4-\frac{x}{2}-\frac{y}{2}, \end{aligned}$$

then, we have

$$\frac{y}{2}-\frac{x}{2} \le k(8-\frac{y}{2}-\frac{3}{2}x), \quad (11)$$

for all $0 \le x \le y < 3$ and $k = 0.9$, which is demonstrated by the following figure,

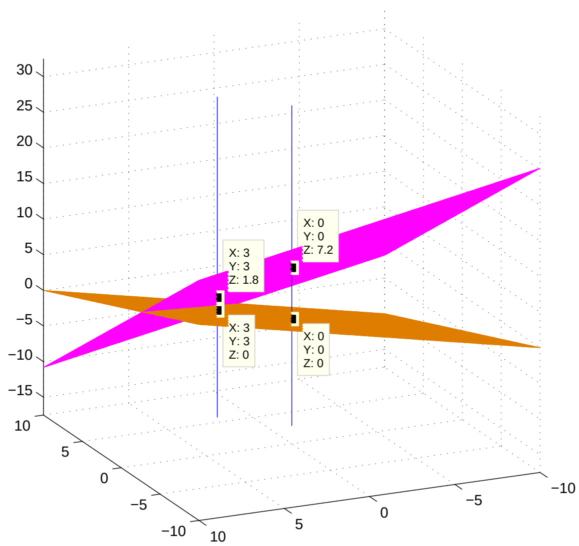

Fig. 1: case I

In above figure, plane with brown color represents the L.H.S. of (11) and plane with purple color represents the R. H. S. of (11). Clearly purple plane is dominating the brown plane within the lines representing our range *i.e.*$0 \le x \le y < 3$. consequently, in this case, condition (10) is satisfied.

Case II. When $3 \le x \le y$. Then, we have

$$H_G(Tx,Sy,Sy) = H_G(\{3\},\{3\},\{3\}) = 0$$

and

$$G(Tx,gy,Sy) = G(\{3\},y,\{3\}) = 2d_G(y,\{3\}) = 0,$$

$$G(fx,Tx,Tx) = G(x,\{3\},\{3\}) = 2d_G(x,\{3\}) = 0.$$

Thus (10) is satisfied for $k = 0.9$.

Case III. When $0 \le x < 3 \le y$. Then, we have

$$\begin{aligned}
H_G(Tx,Sy,Sy) &= H\Big(\Big[0,\frac{x}{4}\Big],\{3\},\{3\}\Big)\\
&= \max\Big[\sup_{a\in[0,\frac{x}{4}]} G\Big(a,\{3\},\{3\}\Big), \sup_{b\in\{3\}} G\Big(b,\{3\},\Big[0,\frac{x}{4}\Big]\Big)\Big]\\
&= \max\Big[\sup_{a\in[0,\frac{x}{4}]} 2d_G\Big(a,\{3\}\Big), \sup_{b\in\{3\}} \Big(d_G\Big(b,\Big[0,\frac{x}{4}\Big]\Big)\\
&\quad + d_G\Big(\{3\},\Big[0,\frac{x}{4}\Big]\Big)\Big)\Big]\\
&= \max\Big[4\Big(3-\frac{x}{4}\Big),4\Big(3-\frac{x}{4}\Big)\Big]\\
&= 12 - x
\end{aligned}$$

and

$$\begin{aligned}
G(fx,Tx,Tx) &= G\Big(1,\Big[0,\frac{x}{4}\Big],\Big[0,\frac{x}{4}\Big]\Big)\\
&= 2d_G\Big(1,\Big[0,\frac{x}{4}\Big]\Big)\\
&= 4\Big(1-\frac{x}{4}\Big)\\
&= 4 - x,
\end{aligned}$$

$$\begin{aligned}
G(Tx,gy,Sy) &= G\Big(\Big[0,\frac{x}{4}\Big],y,\{3\}\Big)\\
&= d_G\Big(\Big[0,\frac{x}{4}\Big],y\Big) + d_G\Big(y,\{3\}\Big) + d_G\Big(\Big[0,\frac{x}{4}\Big],\{3\}\Big)\\
&= 2\Big(y-\frac{x}{4}\Big) + 0 + 2\Big(3-\frac{x}{4}\Big)\\
&= 6 + 2y - \frac{x}{2} - \frac{x}{2}\\
&= 6 + 2y - x.
\end{aligned}$$

Thus, obviously by the following figure, we have

$$12 - x \le k(10 + 2y - 2x), \tag{12}$$

for all $0 \le x < 3 \le y$ and for $k = 0.9$.

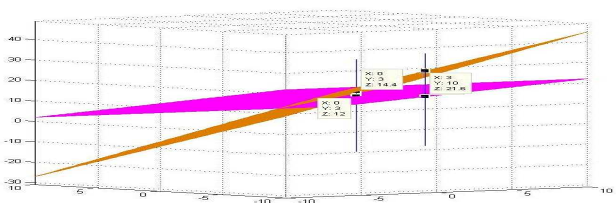

Fig. 2: case III

In presented figure, plane with purple color represents the L.H.S. of (12) and plane with brown color represents the R. H. S. of (12). Evidently brown plane is superimposing the purple plane for $0 \le x < 3 \le y$.

Therefore, we have

$$H_G(Tx,Sy,Sy) \le k[G(fx,Tx,Tx) + G(Tx,gy,Sy)],$$

for all $x,y \in X$ and $k = 0.9 \in (0,1)$. Thus all the conditions of Theorem 21 and Theorem 23(Case I) are satisfied and 3 is a unique common fixed point of pairs (f,T) and (g,S), which is demonstrated by following figure.

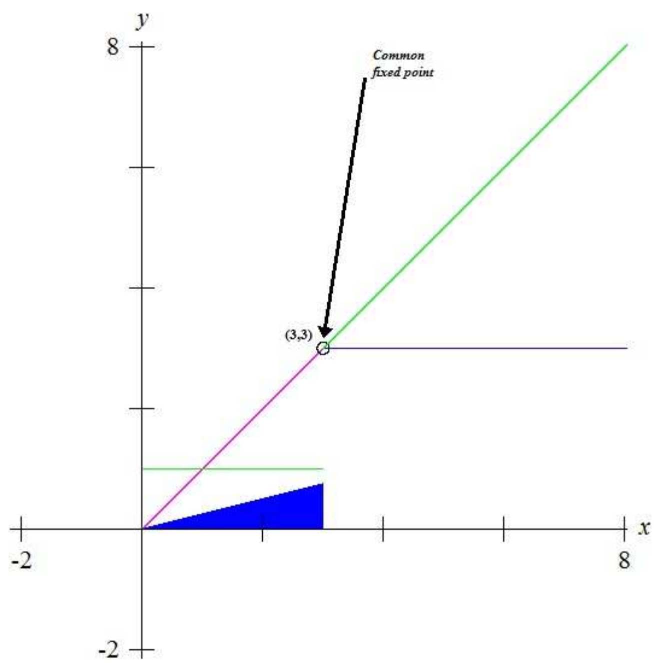

Fig. 3: fixed point

In above figure, lines with green color represent function $fx(=gx)$, blue color represents the multivalued function $Tx(=Sx)$ and purple line represents $y = x$ for fixed point purpose. Clearly, we can see that functions f and T intersect on the line $y = x$ only at $x = 3$, this amounts to say that $x = 3$ is the unique common fixed point of $f(=g)$ and $T(=S)$.

Setting $S = T$ in Theorem 23, we get the following result.

Corollary 25 *Let $f,g : X \to X$ and $T : X \to CB(X)$ be the mappings of G-metric space (X,G). If pairs (f,T) and (g,T) are compatible and subsequentially continuous(Alternately subcompatible and reciprocally continuous), then*

(1) f and T have a coincidence point.

(2) g and T have a coincidence point.

Furthermore, suppose that for all $x,y \in X$ and $\phi \in \Phi$,

$$\begin{aligned}&\phi(\psi(H_G(Tx,Ty,Ty)),\psi(G(fx,gy,Ty)),\\ &\psi(G(fx,Tx,Tx)),\psi(G(gy,Ty,Ty)),\\ &\psi(G(fx,Ty,Ty)),\psi(G(Tx,gy,Ty))) \leq 0.\end{aligned}$$

Then f,g and T have a unique common fixed point in X.

Restricting Theorem 23 to a hybrid pair of mappings (f,T) by setting $f=g$ and $S=T$, then we deduce the following natural result.

Corollary 26 *Let $f : X \to X$ and $T : X \to CB(X)$ be the mappings of G-metric space (X,G). If pair (f,T) is compatible and subsequentially continuous(Alternately subcompatible and reciprocally continuous), then f and T have a coincidence point. Furthermore, suppose that*

$$\begin{aligned}&\phi(\psi(H_G(Tx,Ty,Ty)),\psi(G(fx,fy,Ty)),\\ &\psi(G(fx,Tx,Tx)),\psi(G(fy,Ty,Ty)),\\ &\psi(G(fx,Ty,Ty)),\psi(G(Tx,fy,Ty))) \leq 0,\end{aligned}$$

for all $x,y \in X$ and $\phi \in \Phi$. Then f and T have a unique common fixed point in X.

Invoking to examples in Example 2.1 of implicit function, one can have the following unified corollary covering several new results in the context of G-metric spaces for two hybrid pairs of mappings.

Corollary 27 *The conclusion of Theorem 3.1, Theorem 3.2 and Theorem 3.3 remain true if inequality (1) is replaced by one of the following contraction conditions:*

For all $x,y \in X$ and some $\psi \in \Psi$,

(1) For $a,b,c \geq 0$, $a+2c<1$,

$$\begin{aligned}&\psi(H_G(Tx,Sy,Sy)) \leq a\psi(G(fx,gy,Sy))+\\ &b(\psi(G(fx,Tx,Tx))+\psi(G(gy,Sy,Sy)))+\\ &c(\psi(G(fx,Sy,Sy))+\psi(G(Tx,gy,Sy))),\end{aligned}$$

(2) For $k \in (0,1)$,

$$\begin{aligned}\psi(H_G(Tx,Sy,Sy)) \leq k\max\{&\psi(G(fx,gy,Sy)),\\ &\psi(G(fx,Tx,Tx)),\psi(G(gy,Sy,Sy)),\\ &\psi(G(fx,Sy,Sy)),\psi(G(Tx,gy,Sy))\},\end{aligned}$$

(3) For $k \in (0,1)$,

$$\begin{aligned}\psi(H_G(Tx,Sy,Sy)) \leq k\max\{&\psi(G(fx,gy,Sy)),\\ &\psi(G(fx,Tx,Tx)),\psi(G(gy,Sy,Sy)),\\ &\frac{\psi(G(fx,Sy,Sy))+\psi(G(Tx,gy,Sy))}{2}\},\end{aligned}$$

(4) For $a,b \geq 0$, $a+b<1$,

$$\begin{aligned}\psi(H_G(Tx,Sy,Sy)) \leq a\max\{&\frac{1}{2}\psi(G(fx,gy,Sy))\\ &+\psi(G(fx,Tx,Tx)),\frac{1}{2}\psi(G(gy,Sy,Sy))\\ &+\psi(G(fx,Sy,Sy))\}+b\psi(G(Tx,gy,Sy)),\end{aligned}$$

(5) For $a,b,c \geq 0$, $a+c<1$,

$$\begin{aligned}\psi(H_G(Tx,Sy,Sy)) &\leq a\psi(G(fx,gy,Sy))\\ &+b(\psi(G(fx,Tx,Tx))+\psi(G(gy,Sy,Sy)))+\\ &c\sqrt{\psi(G(fx,Sy,Sy)).\psi(G(Tx,gy,Sy))},\end{aligned}$$

(6) For $a,b,c \geq 0$, $a+c<1$.

$$\begin{aligned}&\psi(H_G(Tx,Sy,Sy)) \leq a\psi(G(fx,gy,Sy))\\ &+b\frac{[\psi(G(fx,Tx,Tx))]^2+[\psi(G(gy,Sy,Sy))]^2}{1+\psi(G(fx,Tx,Tx))+\psi(G(gy,Sy,Sy))}\\ &+c\max\{\psi(G(fx,Sy,Sy)),\psi(G(Tx,gy,Sy))\},\end{aligned}$$

(7) For $a,b,c \geq 0$, $a+b+c<1$,

$$\begin{aligned}&\psi(H_G(Tx,Sy,Sy)) \leq a\psi(G(fx,gy,Sy))\\ &+b\max\{\psi(G(fx,Tx,Tx)),\psi(G(fx,Sy,Sy))\}\\ &+c\max\{\psi(G(gy,Sy,Sy)),\psi(G(Tx,gy,Sy))\},\end{aligned}$$

(8) For $a,b \geq 0$, $b<1$,
$[\psi(H_G(Tx,Sy,Sy))]^2 \leq$
$a\frac{[\psi(G(fx,gy,Sy))]^2[\psi(G(fx,Tx,Tx))]^2+[\psi(G(gy,Sy,Sy))]^2[\psi(G(fx,Sy,Sy))]^2}{1+\psi(G(fx,Tx,Tx))}$
$+b[\psi(G(Tx,gy,Sy))]^2$,

(9) For $k \in (0,1)$,

$$\begin{aligned}H_G(Tx,Sy,Sy) \leq k\max\{&G(fx,gy,Sy)+G(fx,Tx,Tx),\\ &G(fx,Tx,Tx)+G(gy,Sy,Sy),\\ &G(gy,Sy,Sy)+G(fx,Sy,Sy),\\ &G(fx,Tx,Tx)+G(Tx,gy,Sy)\},\end{aligned}$$

Proof: The conclusion follows from Theorem 20, Theorem 21 and Theorem 23 in view of Example 2.1.

Remark 28 Corollaries corresponding to condition (1) to (9) are new results for hybrid mappings in context of G-metric spaces.

On setting $\psi(t)=t$ in above mentioned theorems involving altering distance, one can get some natural and new results in the setting of common fixed point of two pairs of hybrid mappings.

For the sake of simplicity, we only derive the following corollary by putting $\psi(t)=t$ in Theorem 20.

Corollary 29 *Let $f,g : X \to X$ and $S,T : X \to CB(X)$ be mappings of G-metric space (X,G) satisfying the following conditions.*

(1) For all $x,y \in X$, $\phi \in \Phi$,

$$\begin{aligned}&\phi(H_G(Tx,Sy,Sy),G(fx,gy,Sy),\\ &G(fx,Tx,Tx),G(gy,Sy,Sy),\\ &G(fx,Sy,Sy),G(Tx,gy,Sy)) \leq 0.\end{aligned} \tag{13}$$

(2) The pairs (f,T) is reciprocally continuous and sub compatible.

(3) The pairs (g,S) is non-vacuously weakly compatible.

Then f,g,S and T have a unique common fixed point.

4 Conclusion

In this paper, we have introduced the notion of sub-compability for hybrid pairs of mappings in the setting of G- metric spaces. Utilizing this introduced concepts, certain fixed point results are established in the structure of G- metric spaces. During the process, we utilized the points given in E. Karapinar et al. [5] so that our results can not be reduced to any results from existing literature in view of points raised in Jleli et al. [6]. Validity of our results is well demonstrated by examples containing some innovative graphs and figures.

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