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Combined Laplace Transform-Homotopy Perturbation Method for Sine-Gordon Equation

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Abstract: In this paper, the combined Laplace transform-homotopy perturbation method C(LT-HPM) is presented and used to solve the initial value problem for the sine-Gordon equation to obtain the approximate-exact solutions. The results obtained show the reliability and the efficiency of this method.

Keywords: Homotopy perturbation method; Laplace transform method; Sine-Gordon equation; Adomian's polynomials; He's polynomials

1 Introduction

The sine-Gordon equation firstly appeared in the study of the differential geometry of surfaces with Gaussian curvature $K = -1$ has wide applications in the propagation of fluxons in Josephson junctions between two superconductors [1,2,3,4] the motion of a rigid pendulum attached to a stretched wire [4], solid state physics, nonlinear optics, stability of fluid motions, dislocations in crystals [4] and other scientific fields.

In this work, we consider the sine-Gordon equation suggested by Ablowitz et al. in [5], i.e.

$$u_{tt} - u_{xx} + \sin(u) = 0,\ x \in \mathbb{R},\ 0 \le t \le 1 \tag{1}$$

subject to the initial conditions

$$u(x,0) = g_1(x),\ u_t(x,0) = g_2(x), \tag{2}$$

where the subscripts denote the differentiation of u with respect to x and t.

The sine-Gordon equation have been studied in many works [6,7,8,9,10,11,12] to give the approximate solution. Kaya [6] and Wang [7] solved it using modified Adomian decomposition method (MADM). Batiha et al. [8] used variational iteration method (VIM) to solve it. Using homotopy perturbation method (HPM) for solving it were given by Chowdhury and Hashim [9]. Junfeng Lu [10] applied modified homotopy perturbation method (MHPM) to solve the same problem. Also Ugur Yücel [11] used the homotopy analysis method (HAM) for solving the previous mentioned problem. Wazwaz [12] used tanh method for handling the sine-Gordon equation.

The homotopy perturbation method (HPM) proposed by Ji-Huan He in 1998 for addressing nonlinear problems [13,14]. This method has been applied to different linear and nonlinear problems [15,16,17,18,19,20,21,22] The advantage of this method is its capability of combining two powerful methods (namely, Laplace transform and homotopy perturbation method) for obtaining approximate-exact solutions for nonlinear equations.

The main objective of this paper is to use the combined Laplace transform-homotopy perturbation method C(LT-HPM) for solving the initial value problem for the sine-Gordon equation to obtain the approximate-exact solutions.

2 The C(LT-HPM)

In an operator form, Eq. (1) can be written as

$$L_t u - L_x u + Nu = 0, \tag{3}$$

where the differential operators L_t and L_x are defined by

$$L_t = \frac{\partial^2}{\partial t^2},\ \ L_x = \frac{\partial^2}{\partial x^2}.$$

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and $Nu = \sin(u)$ is the nonlinear operator. Taking the Laplace transform $\mathscr{L}$ on both sides of Eq. (3):

$$\mathscr{L}[L_t u] - \mathscr{L}[L_x u] + \mathscr{L}[Nu] = 0. \tag{4}$$

Using the differentiation property of the Laplace transform, we have

$$U(x,s) = \frac{g_1(x)}{s} + \frac{g_2(x)}{s^2} + \frac{1}{s^2}\mathscr{L}[L_x u] - \frac{1}{s^2}\mathscr{L}[Nu], \tag{5}$$

where $U(x,s) = \mathscr{L}[u(x,t)]$. Operating with the Laplace inverse on both sides of Eq. (5) gives

$$u(x,t) = \mathscr{L}^{-1}\left\{\frac{g_1(x)}{s} + \frac{g_2(x)}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}[L_x u] - \frac{1}{s^2}\mathscr{L}[Nu]\right\}, \tag{6}$$

where $u(x,t) = \mathscr{L}^{-1}[U(x,s)](t)$.

Applying the homotopy perturbation method [13,14,15,16]

$$u = \sum_{n=0}^{\infty} p^n u_n, \tag{7}$$

and the nonlinear term Nu can be decomposed as

$$Nu = \sum_{n=0}^{\infty} p^n H_n(u), \tag{8}$$

where the H_n are He's polynomials of $u_0, u_1, \ldots, u_n$ and are calculated by the definitional formula [23,24]

$$H_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \; n = 0,1,2,\ldots .$$

where $p \in [0,1]$ is an embedding parameter. Setting $p = 1$ results in the approximate solution of Eq. (1)

$$u = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + \cdots .$$

In order to obtain the approximate solution of Eq. (1), we consider the Taylor series expansion of $\sin(u)$ in the following form:

$$\begin{aligned}
\sin(u) &= u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \cdots + \frac{(-1)^{n-1}}{(2n-1)!}u^{2n-1} + \cdots \\
&= \sum_{n=0}^{\infty} p^n u_n - \frac{1}{3!}\left(\sum_{n=0}^{\infty} p^n u_n\right)^3 + \frac{1}{5!}\left(\sum_{n=0}^{\infty} p^n u_n\right)^5 - \cdots \\
&= \sum_{n=0}^{\infty} p^n u_n - \frac{1}{3!}\sum_{n=0}^{\infty} p^n A_n(u) + \frac{1}{5!}\sum_{n=0}^{\infty} p^n B_n(u) - \cdots \\
&= \sum_{n=0}^{\infty} p^n H_n(u),
\end{aligned} \tag{9}$$

where A_n and B_n are Adomian polynomials [25,26] given by

$$\begin{aligned}
A_0 &= u_0^3, \\
A_1 &= 3u_0^2 u_1, \\
A_2 &= 3u_0^2 u_2 + 3u_1^2 u_0, \\
A_3 &= u_1^3 + 3u_0^2 u_3 + 6u_0 u_1 u_2, \\
A_4 &= 3u_0^2 u_4 + 3u_1^2 u_2 + 3u_2^2 u_0 + 6u_0 u_1 u_3, \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
B_0 &= u_0^5, \\
B_1 &= 5u_0^4 u_1, \\
B_2 &= 5u_0^4 u_2 + 10u_0^3 u_1^2, \\
B_3 &= 5u_0^4 u_3 + 20u_0^3 u_1 u_2 + 10u_0^2 u_1^3, \\
B_4 &= 5u_0^4 u_4 + 5u_1^4 u_0 + 10u_0^3 u_2^2 + 20u_0^3 u_1 u_3 + 30u_0^2 u_1^2 u_2, \\
&\vdots
\end{aligned}$$

and H_n are He's polynomials given by

$$\begin{aligned}
H_0(u) &= u_0 - \frac{1}{3!}A_0 + \frac{1}{5!}B_0 + \cdots \\
H_1(u) &= u_1 - \frac{1}{3!}A_1 + \frac{1}{5!}B_1 + \cdots \\
H_2(u) &= u_2 - \frac{1}{3!}A_2 + \frac{1}{5!}B_2 + \cdots \\
H_3(u) &= u_3 - \frac{1}{3!}A_3 + \frac{1}{5!}B_3 + \cdots \\
H_4(u) &= u_4 - \frac{1}{3!}A_4 + \frac{1}{5!}B_4 + \cdots \\
&\vdots
\end{aligned}$$

Substituting Eqs. (7), (8) and (9) in Eq. (6) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \mathscr{L}^{-1}\{\frac{g_1(x)}{s} + \frac{g_2(x)}{s^2}\} + p\mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[L_x\sum_{n=0}^{\infty} p^n u_n - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right\}$$

which is the combaination of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of

p, the following approximations are obtained

$$
\begin{aligned}
p^0 &: u_0(x,t) = \mathscr{L}^{-1}\left\{\frac{g_1(x)}{s} + \frac{g_2(x)}{s^2}\right\},\\
p^1 &: u_1(x,t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[L_x u_0 - H_0(u)\right]\right\},\\
p^2 &: u_2(x,t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[L_x u_1 - H_1(u)\right]\right\},\\
p^3 &: u_3(x,t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[L_x u_2 - H_2(u)\right]\right\},\\
p^4 &: u_4(x,t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[L_x u_3 - H_3(u)\right]\right\},\\
&\vdots
\end{aligned}
\tag{10}
$$

3 Application

In this section, we apply the C(LT-HPM) for finding the approximate-exact solutions of two initial value problems examples associated with the sine-Gordon equation. To show the high accuracy of the solution results compared with the exact solution, we give the numerical results and the maximum absolute error. The computations associated with the examples were performed using a Maple 13 package with a precision of 20 dígits.

Example 1 Firstly, let us consider the sine-Gordon equation (1) subject to the initial conditions

$$u(x,0) = 0,\ \ u_t(x,0) = 4sechx. \tag{11}$$

Following the algorithm (10), the first few components are given by

$$
\begin{aligned}
p^0 &: \ u_0(x,t) = 4tsechx,\\
p^1 &: \ u_1(x,t) = -\frac{4}{3}t^3 sech^3 x + \frac{8}{15}t^5 sech^3 x - \frac{64}{315}t^7 sech^5 x\\
p^2 &: \ u_2(x,t) = \frac{4}{5}t^5 sech^5 x - \frac{8}{15}t^5 sech^3 x - \frac{128}{315}t^7 sech^5 x + \frac{32}{315}t^7 sech^3 x\\
&\qquad - \frac{8}{945}t^9 sech^5 x + \frac{160}{567}t^9 sech^7 x - \frac{128}{1925}t^{11} sech^7 x + \frac{512}{36855}t^{13} sech^9 x,\\
p^3 &: \ u_3(x,t) = -\frac{4}{7}t^7 sech^7 x + \text{noise terms}\\
p^4 &: \ u_4(x,t) = \frac{4}{9}t^9 sech^9 x + \text{noise terms}\\
&\vdots
\end{aligned}
$$

Thus the approximate solution in a series form is given by

$$
\begin{aligned}
u_{Approx}(x,t) &= \sum_{i=0}^{n} u_i(x,t)\\
&= 4\left(tsechx - \frac{1}{3}t^3 sech^3 x + \frac{1}{5}t^5 sech^5 x - \frac{1}{7}t^7 sech^7 x + \frac{1}{9}t^9 sech^9 x - \cdots\right)\\
&\quad + \text{noise terms.}
\end{aligned}
$$

This series has the closed form as $n \to \infty$

$$u_{Exact}(x,t) = 4\tan^{-1}(tsechx),$$

which is the exact solution of the problem (1) subject to the initial conditions (11). Notice that the noise terms that appear between various components vanish in the limit.

In Tables 1 and 2, we present the absolute errors between the exact solution and 5-term MADM, 2-iterate VIM, 3-term HPM [9] and 3-term C(LT-HPM). But in Table 3, we present the absolute errors between the exact solution and 4-term C(LT-HPM).

In figures 1 and 2 we show a very good agreement between the exact solution and 3-term of approximate solution C(LT-HPM). In figure 3, we present the absolute errors between the exact solution and 4-term of approximate solution C(LT-HPM).

Example 2 Secondly, we consider the sine-Gordon equation (1) subject to the initial conditions

$$u(x,0) = \pi + \varepsilon\cos(\mu x),\ u_t(x,0) = 0. \tag{12}$$

where $\mu = \dfrac{\sqrt{2}}{2}$ and ε is a constant.

Again, using the algorithm (10), the first few components are given by

$$
\begin{aligned}
p^0 &: \ u_0(x,t) = \pi + \varepsilon\cos(\mu x),\\
p^1 &: \ u_1(x,t) = \frac{1}{2}[-\varepsilon\mu^2\cos(\mu x) + \frac{1}{3!}(\pi + \varepsilon\cos(\mu x))^3 - \frac{1}{5!}(\pi + \varepsilon\cos(\mu x))^5\\
&\qquad -\pi - \varepsilon\cos(\mu x)]t^2,\\
p^2 &: \ u_2(x,t) = \frac{1}{69120}[\varepsilon^9\cos^9(\mu x) + 9\pi\varepsilon^8\cos^8(\mu x) + \left(36\pi^2 - 32\right)\varepsilon^7\cos^7(\mu x)\\
&\qquad + \left(84\pi^3 - 224\pi\right)\varepsilon^6\cos^6(\mu x) + \left(126\pi^4 - 672\pi^2 + 720\mu^2 + 384\right)\varepsilon^5\cos^5(\mu x)\\
&\qquad + \left(126\pi^5 - 1120\pi^3 + 2400\pi\mu^2 + 1920\pi\right)\varepsilon^4\cos^4(\mu x)\\
&\qquad + \left\{84\pi^6 - 1120\pi^4 + \left(3840 + 2880\mu^2\right)\pi^2 - \left(5760 + 480\varepsilon^2\right)\mu^2 - 1920\right\}\varepsilon^3\cos^3(\mu x)\\
&\qquad + \left\{36\pi^7 - 672\pi^5 + \left(1440\mu^2 + 3840\right)\pi^3 - \left(8640 + 1440\varepsilon^2\right)\pi\mu^2 - 5760\pi\right\}\varepsilon^2\cos^2(\mu x)\\
&\qquad + \{9\pi^8 - 224\pi^6 + \left(240\mu^2 + 1920\right)\pi^4 - \left(2880 + 1440\varepsilon^2\right)\pi^2\mu^2\\
&\qquad + 5760\left(\mu^2 - \pi^2\right) + 2880\left(\mu^4 + \varepsilon^2\mu^2 + 1\right)\}\varepsilon\cos(\mu x) + \pi^9 - 32\pi^7 + 384\pi^5\\
&\qquad - \left(480\varepsilon^2\mu^2 + 1920\right)\pi^3 + 2880\pi\left(\varepsilon^2\mu^2 + 1\right)]t^4,\\
&\vdots
\end{aligned}
$$

Table 1: Absolute errors for example 1 at $x = 0.01$

t	$\lvert\text{Exact} - \text{MADM}\rvert$	$\lvert\text{Exact} - \text{VIM}\rvert$	$\lvert\text{Exact} - \text{HPM}\rvert$	$\lvert\text{Exact} - \text{C(LT-HPM)}\rvert$
0.01	$1.320E-06$	$4.999E-07$	$6.341E-16$	$6.341E-16$
0.02	$1.045E-05$	$3.997E-06$	$8.110E-14$	$8.110E-14$
0.03	$3.491E-05$	$1.348E-05$	$1.384E-12$	$1.384E-12$
0.04	$8.191E-05$	$3.192E-05$	$1.035E-11$	$1.035E-11$
0.05	$1.583E-04$	$6.226E-05$	$4.922E-11$	$4.922E-11$
0.06	$2.707E-04$	$1.074E-04$	$1.759E-10$	$1.759E-10$
0.07	$4.253E-04$	$1.702E-04$	$5.155E-10$	$5.155E-10$
0.08	$6.280E-04$	$2.535E-04$	$1.308E-09$	$1.308E-09$
0.09	$8.844E-04$	$3.600E-04$	$2.969E-09$	$2.969E-09$
0.1	$1.200E-03$	$4.924E-04$	$6.175E-09$	$6.175E-09$

Table 2: Absolute errors for example 1 at $x = 0.1$

t	$\lvert\text{Exact} - \text{MADM}\rvert$	$\lvert\text{Exact} - \text{VIM}\rvert$	$\lvert\text{Exact} - \text{HPM}\rvert$	$\lvert\text{Exact} - \text{C(LT-HPM)}\rvert$
0.01	$1.925E-04$	$4.974E-07$	$5.737E-16$	$5.737E-16$
0.02	$3.926E-04$	$3.978E-06$	$7.337E-14$	$7.337E-14$
0.03	$6.079E-04$	$1.341E-05$	$1.252E-12$	$1.252E-12$
0.04	$8.453E-04$	$3.176E-05$	$9.360E-12$	$9.360E-12$
0.05	$1.112E-03$	$6.195E-05$	$4.452E-11$	$4.452E-11$
0.06	$1.413E-03$	$1.069E-04$	$1.590E-10$	$1.590E-10$
0.07	$1.757E-03$	$1.694E-04$	$4.662E-10$	$4.662E-10$
0.08	$2.147E-03$	$2.523E-04$	$1.182E-09$	$1.182E-09$
0.09	$2.591E-03$	$3.583E-04$	$2.683E-09$	$2.683E-09$
0.1	$3.092E-03$	$4.901E-04$	$5.581E-09$	$5.581E-09$

Table 3: Absolute errors for example 1

x / t	0.02	0.04	0.06	0.08	0.10
0.02	$3.23881E-17$	$3.21021E-17$	$3.16307E-17$	$3.09817E-17$	$3.01658E-17$
0.04	$1.65461E-14$	$1.63999E-14$	$1.61591E-14$	$1.58276E-14$	$1.54107E-14$
0.06	$6.33758E-13$	$6.28159E-13$	$6.18932E-13$	$6.06230E-13$	$5.90261E-13$
0.08	$8.39745E-12$	$8.32325E-12$	$8.20097E-12$	$8.03261E-12$	$7.82096E-12$
0.10	$6.21573E-11$	$6.16079E-11$	$6.07024E-11$	$5.94559E-11$	$5.78887E-11$

Therefore, we have the approximate solution is given by

$$
\begin{aligned}
u_{Approx}(x,t) = {} & \pi + \varepsilon\cos(\mu x) + \frac{1}{2}[-\varepsilon\mu^2\cos(\mu x) + \frac{1}{3!}(\pi+\varepsilon\cos(\mu x))^3 \\
& -\frac{1}{5!}(\pi+\varepsilon\cos(\mu x))^5 \\
& -\pi-\varepsilon\cos(\mu x)]t^2 + \frac{1}{69120}[\varepsilon^9\cos^9(\mu x) + 9\pi\varepsilon^8\cos^8(\mu x) \\
& + \left(36\pi^2-32\right)\varepsilon^7\cos^7(\mu x) + \left(84\pi^3-224\pi\right)\varepsilon^6\cos^6(\mu x) \\
& + \left(126\pi^4-672\pi^2+720\mu^2+384\right)\varepsilon^5\cos^5(\mu x) \\
& + \left(126\pi^5-1120\pi^3+2400\pi\mu^2+1920\pi\right)\varepsilon^4\cos^4(\mu x) \\
& + \{84\pi^6-1120\pi^4+\left(3840+2880\mu^2\right)\pi^2-\left(5760+480\varepsilon^2\right)\mu^2 \\
& -1920\}\varepsilon^3\cos^3(\mu x) + \{36\pi^7-672\pi^5+\left(1440\mu^2+3840\right)\pi^3 \\
& -\left(8640+1440\varepsilon^2\right)\pi\mu^2-5760\pi\}\varepsilon^2\cos^2(\mu x) + \{9\pi^8-224\pi^6 \\
& + \left(240\mu^2+1920\right)\pi^4-\left(2880+1440\varepsilon^2\right)\pi^2\mu^2+5760\left(\mu^2-\pi^2\right) \\
& +2880\left(\mu^4+\varepsilon^2\mu^2+1\right)\}\varepsilon\cos(\mu x)+\pi^9-32\pi^7+384\pi^5 \\
& -\left(480\varepsilon^2\mu^2+1920\right)\pi^3+2880\pi\left(\varepsilon^2\mu^2+1\right)]t^4,
\end{aligned}
$$

which is the same as the approximate solution obtained by HPM [9]. In figures 4 and 5, we present 3-term of approximate solution C(LT-HPM) for $\varepsilon = 0.05$ and $\varepsilon = 0.1$ respectively.

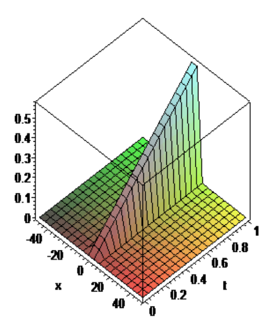

Fig. 1: $u_{Exact}(x,t)$

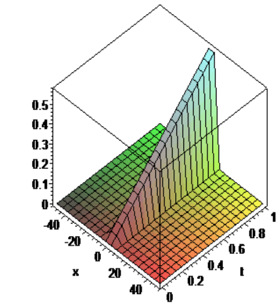

Fig. 2: $u_{Approx}(x,t)$

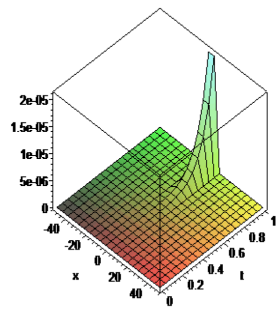

Fig. 3: Absolute error $|u_{Exact}(x,t) - u_{Approx}(x,t)|$

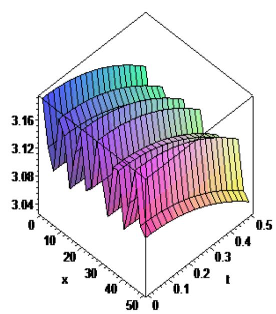

Fig. 4: Approximate solution with $\varepsilon = 0.05$

4 Conclusion

In this work, the combined Laplace transform-homotopy perturbation method C(LT-HPM) has been successfully applied to solve models of sin-Gordon equation with initial conditions to obtain approximate-exact solutions. The C(LT-HPM) has worked effectively to handle these models giving it a wider applicability. The proposed scheme has been applied directly without any need for transformation formulae or restrictive assumptions.

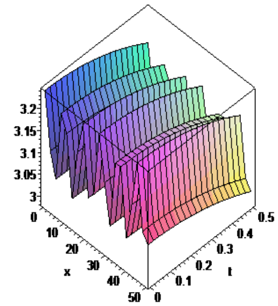

Fig. 5: Approximate solution with $\varepsilon = 0.1$

The approach has been tested by employing the method for two examples with different initial conditions. The results obtained in all cases demonstrate the reliability and the efficiency of this method.

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