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Properties of Solutions to Volterra Integro-Differential Equations with Delay

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Abstract: Using suitable Lyapunov functionals, we study the stability and boundedness of solutions in a class of non-linear Volterra integro-differential equations with delay. Our results give new sufficient conditions for stability and boundedness of solutions for the considered equations, and they improve, complete and extend the results obtained in literature

Keywords: Non-linear, Volterra integro-differential equation, first order, stability, boundedness, Lyapunov functional.

1 Introduction

In the relevant literature, many results have been obtained for the stability and boundedness in Volterra integro-differential equations without delay. We refer to the papers of Becker [2], Burton ([3,4]), Burton and Mahfoud ([6,7]) Diamandescu [9], Hara et al. [13], Miller [14], Staffans [17], Tunc [18], Vanualailai and Nakagiri [21] and the books of Burton [5], Corduneanu [8], Gripenberg et al. [12] and the references cited therein for some works done on the qualitative properties of various Volterra integro-differential equations without delay. An important tool to discuss the qualitative properties of solutions in ordinary, functional and integro-differential equations is the Lyapunov's direct method. Theoretically this method is very appealing, and there are numerous applications where it is natural to use it. The key requirement of the method is to find a positive definite function or functional which is non-increasing along solutions. However, it is a quite difficult task to find a suitable Lyapunov function or functional for a non-linear ordinary or functional differential equation and a non-linear functional Volterra integro- differential equation. The situation becomes more difficult when we replace an ordinary or a functional differential equation with a functional integro-differential equation. By this time, the construction of Lyapunov functions and functionals for non-linear differential and integro-differential systems remains as an open problem in the literature. Besides, in the literature, there are a few papers on the qualitative behaviors of Volterra integro-differential equations with delay. See, for example, the recent papers of Adivar and Raffoul [1], Graef and Tunc [11], Raffoul [15], Raffoul and Unal [16] and Tunc ([19,20]).

In 1982 and 2003, respectively, Burton [4] and Vanualailai and Nakagiri [21] considered the same scalar nonlinear Volterra integro-differential equation without delay given by

$$\frac{d}{dt}[x(t)] = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds. \quad (1)$$

Burton [4] and Vanualailai and Nakagiri [21] studied the stability of zero solution of equation (1) by defining some different suitable Lyapunov functionals.

Later, in a recent paper, the author in [19] discussed the stability and boundedness of solutions to the following Volterra integro-differential equation with delay:

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^t B(t,s)g(x(s))ds + p(t), \quad (2)$$

when $p(t) \equiv 0$ and $p(t) \neq 0$, respectively.
In addition, recently, some qualitative properties of non-linear and non-homogeneous scalar Volterra integro-differential equation with delay

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^t B(t,s)g(x(s))ds + e(t,x(t))$$

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and non-linear and non-homogeneous vector Volterra integro-differential equation with delay

$$x'(t) = -D(t)x(t) + \int_{t-\tau}^{t} B(t,s)E(x(s))(x(s))ds + Q(t,x(t))$$

have been investigated by Tunc [20].

In this paper, we consider the following non-linear Volterra integro-differential equations with constant delay, τ :

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^{t} K(t,s,x(s))g(s,x(s))ds \quad (3)$$

and

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^{t} B(t,s)\psi(s,x(s))ds + h(t,x(t)), \quad (4)$$

respectively, where $t \geq 0$, τ is a positive constant, fixed delay, $x \in \Re$, $a(t) : [0,\infty) \to (0,\infty)$, $g,\psi,h : [0,\infty) \times \Re \to \Re$ and $f : \Re \to \Re$ are continuous functions with $f(0) = 0$, $g(t,0) = 0$, $\psi(t,0) = 0$, and $K(t,s,x(s))$ and $B(t,s)$ are continuous functions on $\Re^+ \times \Re^+ \times \Re$ and $\Re^+ \times \Re^+$ with $0 \leq s \leq t < \infty$, respectively.

We investigate the stability of zero solution of equation (3) and the boundedness of all solutions of equation (4) by defining new suitable Lyapunov functionals, respectively.

It is clear that equation (1) and equation (2) are special cases of equation (3) when $p(t) = 0$, $\tau = 0$ and equation (4) when $\tau = 0$, respectively. Further, if we choose $\tau = 0$, $K(t,s,x(s)) = B(t,s)$ and $g(t,x) = g(x)$, then equation (3) reduces to the equation discussed by Vanualailai and Nakagiri [21], that is, it reduces to equation (1) provided that $p(t) = 0$. Similarly, if we choose $\tau = 0$ and $\tau \neq 0$, respectively, $\psi(t,x) = g(x)$ and $h(t,x) = p(t)$, then equation (4) reduces to the equations discussed by Vanualailai and Nakagiri [21] and Tunc [19], respectively.

Besides, Vanualailai and Nakagiri [21] considered a Volterra integro-differential equation without delay. However, in this paper, we consider a Volterra integero-differential equation with delay. Besides, Vanualailai and Nakagiri [21] discussed the stability of the zero solution of equation (1). However, in addition to the stability of zero solution, we also discuss the boundedness of solutions of equation (4), when $h(t,x(t)) \equiv 0$ and $h(t,x(t)) \neq 0$, respectively.

We use the following notation and basic information throughout this paper.

For any $t_0 \geq 0$ and initial function $\varphi \in [t_0 - \tau, t_0]$, let $x(t) = x(t,t_0,\varphi)$ denote the solution of equation (3) on $[t_0 - \tau, \infty)$ such that $x(t) = \varphi(t)$ on $\varphi \in [t_0 - \tau, t_0]$.

Let $C[t_0,t_1]$ and $C[t_0,\infty)$ denote the set of all continuous real-valued functions on $[t_0,t_1]$ and $[t_0,\infty)$, respectively.

$$For \quad \varphi \in C[0,t_0], |\varphi|_{t_0} := sup\{|\varphi(t)| : 0 \leq t \leq t_0\}.$$

Definition 1. The zero solution of equation (3) is stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon,t_0) > 0$ such that $\varphi \in C[0,t_0]$ with $|\varphi(t)|_{t_0} < \delta$ implies that $|x(t,t_0,\varphi)| < \varepsilon$ for all $t \geq t_0$.

Definition 2. The solutions of equation (4) are bounded if for each $T > 0$, there exists $D > 0$ such that

$$t_0 \geq 0, \varphi \in C[0,t_0], |\varphi(t)|_{t_0} < T \text{ and } t \geq t_0 \text{ imply } |x(t)| < D.$$

The following theorem is need for the stability result of this paper.

Theorem 1. If there exists a functional $V(t,\phi(.))$, defined whenever $t \geq t_0 \geq 0$ and $\phi \in C([0,t],\Re)$, such that

(i) $V(t,0) \equiv 0$, V is continuous in t and locally Lipschitz in ϕ.

(ii) $W : [0,\infty) \to [0,\infty)$ is a continuous function with $W(0) = 0$, $W(r) > 0$ if $r > 0$, and W strictly increasing (positive definite), and

(iii) $V'(t,\phi(.)) \leq 0$,

then the zero solution of equation (3) is stable, and $V(t,\phi(.)) = V(t,\phi(s) : 0 \leq s \leq t)$

is called a Lyapunov functional for equation (3), (see Driver [10]).

2 Stability

We first express some assumptions on the functions that appearing in equation (3).

A. Assumptions

(A1) There are positive constants α, m_1, J, M and N such that

$$f(0) = 0, \; g(t,0) = 0, \; g^2(t,x) \leq \alpha_1^2(t)f^2(x) \quad if \quad |x| \leq M,$$

$$0 < \alpha_1(t) \leq m_1, \; \alpha_1(t) \text{ and } f(x) \text{ are continuous functions},$$

$$xf(x) > 0 \quad for \; all \quad x \neq 0,$$

$$\alpha > 4 \quad such \; that \quad 4x^2 \leq (\alpha - 4)f^2(x) \quad if |x| \leq N.$$

(A2) $a(t)>0$ for $t\geq 0$, $K(t,s,x(s))$ is continuous for $0\leq s\leq t<\infty$ and x,

$$J\geq 1, \tfrac{1}{4a(t)}\textstyle\int_{t-\tau}^{t}|K(t,s,x(s))|ds<J^{-1} \quad for\ \ every\ \ t\geq s-\tau\geq 0,$$

$$\int_{t-\tau}^{\infty}|K(u+\tau,s,x(s))|du$$

$$is\ \ defined\ \ and\ \ continuous\ \ for\ \ \ 0\leq s-\tau\leq t<\infty,$$

$$a(t)-m_1^2(1+\alpha)J^{-1}\int_{t-\tau}^{\infty}|K(u+\tau,t,x(t))|du\geq 0$$

$$for\ \ every\ \ \ t\geq s-\tau\geq 0.$$

We have the following stability result for equation (3).

Theorem 2. Let assumptions $(A1)$ and $(A2)$ hold. If $k=m_1^2(1+\alpha)J^{-1}$, then the zero solution of equation (3) is stable.

Proof. We define a functional $W=W(t)=W(t,x(t))$ by

$$W=\frac{1}{2}x^2+\sqrt{\alpha}\int_0^x\sqrt{f(u)u}du+\frac{1}{2}\alpha\int_0^x f(u)du$$

$$+\lambda\int_0^t\int_{t-\tau}^{\infty}|K(u+\tau,s,x(s))|du f^2(x(s))ds, \qquad (5)$$

where λ is a positive constant to be determined later in the proof.

Clearly, it follows that the functional W is positive definite.

The derivative of functional (5) along the solutions of equation (3) with respect to t leads

$$W'=xx'+\sqrt{\alpha}\sqrt{f(x)x}x'+\frac{1}{2}\alpha f(x)x'$$

$$+\lambda\int_{t-\tau}^{\infty}|K(u+\tau,t,x(t))|du f^2(x)$$

$$-\lambda\int_0^t|K(t,s,x(s))|f^2(x(s))ds. \qquad (6)$$

The first term of (6) and equation (3) yield

$$xx'=-a(t)xf(x)+x\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds$$

$$=-a(t)xf(x)$$

$$-[\sqrt{a(t)}x-\frac{1}{2\sqrt{a(t)}}\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds]^2$$

$$+a(t)x^2+\frac{1}{4a(t)}[\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds]^2$$

$$\leq -a(t)xf(x)+a(t)x^2$$

$$+\frac{1}{4a(t)}[\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds]^2$$

$$\leq -a(t)xf(x)+a(t)x^2$$

$$+\frac{1}{4a(t)}\int_{t-\tau}^{t}|K(t,s,x(s))|ds\int_{t-\tau}^{t}|K(t,s,x(s))|g^2(s,x(s))ds$$

$$\leq -a(t)xf(x)+a(t)x^2$$

$$+\frac{1}{4a(t)}\int_{t-\tau}^{t}|K(t,s,x(s))|ds\int_{t-\tau}^{t}|K(t,s,x(s))|\alpha_1^2(s)f^2(x(s))ds$$

$$\leq -a(t)xf(x)+a(t)x^2$$

$$+\frac{m_1^2}{4a(t)}\int_{t-\tau}^{t}|K(t,s,x(s))|ds\int_{t-\tau}^{t}|K(t,s,x(s))|f^2(x(s))ds$$

$$\leq -a(t)xf(x)+\frac{1}{4}\alpha a(t)f^2(x)-a(t)f^2(x)$$

$$+m_1^2J^{-1}\int_{t-\tau}^{t}|K(t,s,x(s))|f^2(x(s))ds$$

by the assumption of Theorem 2 and the Schwarz inequality, that is, by

$(A2)$ and

$$[\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds]^2$$

$$\leq\int_{t-\tau}^{t}|K(t,s,x(s))|ds\int_{t-\tau}^{t}|K(t,s,x(s))|g^2(s,x(s))ds$$

$$\leq\int_{t-\tau}^{t}|K(t,s,x(s))|ds\int_{t-\tau}^{t}|K(t,s,x(s))|\alpha_1^2(s)f^2(x(s))ds$$

$$\leq m_1^2\int_{t-\tau}^{t}|K(t,s,x(s))|ds\int_{t-\tau}^{t}|K(t,s,x(s))|f^2(x(s))ds.$$

By noting the assumptions of Theorem 2 and the Schwarz inequality, we have from the second and third terms of (6) and equation (3) that

$$\sqrt{\alpha}\sqrt{f(x)x}x'=-[\frac{\sqrt{\alpha}}{2\sqrt{a(t)}}x'-\sqrt{a(t)}\sqrt{f(x)x}]^2$$

$$+\frac{\alpha}{4a(t)}(x')^2+a(t)f(x)x$$

$$\leq\frac{\alpha}{4a(t)}(x')^2+a(t)f(x)x$$

$$=a(t)f(x)x+\frac{\alpha}{4}a(t)f^2(x)$$

$$-\frac{1}{2}\alpha f(x)\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds$$

$$+\frac{\alpha}{4a(t)}[\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds]^2$$

$$\leq a(t)f(x)x+\frac{\alpha}{4}a(t)f^2(x)$$

$$-\frac{1}{2}\alpha f(x)\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds$$

$$+\frac{\alpha}{J}\int_{t-\tau}^{t}|K(t,s,x(s))|\alpha_1^2(s)f^2(x(s))ds$$

$$\leq a(t)f(x)x+\frac{\alpha}{4}a(t)f^2(x)$$

$$-\frac{1}{2}\alpha f(x)\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds$$

$$+\alpha m_1^2J^{-1}\int_{t-\tau}^{t}|K(t,s,x(s))|f^2(x(s))ds$$

and

$$\frac{1}{2}\alpha f(x)x'=-\frac{1}{2}\alpha a(t)f^2(x)$$

$$+\frac{1}{2}\alpha f(x)\int_{t-\tau}^{t}K(t,s,x(s))g(s,x(s))ds.$$

By substituting the obtained estimates into (4), we arrive at

$$W^{'}\leq-[\lambda-m_1^2(1+\alpha)J^{-1}]\int_0^1|K(t,s,x(s))|f^2(x(s))ds$$

$$-m^2J^{-1}\int_0^{t-\tau}|K(t,s,x(s))|f^2(x(s))ds$$

$$-\alpha m_1^2J^{-1}\int_0^{t-\tau}|K(t,s,x(s))|f^2(x(s))ds$$

$$-[a(t)-\lambda\int_{t-\tau}^{\infty}|K(u+\tau,t,x(t))|du]f^2(x)$$

$$\leq-[\lambda-m_1^2(1+\alpha)J^{-1}]\int_0^t|K(t,s,x(s))|f^2(x(s))ds$$

$$-[a(t)-\lambda\int_{t-\tau}^{\infty}|K(u+\tau,t,x(t))|du]f^2(x).$$

Let $\lambda=m_1^2(1+\alpha)J^{-1}$. Then, we arrive at

$$W^{'}\leq-[a(t)-m_1^2(1+\alpha)J^{-1}\textstyle\int_{t-\tau}^{\infty}|K(u+\tau,t,x(t))|du]f^2(x).$$

If

$$a(t)-m_1^2(1+\alpha)J^{-1}\int_{t-\tau}^{\infty}|K(u+\tau,t,x(t))|du\geq 0,$$

then

$$W'\leq 0.$$

Therefore, we can conclude that the zero solution of equation (3) is stable. This completes the proof of the theorem.

3 Boundedness

We have a boundedness result for equation (4).

Let

$\rho(t)=\alpha_0a(t)-\frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|ds-\frac{1}{2}\alpha^2m_2^2\int_{t-\tau}^{\infty}|B(u+\tau,t)|du.$

B. Assumptions

($B1$) There are positive constants α_0, α, m_2 and M such that

$\psi(t,0)=0,\psi^2(t,x)\leq\beta_1^2(t)f_1^2(x)\leq\alpha^2m_2^2x^2\quad if\quad |x|\leq M,$

$\beta_1(t)$ and $f(x)$ are continuous functions,

$0<\beta_1(t)\leq\alpha, f(0)=0, x^{-1}f(x)\geq\alpha_0>0,\quad when\quad x\neq 0,$

($B2$) $a(t)>0$ for $t\geq 0$, $B(t,s)$ is continuous for $0\leq s\leq t<\infty$,

$\int_{t-\tau}^{\infty}|B(u+\tau,s)|du$ is definded and continuous for $0\leq s-\tau\leq t<\infty$, and

$\rho(t)\geq 0$ for every $t\geq s-\tau\geq 0$.

We prove the following boundedness theorem.

Theorem 3. Let assumptions ($B1$) and ($B2$) and, in addition, the assumptions $|h(t,x)|\leq(A+|x|)|\theta(t)|$ and $|\theta(t)|\in L^1(0,\infty)$ hold, where A is a positive constant. Then all solutions of equation (4) are bounded.

Proof.. We define a functional $W_1=W_1(t)=W_1(t,x(t))$ given by

$$W_1=\frac{1}{2}x^2+\mu\int_0^t\int_{t-\tau}^{\infty}|B(u+\tau,s)|dux^2(s)ds,\qquad(7)$$

where μ is a positive constant to be determined later in the proof.

Clearly the functional W_1 is positive definite.

In view of (7), we have, along a trajectory of equation (4),

$$W_1^{'}=-a(t)f(x)x+x\int_{t-\tau}^{t}B(t,s)\psi(s,x(s))ds+xh(t,x)$$

$$+\mu\int_{t-\tau}^{\infty}|B(u+\tau,s)|dux^2-\mu\int_0^t|B(t,s)|x^2(s)ds.\qquad(8)$$

With the aid of the assumptions of Theorem 3 and the inequality $|\alpha\beta| \leq 2^{-1}(\alpha^2+\beta^2)$, we have from (8) that

$$W' \leq -a(t)f(x)x + \frac{1}{2}\int_{t-\tau}^{t} |B(t,s)|(x^2(t)+\psi^2(s,x(s)))ds$$

$$+|h(t,x)||x| + \mu\int_{t-\tau}^{\infty}|B(u+\tau,t)|dux^2 - \mu\int_0^t |B(t,s)|x^2(s)ds$$

$$= -[a(t)x^{-1}f(x) - \tfrac{1}{2}\textstyle\int_{t-\tau}^{t}|B(t,s)|ds - \mu\int_{t-\tau}^{\infty}|B(u+\tau,t)|du]x^2$$

$$+\frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|\psi^2(s,x(s))ds - \mu\int_0^t|B(t,s)|x^2(s)ds$$

$$+A|\theta(t)||x| + |\theta(t)|x^2$$

$$\leq -[\alpha_0 a(t) - \frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|ds - \mu\int_{t-\tau}^{\infty}|B(u+\tau,t)|du]x^2$$

$$+\frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|ds\beta_1^2(s)f_1^2(x(s))ds - \mu\int_0^t|B(t,s)|x^2(s)ds$$

$$+A|\theta(t)| + (1+A)|\theta(t)|x^2$$

$$\leq -[\alpha_0 a(t) - \frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|ds$$

$$-\mu\int_{t-\tau}^{\infty}|B(u+\tau,t)|du]x^2 + \frac{1}{2}\alpha^2 m_2^2\int_{t-\tau}^{t}|B(t,s)|ds$$

$$-\mu\int_0^t|B(t,s)|x^2(s)ds + A|\theta(t)| + (1+A)|\theta(t)|x^2$$

$$= -[\alpha_0 a(t) - \frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|ds - \mu\int_{t-\tau}^{\infty}|B(u+\tau,t)|du]x^2$$

$$-(\mu - \frac{1}{2}\alpha^2 m_2^2)\int_0^t|B(t,s)|x^2(s)ds$$

$$+A|\theta(t)| + (1+A)|\theta(t)|x^2 - \frac{1}{2}\alpha^2 m_2^2\int_0^{t-\tau}|B(t,s)|x^2(s)ds$$

$$\leq -[\alpha_0 a(t) - \frac{1}{2}\int_{t-\tau}^{t}|B(t,s)|ds - \mu\int_{t-\tau}^{\infty}|B(u+\tau,t)|du]x^2$$

$$-(\mu - \frac{1}{2}\alpha^2 m_2^2)\int_0^t|B(t,s)|x^2(s)ds$$

$$+A|\theta(t)| + (1+A)|\theta(t)|x^2.$$

Let $\mu = \frac{1}{2}\alpha^2 m_2^2$. Then

$$W_1' \leq -\rho(t)x^2 + A|\theta(t)| + (1+A)|\theta(t)|x^2$$

$$\leq A|\theta(t)| + (1+A)|\theta(t)|x^2$$

$$\leq A|\theta(t)| + 2(1+A)|\theta(t)|W_1.$$

Integrating the last estimate from zero t_0 to t, we have

$$W_1(t) \leq W_1(t_0) + A\int_0^t|\theta(s)|ds + 2(1+A)|\int_0^t|W_s|\theta(s)|ds.$$

Hence, an application of Gronwalls inequality bounds W_1. Thus, we can conclude that all solutions of equation (4) are bounded.

Remark. By Theorem 2, we improve and extend a stability result obtained for a Volterra integro-differential equation without delay to its delay form (see Vanualailai and Nakagiri [[21], Theorem 3.2]). Besides, Theorem 2 and Theorem 3 complement to the papers in the references, and they have a contribution to the papers of Adivar and Raffoul[1], Becker[2], Burton[4], Raffoul[15] and Tunc[19]. By this way, we also mean that the Volterra integro-differential equation considered and the assumptions established here are different from that in the mentioned papers above and those in the literature. Theorem 3 gives an additional result, the boundedness of solutions, to that of Vanualailai and Nakagiri [[21], Theorem 3.2]. The results of this paperr may be useful for researchers working on the qualitative behaviors of solutions of functional Volterra integro-differential equations. These cases show the novelty and originality of the present paper and its contribution to the literature.

4 Conclusion

A class of non-linear Volterra integro-differential equations of first order with delay is considered. The stability and boundedness of solutions are discussed by using the Lyapunov's functional approach. The obtained results improve and extend some results in the literature, and they also have a contribution to the literature.

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