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The Problem of Mayer for Discrete and Differential Inclusions with Initial Boundary Constraints

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Abstract: In the present paper we consider the Mayer Problem for Second Order Differential Inclusions with initial boundary constraints. We derive the approximation conditions for the problem. Locally adjoint mapping is our basic tool to formulate necessary and sufficient conditions for the optimality of the discrete approximation problem. Then by passing to the limit, sufficient optimality conditions to the optimal problem described by differential inclusions are established.

Keywords: Discrete differential inclusion, Mayer problem, Euler-Lagrange, local tent, adjoint multivalued approximation, dual cone, second order transversality.

1 Introduction

Discrete and continuous time problems with higher order ordinary and partial differential inclusions have wide applications in the field of mathematical economics and in problems of control dynamical system optimization and differential games [2,4], [9,10], [13,20], [25,26,27]. In particular, the problems including the higher order discrete and discrete-approximate differential inclusions and the higher order partial differential inclusions are studied by E.N.Mahmudov [15,16,17]. Especially, the problems including the second order discrete and differential inclusions are studied [1,6]. A lot of investigations on the second order differential inclusions (SODIs) usually are devoted to existence and viability problems [5,8], [11,12],[14,19]. In the classical Mayer problem for the SODIs with initial boundary constraints given by sets make the examined optimal control problem quite complicated. The change of the initial point constraints changes the problem. In [7] the authors obtain sensitivity relations for the Mayer problem associated with the first order differential inclusion and derive optimality conditions.

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space of the state variable x and let $P(\mathbb{R}^n)$ be a family of subsets of $\mathbb{R}^n$. Assume that $F:\mathbb{R}^{2n}\to P(\mathbb{R}^n)$ is a multi-valued mapping and $\varphi:\mathbb{R}^{2n}\to\mathbb{R}$ is proper single valued function, M and N are convex subsets of $\mathbb{R}^n$. Our main intention in this paper is to derive optimality conditions for Mayer problem for the second order differential inclusions

$$\text{minimize } \varphi(x(1),x'(1)) \tag{1}$$
$$x''(t)\in F(x(t),x'(t)),\ \text{a.e. } t\in[0,1], \tag{2}$$
$$x(0)\in M,\ x'(0)\in N. \tag{3}$$

The problem is to find an arc $\tilde{x}(t)$, satisfying (2) almost everywhere (a.e.) on $[0,1]$ and the boundary conditions (3) at $t=0$ that minimizes the Mayer functional $\varphi(x(1),x'(1))$. A feasible trajectory $x(\cdot)$ in the problem is taken to be an absolutely continuous function on time interval $[0,1]$ together with the first order derivatives for which $x''(\cdot)\in L_1^n([0,1])$.

For construction of optimality conditions we begin with the second order discrete problem and then using first and second order difference operators and an auxiliary multifunction, we approximate the convex problem (1)−(3) by the discrete approximation problem. Generally, there are some difficulties in constructing adjoint inclusions and transversality conditions at the endpoints $t=0$ and $t=1$, respectively. We achieve by

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the approximation and formulation of the equivalence theorems.

Setting $\lambda = 1$ and by passing to the formally limit in conditions of the discrete-approximation problem as the discrete step $\delta \to 0$, we establish the sufficient optimality conditions to the convex optimal problem (1)–(3).

Notions that we use in our paper are similar to the notations of Mahmudov in [18]. The multi-valued function $F : \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is convex if its graph is a convex subset of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, where its graph is defined with $gphF = \{(x,u,v) : v \in F(x,u)\}$. The multivalued mapping F is convex closed if its graph is a convex closed set in $\mathbb{R}^{3n}$. It is convex-valued if $F(x,u)$ is a convex set for each $(x,u) \in domF = \{(x,u) : F(x,u) \neq \emptyset\}$. The Hamiltonian function and argmaximum set for multivalued mapping F are defined by

$$H_F(x,u,v^*) = \sup_v \{\langle v,v^*\rangle : v \in F(x,u)\}, v^* \in \mathbb{R}^n,$$
$$F(x,u;v^*) = \{v \in F(x,u) : \langle v,v^*\rangle = H_F(x,u,v^*)\},$$

respectively. For convex F we set $H_F(x,u,v^*) = -\infty$ if $F(x,u) = \emptyset$.

The interior of the set $A \subset \mathbb{R}^{3n}$ is denoted by int$A$ and the relative interior of the set A , i.e. the set of interior points of A with respect to its affine hull Aff A is denoted by ri$A$.

The convex cone $K_A(z_0)$, $z = (x,u,v)$ is called the cone of tangent directions at a point $z_0 \in A$ to the set A if from $\bar{z} = (\bar{x},\bar{u},\bar{v}) \in K_A(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set A at point $z_0 \in A$, i.e., there exists such function $\kappa(\lambda) \in \mathbb{R}^{3n}$ that $z_0 + \lambda\bar{z} + \kappa(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\kappa(\lambda) \to 0$, as $\lambda \downarrow 0$.

If for any $\bar{z}_0 \in$ ri$K_A(z_0)$ there exists a convex cone $K \subseteq K_A(z_0)$ and a continuous mapping $\Psi(\bar{z})$ defined in the neighborhood of the origin such that

(i) $\bar{z}_0 \in$ ri$K$, Lin$K$ = Lin$K_A(z_0)$, where Lin$K$ is linear span of K,

(ii) $\Psi(\bar{z}) = \bar{z} + r(\bar{z})$, $r(\bar{z})\|\bar{z}\|^{-1} \to 0$ as $\bar{z} \to 0$,

(iii) $z_0 + \Psi(\bar{z}) \in A$, $\bar{z} \in K \cap S_\varepsilon(0)$ for some $\varepsilon > 0$, where $S_\varepsilon(0)$ is the ball of radius ε,

then the cone $K_A(z_0)$ is called local tent.

For a convex mapping F at a point $(x,u,v) \in gphF$

$$K_{gphF}(x,u,v) = cone[gphF - (x,u,v)]$$

$$= \{(\bar{x},\bar{u},\bar{v}) : \bar{x} = \lambda(x_1 - x), \bar{u} = \lambda(u_1 - u), \bar{v} = \lambda(v_1 - v)\},$$

$\forall\ (x_1,u_1,v_1) \in gphF$.

For a convex mapping F a multifunction defined by

$$F^*(v^*;(x,u,v)) := \{(x^*,u^*) : (x^*,u^*,-v^*) \in K^*_{gphF}(x,u,v)\}$$

is called a locally adjoint mapping (LAM) to F at point $(x,u,v) \in gphF$, where $K^*_{gphF}(x,u,v)$ is the dual cone to the cone of tangent directions $K_{gphF}(x,u,v)$.

The following multivalued mapping defined by

$$F^*(v^*;(x,u,v)) := \Big\{(x^*,u^*) : H(x_1,u_1,v^*) - H(x,u,v^*)$$
$$\le \langle x^*, x_1 - x\rangle + \langle u^*, u_1 - u\rangle, \forall (x_1,u_1) \in \mathbb{R}^{2n}\Big\},$$
$$v \in F(x,u;v^*)$$

is called the LAM to non-convex mapping F at point $(x,u,v) \in gphF$. Clearly for the convex mapping, $H(\cdot,\cdot,v^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM. Note that, the similar notion is given by Mordukhovich [22], and is called coderivative of multifunctions at a point.

In the following section we deal with second order discrete Mayer problem

$$\text{minimize } g(x_{T-1},x_T) \tag{4}$$
$$x_{t+2} \in F(x_t,x_{t+1}),\ t = 0,\dots,T-2, \tag{5}$$
$$x_0 \in M,\ x_1 - x_0 \in N, \tag{6}$$

where $x_t \in \mathbb{R}^n$, $g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm\infty\}$ is real-valued function, $F : \mathbb{R}^{2n} \to P(\mathbb{R}^n)$ is multivalued mapping and T is fixed natural number, M and N are convex subsets of $\mathbb{R}^n$. A sequence $\{x_t\}_{t=0}^T = \{x_t : t = 0,1,\dots,T\}$ is called the feasible trajectory for the stated problem (4)–(6). If the multivalued function F is convex and $g(\cdot,\cdot)$ is convex proper function, the discrete problem (4)–(6) is said to be convex.

Definition 1.1. If one of the following cases for points $x_t^0 \in \mathbb{R}^n$ is fulfilled

(i) $(x_t^0, x_{t+1}^0, x_{t+2}^0) \in$ri$(gphF)$,

(ii) $(x_t^0, x_{t+1}^0, x_{t+2}^0) \in$int$(gphF)$, $t = 0,\dots T-2$, (with the possible exception of one fixed t_0),

and $g(\cdot,\cdot)$ is continuous at (x_t^0, x_{t+1}^0) , we say that the regularity condition for the convex problem (4)–(6) is satisfied.

Condition I. Suppose that the cones of tangent directions $K_{gphF}(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2})$ to the graph of the mapping F in the problem (4)–(6) are local tents, where $\tilde{x}_t$ are the points of the optimal trajectory $\{\tilde{x}_t\}_{t=0}^T$. Suppose, moreover, that the function $g(x_{T-1},x_T)$ admit a continuous CUA $h_t(\cdot,\tilde{x}_{T-1},\tilde{x}_T)$ [18,23] at the point $(\tilde{x}_{T-1},\tilde{x}_T)$, which ensures that the subdifferential $\partial g(\tilde{x}_{T-1},\tilde{x}_T) = \partial h_t(0,\tilde{x}_{T-1},\tilde{x}_T)$ is defined.

2 Necessary and Sufficient Conditions for Discrete Inclusions

We consider the second order discrete convex problem (4)–(6). Let us introduce a vector $w = (x_0,x_1,\dots,x_T) \in \mathbb{R}^{n(T+1)}$ and define in the space

$\mathbb{R}^{n(T+1)}$ the following convex sets

$$S_t = \{w = (x_0, x_1, \ldots, x_T) \mid (x_t, x_{t+1}, x_{t+2}) \in gphF\},$$
$$t = 0, 1, \ldots, T-2,$$
$$\tilde{M} = \{w = (x_0, x_1, \ldots, x_T) \mid x_0 \in M\} \quad \text{and}$$
$$\tilde{N} = \{w = (x_0, x_1, \ldots, x_T) \mid x_1 - x_0 \in N, x_0 \in M\}.$$

First of all we should compute the dual cones $K^*_{S_t}(w)$, $K^*_{\tilde{M}}(w)$ and $K^*_{\tilde{N}}(w)$.

Lemma 2.1. Let $K_{gphF}(x_t, x_{t+1}, x_{t+2})$ be cones of tangent directions, where $(x_t, x_{t+1}, x_{t+2}) \in gphF$. Then

$$K^*_{S_t}(w) = \Big\{ w^* = (x^*_0, \ldots, x^*_T) \mid (x^*_t, x^*_{t+1}, x^*_{t+2})$$
$$\in K^*_{gphF}(x_t, x_{t+1}, x_{t+2}), x^*_k = 0, k \neq t, t+1, t+2 \Big\}$$

Proof. If for sufficiently small $\lambda > 0$, $w + \lambda \overline{w} \in S_t$, $t = 0, \ldots, T-2$, that is if for sufficiently small $\lambda > 0$, $(x_t + \lambda \overline{x}_t \ ,\ x_{t+1} + \lambda \overline{x}_{t+1} \ ,\ x_{t+2} + \lambda \overline{x}_{t+2}) \in gphF$ then $\overline{w} \in K_{S_t}(w)$. Therefore we can write

$$K_{S_t}(w) = \Big\{ \overline{w} \mid (\overline{x}_t, \overline{x}_{t+1}, \overline{x}_{t+2}) \in K_{gphF}(x_t, x_{t+1}, x_{t+2}) \Big\},$$
$$t = 0, 1, \ldots, T-2.$$

By the definition of dual cone we have $w^* \in K^*_{S_t}(w)$ if and only if

$$\langle w^*, \overline{w} \rangle = \sum_{k=0}^{T} \langle x^*_k, \overline{x}_k \rangle \geq 0 \ ,\ \forall\, \overline{w} \in K_{S_t}(w).$$

Then from the arbitrariness of components $\overline{x}_k, k \neq t, t+1, t+2$, of vectors $\overline{w}$, the inequality is satisfied if $x^*_k = 0, k \neq t, t+1, t+2$. So the last inequality takes the form

$$\langle x^*_t, \overline{x}_t \rangle + \langle x^*_{t+1}, \overline{x}_{t+1} \rangle + \langle x^*_{t+2}, \overline{x}_{t+2} \rangle \geq 0,$$

where $(\overline{x}_t, \overline{x}_{t+1}, \overline{x}_{t+2}) \in K_{gphF}(x_t, x_{t+1}, x_{t+2})$. Hence $(x^*_t, x^*_{t+1}, x^*_{t+2}) \in K^*_{gphF}(x_t, x_{t+1}, x_{t+2})$. This completes the proof of the lemma.

Lemma 2.2. Let $K_M(x_0)$ be the cone of tangent directions at point $x_0 \in M$ to the set M, $K_N(y_0 - x_0)$ be the cone of tangent directions at point $y_0 - x_0 \in N$ to the set N and let the set $\Phi = \big\{(x,y) \mid x \in M, y - x \in N\big\} = M \times (M+N)$ be given. Then we have

$$K^*_{\Phi}(x_0, y_0) = \Big\{ (x^*, y^*) \mid x^* + y^* \in K^*_M(x_0),\ y^* \in K^*_N(y_0 - x_0) \Big\},$$

for a fixed $(x_0, y_0) \in \Phi$.

Proof. Since $(\overline{x}, \overline{y}) \in K_{\Phi}(x_0, y_0)$ if and only if for sufficiently small $\lambda > 0$, $(x_0 + \lambda \overline{x}, y_0 + \lambda \overline{y}) \in \Phi$, i.e. $(y_0 + \lambda \overline{y}) - (x_0 + \lambda \overline{x}) \in N$ and $(x_0 + \lambda \overline{x}) \in M$, then $\overline{x} \in K_M(x_0)$ and $\overline{y} - \overline{x} \in K_N(y_0 - x_0)$. So we obtain the cone of tangent directions to the set Φ at fixed point (x_0, y_0),

$$K_{\Phi}(x_0, y_0) = \Big\{ (\overline{x}, \overline{y}) \mid \overline{x} \in K_M(x_0), \overline{y} - \overline{x} \in K_N(y_0 - x_0) \Big\}.$$

By the definition of dual cone $(x^*, y^*) \in K^*_{\Phi}(x_0, y_0)$ if and only if

$$\Big\langle (x^*, y^*), (\overline{x}, \overline{y}) \Big\rangle \geq 0 \ ,\ \forall\, (\overline{x}, \overline{y}) \in K_{\Phi}(x_0, y_0).$$

The last inequality is equivalent to the following $\langle x^*, \overline{x} \rangle + \langle y^*, \overline{y} \rangle \geq 0$, for all $\overline{x} \in K_M(x_0)$ and $\overline{y} - \overline{x} \in K_N(y_0 - x_0)$ or after some calculations $\langle x^* + y^*, \overline{x} \rangle + \langle y^*, \overline{y} - \overline{x} \rangle \geq 0$ for all $\overline{x} \in K_M(x_0)$ and $\overline{y} - \overline{x} \in K_N(y_0 - x_0)$. Then we obtain the inclusions $x^* + y^* \in K^*_M(x_0)$, $y^* \in K^*_N(y_0 - x_0)$ that x^* and y^* satisfy. So we derive the dual cone

$$K^*_{\Phi}(x_0, y_0) = \Big\{ (x^*, y^*) \mid x^* + y^* \in K^*_M(x_0),\ y^* \in K^*_N(y_0 - x_0) \Big\}.$$

Lemma 2.3. Let $K_M(x_0)$ be the cone of tangent directions at point $x_0 \in M$ to the set M and $K_N(x_1 - x_0)$ be the cone of tangent directions at point $x_1 - x_0 \in N$ to the set N. Then

$$K^*_{\tilde{M}}(w) = \Big\{ w^* = (x^*_0, \ldots, x^*_T) \mid x^*_0 \in K^*_M(x_0),$$
$$x^*_t = 0\ ,\ \ t = 1, \ldots, T \Big\}$$

and

$$K^*_{\tilde{N}}(w) = \Big\{ w^* = (x^*_0, \ldots, x^*_T) \mid x^*_0 + x^*_1 \in K^*_M(x_0),$$
$$x^*_1 \in K^*_N(x_1 - x_0)\ ,\ x^*_t = 0\ ,\ t \neq 0, 1 \Big\}.$$

Proof. Since $w + \lambda \overline{w} \in \tilde{M}$ if and only if $\overline{x}_0 \in K_M(x_0)$, we have

$$K_{\tilde{M}}(w) = \Big\{ \overline{w} = (\overline{x}_0, \ldots, \overline{x}_T) \mid \overline{x}_0 \in K_M(x_0) \Big\}$$

and hence

$$K^*_{\tilde{M}}(w) = \Big\{ w^* = (x^*_0, \ldots, x^*_T) \mid x^*_0 \in K^*_M(x_0),$$
$$x^*_t = 0\ ,\ \ t = 1, \ldots, T \Big\}.$$

If $w + \lambda \overline{w} \in \tilde{N}$ for sufficiently small $\lambda > 0$, i.e. $(x_1 + \lambda \overline{x}_1) - (x_0 + \lambda \overline{x}_0) \in N$ and $(x_0 + \lambda \overline{x}_0) \in M$, then $\overline{x}_0 \in K_M(x_0)$ and $\overline{x}_1 - \overline{x}_0 \in K_N(x_1 - x_0)$. If we use Lemma 2.2, we obtain

$$K_{\tilde{N}}(w) = \Big\{ \overline{w} = (\overline{x}_0, \ldots, \overline{x}_T) \mid \overline{x}_0 \in K_M(x_0),$$
$$\overline{x}_1 - \overline{x}_0 \in K_N(x_1 - x_0) \Big\}.$$

By the definition of dual cone $w^* \in K^*_{\tilde{N}}(w)$ if and only if

$$\langle w^*, \overline{w} \rangle = \sum_{k=0}^{T} \langle x^*_k, \overline{x}_k \rangle \geq 0 \ ,\ \forall\, \overline{w} \in K_{\tilde{N}}(w).$$

The last inequality is equivalent to the following $\langle x_0^*+x_1^*,\overline{x}_0\rangle+\langle x_1^*,\overline{x}_1-\overline{x}_0\rangle+\langle x_2^*,\overline{x}_2\rangle+\cdots+\langle x_T^*,\overline{x}_T\rangle\geq 0$, for all $\overline{x}_0\in K_M(x_0)$ and $\overline{x}_1-\overline{x}_0\in K_N(x_1-x_0)$, and $\overline{x}_k$ arbitrary for $k\neq 0,1$. From the arbitrariness of components $\overline{x}_k$, $k\neq 0,1$, and Lemma 2.2, we derive

$$K_{\tilde{N}}^*(w)=\Big\{w^*=(x_0^*,\ldots,x_T^*)\mid x_0^*+x_1^*\in K_M^*(x_0),\\ x_1^*\in K_N^*(x_1-x_0)\ ,\ x_t^*=0\ ,\ t\neq 0,1\Big\}.$$

Now we give the necessary and sufficient conditions for the problem (4)–(6) in the sense of the terminology of first order discrete inclusions [18,20,23].

Theorem 2.1. Let F be convex mapping and $g(\cdot,\cdot)$ be convex continuous function at the points of some feasible trajectory $\{x_t^0\}_{t=0}^T$. Then for $\{\tilde{x}_t\}_{t=0}^T$ to be an optimal trajectory of the problem (4)–(6), it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors $x_t^*,x_T^*,u_t^*,t=0,\ldots,T-1$, simultaneously not all equal to zero satisfying the discrete Euler-Lagrange and transversality inclusions

(i) $(x_t^*-u_t^*,u_{t+1}^*)\in F^*(x_{t+2}^*;(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}))$, $t=0,1,\ldots,T-2$,

(ii) $(u_{T-1}^*-x_{T-1}^*,-x_T^*)\in\lambda\partial g(\tilde{x}_{T-1},\tilde{x}_T)$,

(iii) $u_0^*\in K_M^*(\tilde{x}_0),-x_0^*-x_1^*\in K_M^*(\tilde{x}_0),-x_1^*\in K_N^*(\tilde{x}_1-\tilde{x}_0)$.

And if the regularity condition is satisfied these conditions are sufficient for the optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$.

Proof. Denoting $f(w)=g(x_{T-1},x_T)$, we will reduce this problem to the problem with geometric constraints. Indeed it can be easily seen that our basic problem (4)–(6) is equivalent to the following one

$$\begin{aligned}&\text{minimize } f(w)\\ &\text{subject to}\quad P=(\bigcap_{t=0}^{T-2}S_t)\cap\tilde{M}\cap\tilde{N},\end{aligned}\tag{7}$$

where P is a convex set.

By the hypothesis of the theorem, $\{\tilde{x}_t\}_{t=0}^T$ is an optimal trajectory, consequently, $\tilde{w}=(\tilde{x}_0,\ldots,\tilde{x}_T)$ is a solution of the problem (7). The result taken from Theorem 3.4 in [18], provides necessary optimality conditions for the convex mathematical programming (7). According to this theorem there exist vectors $w^*(t)\in K_{S_t}^*(\tilde{w}),t=0,1,\ldots,T-2,w_0^*\in K_{\tilde{M}}^*(\tilde{w}),w_1^*\in K_{\tilde{N}}^*(\tilde{w})$, not all zero, and the number $\lambda\in\{0,1\}$, such that

$$\lambda w^{0*}=\sum_{t=0}^{T-2}w^*(t)+w_0^*+w_1^*,\quad w^{0*}\in\partial_w f(\tilde{w}).\tag{8}$$

From the definition of the function f it is easy to see that vector $w^{0*}\in\partial_w f(\tilde{w})$ has a form

$$w^{0*}=(0,\ldots,0,\overline{x}_{T-1}^*,\overline{x}_T^*),$$

where $(\overline{x}_{T-1}^*,\overline{x}_T^*)\in\partial g(\tilde{x}_{T-1},\tilde{x}_T)$ and for $t=0,1,\ldots,T-2$, $\overline{x}_t^*=0$ by the fact that $g(\tilde{x}_t,t)=0$. By Lemma 2.1 and Lemma 2.3 we have

$$w^*(t)=(0,\ldots,0,x_t^*(t),x_{t+1}^*(t),x_{t+2}^*(t),0,\ldots,0),$$

$$\begin{gathered}(x_t^*(t),x_{t+1}^*(t),x_{t+2}^*(t))\in K_{gphF}^*(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}),\\ t=0,1,\ldots,T-2,\end{gathered}\tag{9}$$

$$w_0^*=(x_a^*,0,\ldots,0),w_1^*=(x_b^*,x_c^*,0,\ldots,0),$$

where $x_a^*\in K_M^*(\tilde{x}_0),x_b^*+x_c^*\in K_M^*(\tilde{x}_0),x_c^*\in K_N^*(\tilde{x}_1-\tilde{x}_0)$. Now, using the component-wise representation of (8) we deduce that

$$\begin{aligned}0&=x_a^*+x_b^*+x_0^*(0),\\ 0&=x_c^*+x_1^*+x_1^*(0),\\ 0&=x_t^*(t)+x_t^*(t-1)+x_t^*(t-2),\\ &t=2,\ldots,T-2.\end{aligned}\tag{10}$$

By the definition of LAM and from the second formula of (9) we derive that

$$\begin{gathered}(x_t^*(t),x_{t+1}^*(t))\in F^*(-x_{t+2}^*(t);(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2})),\\ t=0,1,\ldots,T-2.\end{gathered}\tag{11}$$

Introducing the new notations $x_{t+1}^*(t)\equiv u_{t+1}^*$ and $-x_{t+2}^*(t)\equiv x_{t+2}^*$, $t=0,1,\ldots,T-2$ in the third formula of (10), we obtain by (11) that

$$\begin{gathered}(-x_a^*-x_b^*,\ u_1^*)\in F^*(x_2^*;(\tilde{x}_0,\tilde{x}_1,\tilde{x}_2))\\ (-x_c^*-u_1^*,\ u_2^*)\in F^*(x_3^*;(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)),\\ (x_t^*-u_t^*,\ u_{t+1}^*)\in F^*(x_{t+2}^*;(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}))\\ t=2,\ldots,T-2.\end{gathered}\tag{12}$$

On the other hand if we denote $x_a^*=u_0^*$, $x_b^*=-x_0^*$ and $x_c^*=-x_1^*$ in the first and second inclusions, respectively, we can generalize the formula (12) as

$$(x_t^*-u_t^*,u_{t+1}^*)\in F^*(x_{t+2}^*;(\tilde{x}_t,\tilde{x}_{t+1},\tilde{x}_{t+2}))$$

$$t=0,1,\ldots,T-2,$$

where $u_0^*\in K_M^*(\tilde{x}_0),-x_0^*-x_1^*\in K_M^*(\tilde{x}_0)$, $-x_1^*\in K_N^*(\tilde{x}_1-\tilde{x}_0)$ and these are (i) and (iii) in the result of the theorem, respectively.

Finally, for $t=T-1$ and $t=T$ we have

$$\lambda\overline{x}_{T-1}^*=x_{T-1}^*(T-2)+x_{T-1}^*(T-3)$$

$$\lambda\overline{x}_T^*=x_T^*(T-2),$$

or on the accepted notations

$$\begin{aligned}\lambda\overline{x}_{T-1}^*&=\ u_{T-1}^*-x_{T-1}^*\\ \lambda\overline{x}_T^*&=\ -x_T^*.\end{aligned}\tag{13}$$

Therefore $(u_{T-1}^*-x_{T-1}^*,-x_T^*)\in\lambda\partial g(\tilde{x}_{T-1},\tilde{x}_T)$. Thus taking into account the formulas (12) and (13), we complete the first part of the proof of the theorem.

As for the sufficiency of the obtained conditions, it is clear that by Theorem 3.3[18] under the regularity condition, the representation (10) holds with parameter $\lambda = 1$ for the point $w^{0*} \in \partial_w f(\tilde{w}) \cap K_P^*(\tilde{w})$.

Theorem 2.2. Suppose that for the non-convex problem (4)$-$(6) Condition I holds. Then the necessary condition for the optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$ for this non-convex problem is that there exist a number $\lambda \in \{0,1\}$ and pair of vectors $\{x_t^*\}, \{u_t^*\}$, simultaneously not all equal to zero, satisfying the conditions of Theorem 2.1.

Proof. In this case Condition I ensures the conditions of Theorem 3.24[18] for the problem (7). Therefore, according to this theorem, the necessary condition is obtained as in Theorem 2.1 by starting from the relation (8), written for the non-convex problem.

3 Necessary and Sufficient Conditions of Optimality for Discrete-approximation Problem

Let δ be a step on the t-axis and $x(t) \equiv x_\delta(t)$ be a grid function on a uniform grid on $[0,1]$. The first and second order difference operators are as following

$$\Delta x(t) = \frac{1}{\delta}[x(t+\delta) - x(t)], \Delta^2 x(t) = \frac{1}{\delta}[\Delta x(t+\delta) - \Delta x(t)],$$

and in special case $\Delta x(0) = \dfrac{1}{\delta}(x(\delta) - x(0))$.

Using difference operators, given above, with the problem (1)$-$(3) we now associate the following second order discrete-approximation problem

$$\text{minimize } \varphi(x(1-\delta), \Delta x(1-\delta)), \tag{14}$$
$$\Delta^2 x(t) \in F(x(t), \Delta x(t)) \ , \ t = 0, \delta, \dots, 1-2\delta, \tag{15}$$
$$x(0) \in M, \ \Delta x(0) \in N. \tag{16}$$

Let us use the following straightforward auxiliary mapping

$$Q(x,u) = 2u - x + \delta^2 F\left(x, \frac{u-x}{\delta}\right) \tag{17}$$

to reduce the problem (9) and (10) to a problem of the form (4)$-$(6) and so rewrite the problem (14)$-$(16) as following

$$\text{minimize } \varphi(x(1-\delta), \Delta x(1-\delta)), \tag{18}$$
$$x(t+2\delta) \in Q(x(t), x(t+\delta)), \tag{19}$$
$$t = 0, \delta, 2\delta, \dots, 1-2\delta,$$
$$x(0) \in M, \ x(\delta) \in x(0) + \delta N. \tag{20}$$

By Theorem 2.1 for the optimality of the trajectory $\{\tilde{x}(t)\} := \{\tilde{x}(t) : t = 0, \delta, \dots, 1\}$, in problem (18)$-$(20) it is necessary that there exist a pair of vectors $\{u^*(t)\}, \{x^*(t)\}$ and a number $\lambda \in \{0,1\}$, not all zero, such that

$$\begin{gathered}(x^*(t) - u^*(t), u^*(t+\delta)) \\ \in Q^*(x^*(t+2\delta); (\tilde{x}(t), \tilde{x}(t+\delta), \tilde{x}(t+2\delta))), \\ t = 0, \delta, 2\delta, \dots, 1-2\delta.\end{gathered} \tag{21}$$

By Theorem 2.1 the transversality condition at the starting point takes the form

$$\begin{aligned} u^*(0) &\in K_M^*(\tilde{x}(0)), \\ -x^*(0) - x^*(\delta) &\in K_M^*(\tilde{x}(0)), \\ -x^*(\delta) &\in K_{\delta N}^*(\tilde{x}(\delta) - \tilde{x}(0)). \end{aligned} \tag{22}$$

The transversality condition of second order discrete approximation problem with objective function $\delta g(x_{T-1}, x_T)$ by Theorem 2.1 in extended form is as following

$$(-x^*(1-\delta) + u^*(1-\delta), -x^*(1)) \in \lambda \delta \partial g(\tilde{x}(1-\delta), \tilde{x}(1)). \tag{23}$$

Notice that the function φ in problems (14)$-$(16) and (18)$-$(20) is in the form

$$\varphi(x(1-\delta), \Delta x(1-\delta)) = \delta g(x(1-\delta), x(1)).$$

Therefore we may rewrite tansversality condition (23) as following

$$(-x^*(1-\delta) + u^*(1-\delta), -x^*(1)) \in \lambda \partial \overline{\varphi}(\tilde{x}(1-\delta), \tilde{x}(1)), \tag{24}$$

where

$$\overline{\varphi}(x(1-\delta), x(1))) \equiv \varphi(x(1-\delta), \Delta x(1-\delta)). \tag{25}$$

In order to write (24) in terms of subdifferential of φ , we prove the following theorem:

Theorem 3.1. Suppose $\overline{\varphi}(\cdot,\cdot)$ is a proper convex function given by the relation (25) that is $\overline{\varphi}(x,v) \equiv \varphi(x, \frac{v-x}{\delta})$. Then the following inclusions are equivalent:

$$(\overline{x}^*, \overline{v}^*) \in \partial_{x,v} \overline{\varphi}(x^0, v^0), (x^0, v^0) \in \operatorname{dom}\varphi, \tag{26}$$
$$(\overline{x}^* + \overline{v}^*, \delta \overline{v}^*) \in \partial \varphi\left(x^0, \frac{v^0 - x^0}{\delta}\right). \tag{27}$$

Proof. The subdifferential $\partial_{x,v}\overline{\varphi}(x^0, v^0)$ of proper convex function φ is a convex closed set and is bounded for $(x^0, v^0) \in ri(\operatorname{dom}\varphi)$ [18,21,23,24]. By using the subdifferential definition we obtain the subdifferentials

$$\begin{aligned}\partial_{x,v}\overline{\varphi}(x^0, v^0) = \Big\{ &(\overline{x}^*, \overline{v}^*) \mid \overline{\varphi}(x,v) - \overline{\varphi}(x^0, v^0) \\ &\geq \langle \overline{x}^*, x - x^0 \rangle + \langle \overline{v}^*, v - v^0 \rangle \ , \ \forall \, (x,v) \in \mathbb{R}^{2n} \Big\}\end{aligned} \tag{28}$$

and

$$\partial \varphi\left(x^0, \frac{v^0 - x^0}{\delta}\right) = \left\{ (x^*, v^*) \mid \varphi(x, \frac{v-x}{\delta}) - \varphi(x^0, \frac{v^0 - x^0}{\delta}) \right.$$

$$\geq \langle x^*, x-x^0\rangle + \langle v^*, \frac{v-x}{\delta} - \frac{v^0-x^0}{\delta}\rangle, \forall (x,v)\in \mathbb{R}^{2n}\Big\}$$

of functions $\overline{\varphi}$ and φ , respectively. The last relation can be rewritten as following

$$\partial\varphi\Big(x^0, \frac{v^0-x^0}{\delta}\Big) = \Big\{(x^*,v^*)\mid \varphi\big(x,\frac{v-x}{\delta}\big) - \varphi\big(x^0,\frac{v^0-x^0}{\delta}\big)$$

$$\geq \langle x^* - \frac{v^*}{\delta}, x-x^0\rangle + \langle \frac{v^*}{\delta}, v-v^0\rangle, \forall (x,v)\in\mathbb{R}^{2n}\Big\}. \quad (29)$$

From the equivalence $\overline{\varphi}(x,v) \equiv \varphi\big(x,\frac{v-x}{\delta}\big)$, we derive from (28) and (29) that

$$\overline{x}^* = x^* - \frac{v^*}{\delta}\ ,\ \ \overline{v}^* = \frac{v^*}{\delta},$$

or simply

$$x^* = \overline{x}^* + \overline{v}^*\ ,\ \ v^* = \delta\overline{v}^*.$$

Then $(\overline{x}^*,\overline{v}^*) \in \partial_{x,v}\overline{\varphi}(x^0,v^0)$ if and only if $(\overline{x}^*+\overline{v}^*, \delta\overline{v}^*) \in \partial\varphi(x^0, \frac{v^0-x^0}{\delta})$.

We should express the LAM Q^* in (21) in terms of LAM F^*.

Theorem 3.2. Let $F:\mathbb{R}^n\times\mathbb{R}^n \to P(\mathbb{R}^n)$ be a convex multivalued mapping and Q be mapping defined as $Q(x,u) = 2u - x + \delta^2 F(x,\frac{u-x}{\delta})$. Then the following inclusions are equivalent

$$\Big(\frac{x^*+u^*-v^*}{\delta^2}, \frac{u^*-2v^*}{\delta}\Big) \in F^*\Big(v^*;\big(x,\frac{u-x}{\delta},\frac{x+u-v}{\delta^2}\big)\Big), \quad (30)$$

$$\frac{x+u-v}{\delta^2} \in F\big(x,\frac{u-x}{\delta};v^*\big), v^*\in\mathbb{R}^n$$

$$(x^*,u^*) \in Q^*(v^*;(x,u,v)), v\in Q(x,u;v^*), \quad (31)$$

where $Q(x,u;v^*)$ is the argmaximum set for mapping Q and

$$Q(x,u;v^*) = \{v\in Q(x,u) \mid \langle v,v^*\rangle = H_Q(x,u,v^*)\}.$$

Proof. Taking into account that $v\in Q(x,u;v^*)$ and $\frac{x+u-v}{\delta^2} \in F\big(x,\frac{u-x}{\delta};v^*\big)$, we ensure that the LAMs are nonempty at a given point.

By [18], $F^*(v^*;(x,u,v)) = \partial_{x,u}H(x,u,v^*)$, $v\in F(x,u;v^*)$, holds for convex mapping F, where $\partial_{x,u}H(x,u,v^*) = -\partial_{x,u}[-H(x,u,v^*)]$. Using this formula we find that

$$F^*\Big(v^*;\big(x,\frac{u-x}{\delta},\frac{x+u-v}{\delta^2}\big)\Big) = \partial_{x,u}H_F\big(x,\frac{u-x}{\delta},v^*\big)$$

$$\frac{x+u-v}{\delta^2} \in F\big(x,\frac{u-x}{\delta};v^*\big).$$

Therefore we may rewrite (30) as

$$\Big(\frac{x^*+u^*-v^*}{\delta^2}, \frac{u^*-2v^*}{\delta}\Big) \in \partial_{x,u}H_F\big(x,\frac{u-x}{\delta},v^*\big). \quad (32)$$

Then by the definition of subdifferential, we have

$$H_F\big(x_1,\frac{u_1-x_1}{\delta},v^*\big) - H_F\big(x,\frac{u-x}{\delta},v^*\big)$$

$$\leq \langle x_1-x, \frac{x^*+u^*-v^*}{\delta^2}\rangle + \langle \frac{u_1-x_1}{\delta} - \frac{u-x}{\delta}, \frac{u^*-2v^*}{\delta}\rangle.$$

After some necessary arrangements we obtain that

$$\langle 2u_1-x_1,v^*\rangle + \delta^2 H_F\big(x_1,\frac{u_1-x_1}{\delta},v^*\big) - \langle 2u-x,v^*\rangle$$

$$-\delta^2 H_F\big(x,\frac{u-x}{\delta},v^*\big) \leq \langle x_1-x,x^*\rangle + \langle u_1-u,u^*\rangle. \quad (33)$$

Using the connection between the Hamiltonian functions H_Q and H_F [17]

$$H_Q(x,u,v^*) = \langle 2u-x,v^*\rangle + \delta^2 H_F(x,\frac{u-x}{\delta},v^*), \quad (34)$$

the inequality (33) is replaced with the inequality

$$H_Q(x_1,u_1,v^*) - H_Q(x,u,v^*) \leq \langle x_1-x,x^*\rangle + \langle u_1-u,u^*\rangle,$$

which implies $(x^*,u^*)\in\partial_{x,u}H_Q(x,u,v^*)$. Since $Q^*(v^*;(x,u,v)) = \partial_{x,u}H_Q(x,u,v^*)$, $v\in Q(x,u;v^*)$, we conclude that (31) holds.

Conversely, let (31) holds then similarly, using (34) we obtain inclusion (33) and therefore (30).

Theorem 3.2 can be generalized to the non-convex case; if the problem (4)–(6) is non-convex and consequently the mapping F is non-convex then using the definition of a local tent we can establish the equivalence of the inclusions in Theorem 3.2 for non-convex function F.

Theorem 3.3. Let F be a convex-valued mapping such that the cone $K_{gphQ}(x,u,v)$, $(x,u,v)\in gphQ$ of tangent directions for the mapping defined as $Q(x,u) = 2u - x + \delta^2 F(x,\frac{u-x}{\delta})$ determine a local tent. Then the inclusions (30) and (31) are equivalent.

Proof. See [17].

Lemma 3.1. Let $K_{\delta N}(\tilde{x}(\delta)-\tilde{x}(0))$ be the cone of tangent directions of the set δN at point $\tilde{x}(\delta)-\tilde{x}(0)$ and $K_N(\Delta\tilde{x}(0))$ be the cone of tangent directions of the set N at $\Delta\tilde{x}(0)$, then $K_{\delta N}(\tilde{x}(\delta)-\tilde{x}(0)) = K_N(\Delta\tilde{x}(0))$. Furthermore the relation between the dual cones of these cones $K^*_{\delta N}(\tilde{x}(\delta)-\tilde{x}(0)) = K^*_N(\Delta\tilde{x}(0))$ holds.

Proof. Observe, first, that for arbitrary $\overline{y}\in K_{\delta N}(\tilde{x}(\delta)-\tilde{x}(0))$ and for sufficiently small $\lambda>0$ relation $\tilde{x}(\delta)-\tilde{x}(0)+\lambda\overline{y}\in\delta N$ or in other words, relation $\frac{\tilde{x}(\delta)-\tilde{x}(0)}{\delta} + \frac{\lambda\overline{y}}{\delta} \in N$ holds. Since $K_N(\Delta\tilde{x}(0))$ is a cone of tangent directions and $\delta>0$, we have $\overline{y}\in K_N(\Delta\tilde{x}(0))$. Consequently, $K_{\delta N}(\tilde{x}(\delta)-\tilde{x}(0)) \subseteq K_N(\Delta\tilde{x}(0))$.

Conversely, if $\bar{y} \in K_N(\Delta\tilde{x}(0))$, then for $\delta > 0$, $\bar{y} \in \delta K_N(\Delta\tilde{x}(0))$ and hence $\Delta\tilde{x}(0) + \frac{\lambda\bar{y}}{\delta} \in N$. By the difference formula, we obtain $\tilde{x}(\delta) - \tilde{x}(0) + \lambda\bar{y} \in \delta N$. Hence by the definition of cone of tangent directions $\bar{y} \in K_{\delta N}(\tilde{x}(\delta) - \tilde{x}(0))$. That completes the first part of the proof of the theorem.

On the other hand $y^* \in K^*_{\delta N}(\tilde{x}(\delta) - \tilde{x}(0))$ if and only if the inequality $\langle \bar{y}, y^* \rangle \geq 0$ is satisfied for all $\bar{y} \in K_{\delta N}(\tilde{x}(\delta) - \tilde{x}(0))$. Therefore from the first relation of the theorem, the last inequality holds for all $\bar{y} \in K_N(\Delta\tilde{x}(0))$. Hence it follows that $y^* \in K^*_N(\Delta\tilde{x}(0))$. Going in the reverse direction, by the same way we obtain that $y^* \in K^*_{\delta N}(\tilde{x}(\delta) - \tilde{x}(0))$. So the relation between the dual cones holds. That completes the proof.

Theorem 3.4. Let F be a convex function and φ be proper function that is convex with respect to x and continuous at points of some feasible trajectory $\{x^0(t)\}, t = 0, \delta, \ldots, 1$. Then for the optimality of the trajectory $\{\tilde{x}(t)\}$ in the discrete approximation problem (18)$-$(20) it is necessary that there exist a number $\lambda \in \{0,1\}$ and a pair $\{x^*(t), v^*(t)\}$ simultaneously not all equal to zero, satisfying the approximate Euler-Lagrange and transversality inclusions and adjoint boundary condition for $t = 0$:

$$\begin{aligned} &\left(\Delta^2 x^*(t) + \Delta v^*(t), v^*(t)\right) \\ &\in F^*\left(x^*(t+2\delta); \left(\tilde{x}(t), \Delta\tilde{x}(t), \Delta^2\tilde{x}(t)\right)\right), \\ &\qquad t = 2\delta, 3\delta, \ldots, 1-2\delta; \end{aligned} \tag{35}$$

$$\left(v^*(1-\delta) + \Delta x^*(1-\delta), -x^*(1)\right) \in \lambda\partial\varphi\left(\tilde{x}(1-\delta), \Delta\tilde{x}(1-\delta)\right), \tag{36}$$

$$\begin{aligned} -x^*(\delta) &\in K^*_N(\Delta\tilde{x}(0)), \\ v^*(0) + \Delta x^*(0) &\in K^*_M(\tilde{x}(0)) \end{aligned} \tag{37}$$

respectively, where $v^*(t) = \frac{u^*(t) - 2x^*(t+\delta)}{\delta}$, and $v^*(0)$ is the value of $v^*(t)$ for $t = 0$.

Proof. By Theorem 3.1 and Lemma 3.1 the conditions (21), (22) and (24) for convex problem takes the form

$$\begin{aligned} &\left(\frac{x^*(t) - u^*(t) + u^*(t+\delta) - x^*(t+2\delta)}{\delta^2}, \frac{u^*(t+\delta) - 2x^*(t+2\delta)}{\delta}\right) \\ &\in F^*(x^*(t+2\delta); (\tilde{x}(t), \Delta\tilde{x}(t), \Delta^2\tilde{x}(t))), \\ &\qquad t = 0, \delta, 2\delta, 3\delta, \ldots, 1-2\delta; \end{aligned} \tag{38}$$

$$\left(\frac{u^*(1-\delta) - x^*(1-\delta)}{\delta}, \frac{-x^*(1)}{\delta}\right) \in \lambda\partial\overline{\varphi}(\tilde{x}(1-\delta), \tilde{x}(1)), \tag{39}$$

$$\begin{aligned} u^*(0) &\in K^*_M(\tilde{x}(0)), \\ -x^*(0) - x^*(\delta) &\in K^*_M(\tilde{x}(0)), \\ -x^*(\delta) &\in K^*_N(\Delta\tilde{x}(0)), \end{aligned} \tag{40}$$

respectively, only it is taken into account that LAM is positive homogeneous on the first argument.

Transversality condition (39) is obtained from (24) by denoting $\delta x^*(t)$ and $\delta u^*(t)$ again with $x^*(t)$ and $u^*(t)$, respectively.

The third inclusion in (40) is in the result of Lemma 3.1 applied to the third inclusion of (22).

Under the regularity condition, conditions (38)$-$(40) are also sufficient for optimality of $\{\tilde{x}(t)\}$. Let us denote $v^*(t+\delta) = \frac{u^*(t+\delta) - 2x^*(t+2\delta)}{\delta}$, then it is obvious that

$$\begin{aligned} &\frac{x^*(t) - u^*(t) + u^*(t+\delta) - x^*(t+2\delta)}{\delta^2} \\ &= \frac{x^*(t) - \delta v^*(t) - 2x^*(t+2\delta) + \delta v^*(t+\delta) + x^*(t+2\delta)}{\delta^2} \\ &= \Delta^2 x^*(t) + \Delta v^*(t). \end{aligned} \tag{41}$$

Therefore from (38) and (41), the inclusion (35) holds.

On the other hand by condition (39) and Theorem 3.1 we have

$$\begin{aligned} &\left(\frac{u^*(1-\delta) - 2x^*(1) + x^*(1) - x^*(1-\delta)}{\delta}, \frac{-\delta x^*(1)}{\delta}\right) \\ &\in \lambda\partial\varphi\left(\tilde{x}(1-\delta), \Delta\tilde{x}(1-\delta)\right). \end{aligned}$$

and hence, by the notations given above this inclusion is simply (36), that is

$$\left(v^*(1-\delta) + \Delta x^*(1-\delta), -x^*(1)\right) \in \lambda\partial\varphi\left(\tilde{x}(1-\delta), \Delta\tilde{x}(1-\delta)\right).$$

The first inclusion of (37) is not different from the third inclusion of (40). Also since the cone $K^*_M(\tilde{x}(0))$ is convex, then by the first and the second inclusions of (40) the second inclusion of (37) follows, where $v^*(0) = \frac{u^*(0) - 2x^*(\delta)}{\delta}$.

Theorem 3.5. Suppose that Condition I is satisfied for the non-convex problem, then $\{\tilde{x}(t)\}$ is an optimal trajectory of this problem if there exist a number $\lambda \in \{0,1\}$ and a pair $\{x^*(t), v^*(t)\}$ simultaneously not all equal to zero, satisfying (35), (36) and (37) for non-convex case.

4 Sufficient Conditions of Optimality for the Mayer Problem

Theorem 4.1. For the optimality of the trajectory $\tilde{x}(t)$ in the convex problem (1)$-$(3), it is sufficient that there exists a pair of absolutely continuous functions $\{x^*(t), v^*(t)\}, t \in [0,1]$, satisfying the second order Euler-Lagrange differential inclusion

(i) $\left(\frac{d^2x^*(t)}{dt^2}+\frac{dv^*(t)}{dt},v^*(t)\right)\in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t))\right)$ a.e. $t\in[0,1]$,

the transversality conditions at the endpoints $t=1$ and $t=0$

(ii) $\left(v^*(1)+\frac{dx^*(1)}{dt},-x^*(1)\right)\in\partial\varphi\left(\tilde{x}(1),\tilde{x}'(1)\right)$,

(iii) $-x^*(0)\in K^*_N(\tilde{x}'(0))$, $v^*(0)+\frac{dx^*(0)}{dt}\in K^*_M(\tilde{x}(0))$, respectively,

and the condition ensuring that the locally adjoint mapping F^* is nonempty at a given point

(iv) $\frac{d^2\tilde{x}(t)}{dt^2}\in F\left(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t)\right)$, a.e. $t\in[0,1]$, where $F(x,u;v^*)=\{v\in F(x,u)\mid\langle v,v^*\rangle=H(x,u,v^*)\}$ is the argmaximum set for multivalued mapping F.

Here we assume $x^*(t),t\in[0,1]$, to be absolutely continuous function together with the first order derivative and $\frac{d^2x^*(\cdot)}{dt^2}\in L^n_1([0,1])$. Besides $v^*(t)$, $t\in[0,1]$ is absolutely continuous and $\frac{dv^*(\cdot)}{dt}\in L^n_1([0,1])$.

Proof. From condition (i) of the theorem we have

$$\left(\frac{d^2x^*(t)}{dt^2}+\frac{dv^*(t)}{dt},v^*(t)\right)\in\partial_{(x,v)}H\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right). \quad (42)$$

Thus, using the definition of subdifferential set of the Hamiltonian function H_F, (42) can be replaced by the inequality

$$\left\langle\frac{d^2x^*(t)}{dt^2}+\frac{dv^*(t)}{dt},x(t)-\tilde{x}(t)\right\rangle+\left\langle v^*(t),\frac{dx(t)}{dt}-\frac{d\tilde{x}(t)}{dt}\right\rangle$$

$$\geq H_F(x(t),x'(t),x^*(t))-H_F(\tilde{x}(t),\tilde{x}'(t),x^*(t)). \quad (43)$$

Moreover, by the definition of the Hamiltonian function, we can rewrite (43) in the form

$$\left\langle\frac{d^2x^*(t)}{dt^2},x(t)-\tilde{x}(t)\right\rangle+\frac{d}{dt}\left\langle v^*(t),x(t)-\tilde{x}(t)\right\rangle$$

$$\geq\left\langle\frac{d^2x(t)}{dt^2},x^*(t)\right\rangle-\left\langle\frac{d^2\tilde{x}(t)}{dt^2},x^*(t)\right\rangle.$$

This means that

$$0\leq\left\langle\frac{d^2x^*(t)}{dt^2},x(t)-\tilde{x}(t)\right\rangle-\left\langle\frac{d^2(x(t)-\tilde{x}(t))}{dt^2},x^*(t)\right\rangle+\frac{d}{dt}\left\langle v^*(t),x(t)-\tilde{x}(t)\right\rangle. \quad (44)$$

First two inner products in (44) can be shown as difference of two derivatives, that is

$$\left\langle\frac{d^2x^*(t)}{dt^2},x(t)-\tilde{x}(t)\right\rangle-\left\langle\frac{d^2(x(t)-\tilde{x}(t))}{dt^2},x^*(t)\right\rangle$$

$$=\frac{d}{dt}\left\langle\frac{dx^*(t)}{dt},x(t)-\tilde{x}(t)\right\rangle-\frac{d}{dt}\left\langle\frac{d(x(t)-\tilde{x}(t))}{dt},x^*(t)\right\rangle.$$

Integrating (44) over the interval $[0,1]$ and taking into account that $x(\cdot),\tilde{x}(\cdot)$ are feasible we obtain

$$\int_0^1\left[\frac{d}{dt}\left\langle\frac{dx^*(t)}{dt},x(t)-\tilde{x}(t)\right\rangle-\frac{d}{dt}\left\langle\frac{d(x(t)-\tilde{x}(t))}{dt},x^*(t)\right\rangle\right]dt$$

$$+\langle v^*(1),x(1)-\tilde{x}(1)\rangle-\langle v^*(0),x(0)-\tilde{x}(0)\rangle\geq 0. \quad (45)$$

If we compute the integral on the right hand side of (45), then it follows that

$$0\leq\left\langle\frac{dx^*(1)}{dt},x(1)-\tilde{x}(1)\right\rangle-\left\langle\frac{dx^*(0)}{dt},x(0)-\tilde{x}(0)\right\rangle$$

$$+\left\langle\frac{dx(0)}{dt}-\frac{d\tilde{x}(0)}{dt},x^*(0)\right\rangle-\left\langle\frac{dx(1)}{dt}-\frac{d\tilde{x}(1)}{dt},x^*(1)\right\rangle$$

$$+\langle v^*(1),x(1)-\tilde{x}(1)\rangle-\langle v^*(0),x(0)-\tilde{x}(0)\rangle,$$

and hence

$$0\leq\left\langle\frac{dx^*(1)}{dt}+v^*(1),x(1)-\tilde{x}(1)\right\rangle$$

$$-\left\langle\frac{dx(1)}{dt}-\frac{d\tilde{x}(1)}{dt},x^*(1)\right\rangle+\left\langle\frac{dx(0)}{dt}-\frac{d\tilde{x}(0)}{dt},x^*(0)\right\rangle$$

$$-\left\langle\frac{dx^*(0)}{dt}+v^*(0),x(0)-\tilde{x}(0)\right\rangle. \quad (46)$$

Using transversality conditions (ii) and (iii), for all feasible arcs $x(t),t\in[0,1]$, the relations

$$\left\langle v^*(1)+\frac{dx^*(1)}{dt},x(1)-\tilde{x}(1)\right\rangle-\left\langle\frac{dx(1)}{dt}-\frac{d\tilde{x}(1)}{dt},x^*(1)\right\rangle$$

$$\leq\varphi(x(1),x'(1))-\varphi(\tilde{x}(1),\tilde{x}'(1)) \quad (47)$$

and

$$\left\langle v^*(0)+\frac{dx^*(0)}{dt},x(0)-\tilde{x}(0)\right\rangle\geq 0,$$

$$\left\langle\frac{dx(0)}{dt}-\frac{d\tilde{x}(0)}{dt},-x^*(0)\right\rangle\geq 0$$

hold. Thus (46), (47) and last inequalities imply

$$0\leq\varphi(x(1),x'(1))-\varphi(\tilde{x}(1),\tilde{x}'(1)), \quad (48)$$

then it follows that $\tilde{x}(t),t\in[0,1]$ is optimal.

Theorem 4.2. Let problem (1)−(3) be non-convex problem, that is function $\varphi:\mathbb{R}^{2n}\to\mathbb{R}$ is non-convex with respect to x, and F is a non-convex mapping. Then for the optimality of arc $\tilde{x}(t),t\in[0,1]$, among all feasible solutions of the problem (1)−(3) it is sufficient that there exists a pair of absolutely continuous functions $\{x^*(t),v^*(t)\}$, $t\in[0,1]$, satisfying the conditions:

(i) $\left(\frac{d^2x^*(t)}{dt^2}+\frac{dv^*(t)}{dt}+x^*(t),v^*(t)\right)\in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t))\right)$, a.e. $t\in[0,1]$,

(ii) $\varphi(x,v) - \varphi(\tilde{x}(1),\tilde{x}'(1)) \geq \langle v^*(1) + \frac{dx^*(1)}{dt}, x - \tilde{x}(1)\rangle - \langle x^*(1), v - \tilde{x}'(1)\rangle$, $\forall (x,v) \in \mathbb{R}^{2n}$,

(iii) $-x^*(0) \in K_N^*(\tilde{x}'(0))$, $v^*(0) + \frac{dx^*(0)}{dt} \in K_M^*(\tilde{x}(0))$,

(iv) $\langle \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\rangle = H_F(\tilde{x}(t),\tilde{x}'(t);x^*(t))$, a.e. $t \in [0,1]$.

Proof. By condition (i) and definition of LAM in the non-convex case (see Section 1)

$$H_F\left(x(t),x'(t),x^*(t)\right) - H_F\left(\tilde{x}(t),\tilde{x}'(t),x^*(t)\right)$$
$$\leq \left\langle \frac{d^2x^*(t)}{dt^2} + \frac{dv^*(t)}{dt} + x^*(t), x(t) - \tilde{x}(t)\right\rangle + \left\langle v^*(t), \frac{dx(t)}{dt} - \frac{d\tilde{x}(t)}{dt}\right\rangle$$

or similarly

$$\left\langle \frac{d^2x(t)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\right\rangle$$
$$\leq \left\langle \frac{d^2x^*(t)}{dt^2} + x^*(t), x(t) - \tilde{x}(t)\right\rangle + \frac{d}{dt}\langle v^*(t), x(t) - \tilde{x}(t)\rangle.$$

From the latter inequality is justified (44). Thus the continuation of the proof of the theorem is similar to the one for Theorem 4.1.

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