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Féjer Type Inequalities for (s,m)-Convex Functions in Second Sense

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Abstract: In this paper, we consider (s,m)-convex functions in second sense which were introduced and studied by N. Eftekhari . We prove several Féjer-Hermite-Hadamard type integral inequalities for (s,m)-convex functions in second sense. Our results include several new and known results as special cases. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

Keywords: Féjer-Hermite-Hadamard inequality, min a/m, a, max b, b/m

1 Introduction

In recent years, theory of convex functions has received special attention by many researchers because of its importance in different fields of science like biology, economy and optimization, in particular, in recent years various generalizations and extensions of the classical convexity have been introduced and so the theory of inequalities has made significant contributions in various areas of mathematics.

This research is dedicated to generalize some results related to convex functions, attributed to Charles Hermite [10], Jaques Hadamard [9] and Lipót Fejér [8].

The Hermite-Hadamard and Fejér inequalities have been under intense investigation. Many applications and generalizations; its proofs can be found in the literature, for example, [1,3,13,15,17,18,23], and their references.

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function, if

$$f(tx+(1-t)y) \leq t f(x)+(1-t) f(y),$$

for all $x,y \in I$ and $t \in [0,1]$.

The following double inequality, which is well known as the Hermite-Hadamard inequality.

Theorem 1(See [9]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a,b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)\,dx \tag{1}$$

$$\leq \frac{f(a)+f(b)}{2}. \tag{2}$$

In the year 1905 Leopold Féjer [8] gave a generalization of the inequality (1) as the following:

Theorem 2.*If $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a convex function, and $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx$$
$$\leq \int_a^b f(x)g(x)dx$$
$$\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{3}$$

For some results which generalize, improve, and extend the inequalities (1), see [4,5].

In the year 1984, G. Toader, [20], defined $m-$convexity, an intermediate between usual convexity and the starshaped property, as the following:

Definition 1(see [20]). *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if we have*

$$f(tx+m(1-t)y) \leq t f(x)+m(1-t) f(y) \tag{4}$$

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for all $x,y \in [0,b]$ *and* $t \in [0,1]$.

Denote by $K_m(b)$ the set of m-convex functions on $[0,b]$ for which $f(0) \le 0$.

*Remark.*For $m=1$, we recapture the concept of convex functions defined on $[0,b]$, and $m=0$ the concept of starshaped functions on $[0,b]$, i.e. $f(tx) \le t f(x)$, for all $x \in [0,b]$ and $t \in [0,1]$.

In the same article we can find the following results.

Lemma 1.*[21]*

1.If $f \in K_m(b)$, then f it is starshaped.
2.If f is m-convex and $0<n<m\le 1$, then f is n-convex.

It is important to note that for $m \in (0,1)$ there are continuous and differentiable functions that are m-convex, but they aren't convex in the standard sense (Ver [22]).

In [6], S.S. Dragomir and G. Toader prove the following inequality type Hermite-Hadamard:

Theorem 3([6]). *Let $f:[0,+\infty) \to \mathbb{R}$ to be a m-convex function with $m \in (0,1]$. If $0 \le a < b < +\infty$ and $f \in L^1([a,b])$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)dx \tag{5}$$

$$\le \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{6}$$

There are some generalizations of this result, which can be found in [11].

Also in the year 2008, prove the following theorem of Hermite-Hadamard type for differentiable functions.

Theorem 4([12]). *Let I be an open real interval such that $[0,+\infty) \subseteq I$. Let $f: I \to \mathbb{R}$ be a differentiable on I such that $f' \in L^1[a,b]$ where $0 \le a < b < +\infty$. If $|f'|^q$ is $m-$convex on $[a,b]$ for some fixed $m \in (0,1]$ and $q \in [1,+\infty)$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx$$

$$\le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m|f'(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}, \left(\frac{|f'(b)|^q + m|f'(\frac{a}{m})|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

In the year 1978 , W.W. Breckner (see [2]), introduced the definition of $s-$ convex of real valued functions in second sense:

Definition 2(see [2]). *Let $0 < s \le 1$. A function $f:[0,b] \to \mathbb{R}$, is said to be $s-$convex in the second sense if:*

$$f(tx+(1-t)y) \le t^s f(x) + (1-t)^s f(y),$$

for all $x,y \in [0,b]$, and $t \in [0,1]$.

S. S. Dragomir and S. Fitzpatrik (see [5]) prove the following theorem:

Theorem 5.*Let $f:[0,+\infty) \to [0,+\infty)$ be a $s-$convex mapping in the second sense with $s \in (0,1)$. If $a,b \in [0,+\infty)$, with $a<b$, $f \in L_1([a,b])$, then:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}.$$

In the year 1993 V. Mihesan (see [16]), introduced the class of $(s,m)-$ convex functions as the following:

Definition 3([16]). *The function $f:[0,b] \to \mathbb{R}$ is said to be $(s,m)-$ convex, where $s,m \in (0,1]$, if for every $x,y \in [a,b]$ and $t \in [0,1]$ we have*

$$f(tx+m(1-t)y) \le t^s f(x) + m(1-t^s) f(y).$$

Recently, in the year 2014, combining notions of s - convexity in the second sense of Breckner with m-convexity, N. Eftekhari (see [7]) introduced the class of (s,m) - convexity in the second sense.

Definition 4([7]). *A function $f:[0,b] \to \mathbb{R}$ is said to be $(s,m)-$convex in the second sense, where $(s,m) \in (0,1]^2$ if for every $x,y \in [0,b]$ and $t \in [0,1]$ we have*

$$f(tx+m(1-t)y) \le t^s f(x) + m(1-t)^s f(y).$$

Now gives way to the main objective of this paper, establish inequality of the Hermite-Hadamard-Fejér type for functions (s,m)-convex in the second sense.

2 Some basic properties

Proposition 1.*Let $f:[0,+\infty) \to \mathbb{R}$ be a (s,m)-convex function in second sense, where $s,m \in (0,1]$, and let a,b be nonnegative real numbers with $a<b$. Then for any $x \in [a,b]$ there is $t \in [0,1]$ such that*

$$f(a+b-x)$$
$$\le t^s(f(b)+f(a)) + m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right) - f(x).$$

*Proof.*Since any $x\in[a,b]$ can be represented as $x=ta+(1-t)b$, $t\in[0,1]$, then

$$\begin{aligned}
&f(a+b-x)\\
&=f(a+b-(ta+(1-t)b))=f((1-t)a+tb)\\
&=f\left(tb+m(1-t)\frac{a}{m}\right)\\
&\le t^s f(b)+m(1-t)^s f\left(\frac{a}{m}\right)\\
&=t^s f(b)+m(1-t)^s f\left(\frac{a}{m}\right)-t^s f(a)+t^s f(a)\\
&\quad+m(1-t)^s f\left(\frac{b}{m}\right)-m(1-t)^s f\left(\frac{b}{m}\right)\\
&\le t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)\\
&\quad-t^s f(a)-m(1-t)^s f\left(\frac{b}{m}\right)\\
&\le t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)\\
&\quad-f\left(ta+m(1-t)\frac{b}{m}\right)\\
&=t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)-f(x).
\end{aligned}$$

Proposition 2.*If $f_i:[0,+\infty)\to\mathbb{R}$, $i=1,\dots,n$, are (s,m)-convex functions in second sense, where $(s,m)\in(0,1]^2$, then the function given by $f:=\max\limits_{i=1,2,\dots,n}\{f_i\}$ is also (s,m)-convex in second sense.*

*Proof.*If $x,y\in[0,+\infty)$ and $t\in[0,1]$ then

$$\begin{aligned}
&f_i(tx+m(1-t)y)\\
&\le t^s f_i(x)+m(1-t)^s f_i(y)\\
&\le t^s\max_{i=1,2,\dots,n}\{f_i(x)\}+m(1-t)^s\max_{i=1,2,\dots,n}\{f_i(y)\}\\
&=t^s f(x)+m(1-t)^s f(y).
\end{aligned}$$

Proposition 3.*Let $f_n:[0,+\infty)\to\mathbb{R}$ be is a sequence of functions. If f_n is (s,m)-convex function in second sense, for all $n\ge k$ and $f_n(x)\to f(x)$ (on $[0,+\infty)$), then f is (s,m)-convex function in second sense.*

*Proof.*If $x,y\in[0,+\infty)$ and $t\in[0,1]$, we have, for $n\ge k$.

$$\begin{aligned}
f_n(tx+m(1-t)y)&\le t^s f_n(x)+m(1-t)^s f_n(y),\qquad\forall n\ge k\\
\lim_{n\to\infty}f_n(tx+m(1-t)y)&\le\lim_{n\to\infty}\{t^s f_n(x)+m(1-t)^s f_n(y)\}\\
f(tx+m(1-t)y)&\le t^s f(x)+m(1-t)^s f(y),
\end{aligned}$$

this completes the proof.

Proposition 4.*Let $f:[0,+\infty)\to\mathbb{R}$ be a $(s_1,m)-$convex function in second sense and let $g:[0,+\infty)\to\mathbb{R}$ be a (s_2,m)-convex function in second sense, where $s_1,s_2,m\in(0,1]$. Then $f+g$ is (s,m)-convex function in second sense, where $s=\min\{s_1,s_2\}$.*

Proof.

$$\begin{aligned}
&(f+g)(tx+m(1-t)y)\\
&=f(tx+m(1-t)y)+g(tx+m(1-t)y)\\
&\le t^{s_1}f(x)+m(1-t)^{s_1}f(y)+t^{s_2}g(x)+m(1-t)^{s_2}g(y)\\
&\le t^s f(x)+m(1-t)^s f(y)+t^s g(x)+m(1-t)^s g(y)\\
&=t^s(f(x)+g(x))+m(1-t)^s(f(x)+g(x))\\
&=t^s(f+g)(x)+m(1-t)^s(f+g)(y).
\end{aligned}$$

Proposition 5.*Let $f:[0,+\infty)\to\mathbb{R}$ be a (s,m)-convex function in second sense, where $(s,m)\in(0,1]^2$.*

If $\lambda\ge0$, then λf is (s,m)-convex function in second sense.

*Proof.*For $x,y\in[0,+\infty)$ and $t\in[0,1]$, then

$$\begin{aligned}
\lambda f(tx+m(1-t)y)&\le\lambda\,(t^s f(x)+m(1-t)^s f(y))\\
&=t^s\lambda f(x)+m(1-t)^s\lambda f(y)\\
&=t^s(\lambda f)(x)+m(1-t)^s(\lambda f)(y).
\end{aligned}$$

This completes the proof.

Proposition 6.*Let $f:[0,b]\to\mathbb{R}$, $b>0$ be an m-convex functions with $m\in(0,1]$, and $g:I\subseteq f([0,b])\to\mathbb{R}$ be nondecreasing and (s,m)-convex in second sense on I for some fixed $s\in(0,1]$, then $g\circ f$ is $(s,m)-$convex in second sense on $[0,b]$.*

*Proof.*Let $x,y\in[0,b]$ and $t\in[0,1]$. We have successively

$$\begin{aligned}
f(tx+m(1-t)y)&\le tf(x)+m(1-t)f(y)\\
g(f(tx+m(1-t)y))&\le g(tf(x)+m(1-t)f(y))\\
&\le t^s g(f(x))+m(1-t)^s g(f(y)),
\end{aligned}$$

which shows that $g\circ f$ is $(s,m)-$convex in second sense on $[0,b]$.

Theorem 6.*Let $f:[0,+\infty)\to\mathbb{R}$ a finite function on $\left[ma,\dfrac{b}{m}\right]\subseteq[0,+\infty)$, $(s,m)-$convex in second sense with $m\in(0,1]$. Then f is bounded on any closed interval $[a,b]$.*

*Proof.*Let $x\in[a,b]$, then there is a $t\in[0,1]$ such that $x=ta+(1-t)b$, we have

$$\begin{aligned}
f(x)&=f(ta+(1-t)b)\\
&\le t^s f(a)+m(1-t)^s f\left(\frac{b}{m}\right)\\
&\le f(a)+f\left(\frac{b}{m}\right).
\end{aligned}$$

Thus, f is upper bounded on $[a,b]$.

Now we notice that any $x\in[a,b]$ can be written as $\dfrac{a+b}{2}+t$ for $|t|\le\dfrac{b-a}{2}$ (see [14]),

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)&=f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right)\\
&\le\frac{1}{2^s}f\left(\frac{a+b}{2}+t\right)+\frac{m}{2^s}f\left(\frac{\frac{a+b}{2}-t}{m}\right).
\end{aligned}$$

Applying the first part of this theorem in the interval $\left[\frac{a}{m},\frac{b}{m}\right]$, we obtain that

$$f\left(\frac{a+b}{2}\right)-\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m^2}\right)\right]$$
$$\leq f\left(\frac{a+b}{2}\right)-\frac{m}{2^s}f\left(\frac{\frac{a+b}{2}-t}{m}\right)$$
$$\leq \frac{1}{2^s}f\left(\frac{a+b}{2}+t\right)=\frac{1}{2^s}f(x).$$

Which we obtain that

$$2^sf\left(\frac{a+b}{2}\right)-2^s\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m^2}\right)\right]\leq f(x),$$

3 Main results

The proof of the following theorem is a technique similar to the theorem 5 in [19].

Theorem 7. *Let $f:I\subseteq[0,+\infty)\to[0,+\infty)$ be a $(s,m)-$convex in second sense function, $a,b\in I$ with $a<b$, $f\in L_1([a,b])$ and $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$\int_a^b f(x)g(x)dx$$
$$\leq \frac{f(a)+f(b)}{2}\int_a^b\left(\frac{b-x}{b-a}\right)^s g(x)dx$$
$$+\frac{m}{2}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\int_a^b\left(\frac{x-a}{b-a}\right)^s g(x)dx.$$

Proof. Since f and g are real nonnegative functions, g is integrable and symmetric about $\frac{a+b}{2}$, we will have that

$$\int_a^b f(x)g(x)dx$$
$$=\frac{1}{2}\left[\int_a^b f(x)g(x)dx+\int_a^b f(a+b-x)g(a+b-x)dx\right]$$
$$=\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]g(x)dx$$
$$=\frac{1}{2}\int_a^b\left\{f\left(a\left(\frac{b-x}{b-a}\right)+b\left(\frac{x-a}{b-a}\right)\right)\right.$$
$$\left.+f\left(a\left(\frac{x-a}{b-a}\right)+b\left(\frac{b-x}{b-a}\right)\right)\right\}g(x)dx$$
$$\leq\frac{1}{2}\int_a^b\left\{\left(\frac{b-x}{b-a}\right)^s f(a)+m\left(\frac{x-a}{b-a}\right)^s f\left(\frac{b}{m}\right)\right.$$
$$\left.+m\left(\frac{x-a}{b-a}\right)^s f\left(\frac{a}{m}\right)+\left(\frac{b-x}{b-a}\right)^s f(b)\right\}g(x)dx$$
$$=\frac{f(a)+f(b)}{2}\int_a^b\left(\frac{b-x}{b-a}\right)^s g(x)dx$$
$$+\frac{m}{2}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\int_a^b\left(\frac{x-a}{b-a}\right)^s g(x)dx.$$

The proof is complete.

Remark. Note that if we do $m=1$ in the previous theorem we obtain inequality of the Hermite-Hadamard-Féjer type for s-convex functions, i.e.:

$$\int_a^b f(x)g(x)dx$$
$$\leq\frac{f(a)+f(b)}{2}\int_a^b\left[\left(\frac{b-x}{b-a}\right)^s+\left(\frac{x-a}{b-a}\right)^s\right]g(x)dx.$$

Corollary 1. *Under the same hypotheses of theorem 7, if $g(x)=1$, we have:*

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx$$
$$\leq\left(\frac{f(a)+f(b)}{2}+\frac{m}{2}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\right)\left(\frac{1}{2^s}\right).$$

Proof. If we take $g(x)=1$ in theorem 7, we have:

$$\int_a^b f(x)dx$$
$$\leq\frac{f(a)+f(b)}{2}\int_a^b\left(\frac{b-x}{b-a}\right)^s dx$$
$$+\frac{m}{2}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\int_a^b\left(\frac{x-a}{b-a}\right)^s dx.$$

Given that the function $\varphi(x)=x^s$ is concave if $0<s\leq 1$, then from Jensen's inequality:

$$\frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq\frac{f(a)+f(b)}{2}\left(\frac{1}{b-a}\int_a^b\frac{b-x}{b-a}dx\right)^s$$
$$+\frac{m}{2}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\left(\frac{1}{b-a}\int_a^b\frac{x-a}{b-a}dx\right)^s$$
$$=\frac{1}{2^s}\left(\frac{f(a)+f(b)}{2}+\frac{m}{2}\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\right)$$
$$=\frac{1}{2^{s+1}}\left(f(a)+f(b)+m\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]\right).$$

The following theorem will serve for a lower level to the left inequality (3), for functions (s,m)-convex in the second sense.

Theorem 8. *Let $f:I\subseteq\mathbb{R}\to[0,+\infty)$ be is $(s,m)-$convex in second sense function, $a,b\in I$ with $a<b$, $f\in L_1[c,d]$, where $c=min\{a/m,a\}$ and $d=max\{b,b/m\}$ and $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx$$
$$\leq\frac{1}{2^s}\int_a^b f(x)g(x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx.$$

Proof. Since $f : I \subseteq \mathbb{R} \to [0,+\infty)$ be is $(s,m)-$convex in second sense function and $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\dfrac{a+b}{2}$.

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\
&= \int_a^b f\left(\frac{a+b-x+x}{2}\right)g(x)dx \\
&\leq \int_a^b \left[\frac{1}{2^s} f(a+b-x)+\frac{m}{2^s} f\left(\frac{x}{m}\right)\right]g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(a+b-x)\,g(x)dx+\left(\frac{m}{2^s}\right)\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(a+b-x)\,g(a+b-x)dx+\left(\frac{m}{2^s}\right)\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(x)\,g(x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx.
\end{aligned}
$$

Remark. In Theorem 8, if we take $m = 1$, then

$$
f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \frac{1}{2^{s-1}}\int_a^b f(x)g(x)dx,
$$

which is an inequality of the Hermite-Hadamard-Féjer type for s-convex functions.

Corollary 2. *Under the same hypotheses of theorem 8, if $g(x) = 1$ and $m = 1$, we have:*

$$
f\left(\frac{a+b}{2}\right)(b-a) \leq \frac{1}{2^{s-1}}\int_a^b f(x)dx.
$$

In this case we obtain inequality on the side left of the theorem 5.

We finally establish a Hermite-Hadamard type inequalities for the product of functions when the product is a function (s,m) - convex in the second sense.

Theorem 9. *Let $f : [0,+\infty) \to \mathbb{R}$ be a (s,m)-convex in second sense, where $s,m \in (0,1]$, and let $a,b \in [0,+\infty)$ with $a < b$. Suppose that $f \in L_1[a,b/m]$, and that $g : [a,b] \to \mathbb{R}$ is a nonnegative, integrable function which is symmetric with respect to $\dfrac{a+b}{2}$. Then*

$$
\begin{aligned}
& 2^s f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - m\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&\leq \int_a^b f(x)g(x)dx.
\end{aligned}
$$

Proof. In this case we have

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\
&= \int_a^b f\left(\frac{a+b-x}{2}+\frac{m}{2}\frac{x}{m}\right)g(x)dx \\
&\leq \int_a^b \left(\frac{1}{2^s} f(a+b-x)+\frac{m}{2^s} f\left(\frac{x}{m}\right)\right)g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(a+b-x)g(x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(a+b-x)g(a+b-x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(x)g(x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx,
\end{aligned}
$$

thus obtaining the required inequality.

Theorem 10. *Let $f,g : [0,+\infty) \to [0,+\infty)$ be such that $fg \in L_1([a,b])$, where $0 \leq a < b < \infty$. If f is (s_1,m_1)-convex function and g is (s_2,m_2)-convex function on $[a,b]$, for some fixed $m_1,m_2,s_1,s_2 \in (0,1]$, then*

$$
\frac{1}{b-a}\int_a^b f(t)g(t)dt \leq \min\{M_1,M_2\},
$$

where

$$
\begin{aligned}
M_1 :=& \frac{1}{s_1+s_2+1}\left(f(a)g(a)+m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right) \\
&+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
&\cdot\left(m_2 f(a)g\left(\frac{b}{m_2}\right)+m_1 f\left(\frac{b}{m_1}\right)g(a)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
M_2 :=& \frac{1}{s_1+s_2+1}\left(f(b)g(b)+m_1m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right) \\
&+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
&\cdot\left(m_2 f(b)g\left(\frac{a}{m_2}\right)+m_1 f\left(\frac{a}{m_1}\right)g(b)\right).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
f(tx+(1-t)y) &\leq t^{s_1} f(x)+m_1(1-t)^{s_1} f\left(\frac{y}{m_1}\right) \\
g(tx+(1-t)y) &\leq t^{s_2} g(x)+m_2(1-t)^{s_2} g\left(\frac{y}{m_2}\right),
\end{aligned}
$$

for all $t \in [0,1]$. f and g are nonnegative, hence

$$
\begin{aligned}
& f(ta+(1-t)b)\cdot g(ta+(1-t)b) \\
&\leq t^{s_1+s_2} f(a)g(a)+m_2 t^{s_1}(1-t)^{s_2} f(a)g\left(\frac{b}{m_2}\right) \\
&+m_1(1-t)^{s_1} f\left(\frac{b}{m_1}\right)t^{s_2}g(a) \\
&+m_1(1-t)^{s_1} f\left(\frac{b}{m_1}\right)m_2(1-t)^{s_2}g\left(\frac{b}{m_2}\right).
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)g(t)dt = \int_0^1 f(ta-(1-t)b)g(ta+(1-t)b)dt\\
&\le f(a)g(a)\int_0^1 t^{s_1+s_2}dt + m_2 f(a) g\left(\frac{b}{m_2}\right)\int_0^1 t^{s_1}(1-t)^{s_2}\\
&\quad +m_1 f\left(\frac{b}{m_1}\right) g(a)\int_0^1 (1-t)^{s_1}t^{s_2}dt\\
&\quad +m_1m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\int_0^1 (1-t)^{s_1+s_2}dt\\
&= f(a)g(a)B(s_1+s_2+1,1)+m_2 f(a) g\left(\frac{b}{m_2}\right)B(s_1+1,s_2+1)\\
&\quad +m_1 f\left(\frac{b}{m_1}\right) g(a)B(s_2+1,s_1+1)\\
&\quad +m_1m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)B(1,s_1+s_2+1),
\end{aligned}
$$

where B is the Euler Beta-function. Then,

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)g(t)dt\\
&\le f(a)g(a)\frac{\Gamma(s_1+s_2+1)\Gamma(1)}{\Gamma(s_1+s_2+2)}\\
&\quad +m_2 f(a) g\left(\frac{b}{m_2}\right)\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}\\
&\quad +m_1 f\left(\frac{b}{m_1}\right) g(a)\frac{\Gamma(s_2+1)\Gamma(s_1+1)}{\Gamma(s_1+s_2+2)}\\
&\quad +m_1m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\frac{\Gamma(1)\Gamma(s_1+s_2+1)}{\Gamma(s_1+s_2+2)}\\
&= f(a)g(a)\frac{(s_1+s_2)\Gamma(s_1+s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad +m_2 f(a) g\left(\frac{b}{m_2}\right)\frac{s_1\Gamma(s_1)s_2\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad +m_1 f\left(\frac{b}{m_1}\right) g(a)\frac{s_2\Gamma(s_2)s_1\Gamma(s_1)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad +m_1m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\frac{(s_1+s_2)\Gamma(s_1+s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&= f(a)g(a)\frac{1}{(s_1+s_2+1)}\\
&\quad +m_2 f(a) g\left(\frac{b}{m_2}\right)\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad +m_1 f\left(\frac{b}{m_1}\right) g(a)\frac{s_1s_2\Gamma(s_2)\Gamma(s_1)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad +m_1m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\frac{1}{(s_1+s_2+1)}\\
&= \frac{1}{s_1+s_2+1}\left(f(a)g(a)+m_1m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right)\\
&\quad +\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad \cdot\left(m_2 f(a) g\left(\frac{b}{m_2}\right)+m_1 f\left(\frac{b}{m_1}\right) g(a)\right).
\end{aligned}
$$

Analogously we obtain

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)g(t)dt\\
&\le \frac{1}{s_1+s_2+1}\left(f(b)g(b)+m_1m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\right)\\
&\quad +\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\\
&\quad \cdot\left(m_2 f(b) g\left(\frac{a}{m_2}\right)+m_1 f\left(\frac{a}{m_1}\right) g(b)\right),
\end{aligned}
$$

hence

$$\frac{1}{b-a}\int_a^b f(t)g(t)dt \le \min\{M_1,M_2\}.$$

Theorem 11. *Let $f: I\subseteq\mathbb{R}\to[0,+\infty)$ be is $(s,m)-$convex in second sense function, $a,b\in I$ with $a<b$, $f\in L_1[c,d]$, where $c=\min\{a/m,a\}$ and $d=\max\{b,b/m\}$. Then*

$$
\begin{aligned}
&2^s f\left(\frac{a+b}{2}\right)-\frac{m}{(b-a)}\int_a^b f\left(\frac{t}{m}\right)dt\\
&\le \frac{1}{b-a}\int_a^b f(t)dt\\
&\le \frac{1}{s+1}\min\left\{f(a)+mf\left(\frac{b}{m}\right), f(b)+mf\left(\frac{a}{m}\right)\right\}. \qquad (7)
\end{aligned}
$$

Proof. Since f is $(s,m)-$convex in second sense on I, we get

$$f(tx+(1-t)y)\le t^s f(x)+m(1-t)^s f\left(\frac{y}{m}\right). \qquad (8)$$

It is easy to observe that

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}\right)\\
&\le \frac{1}{2^s}f(ta+(1-t)b)+\frac{m}{2^s}f\left(\frac{(1-t)a+tb}{m}\right)
\end{aligned}
$$

Integrating over the interval $(0,1)$, we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{2^s}\int_0^1 f(ta+(1-t)b)dt+\frac{m}{2^s}\int_0^1 f\left(\frac{(1-t)a+tb}{m}\right)dt\\
&= \frac{1}{2^s(b-a)}\int_a^b f(t)dt+\frac{m}{2^s(b-a)}\int_a^b f\left(\frac{t}{m}\right)dt
\end{aligned}
$$

We obtain

$$2^s f\left(\frac{a+b}{2}\right)-\frac{m}{(b-a)}\int_a^b f\left(\frac{t}{m}\right)dt \le \frac{1}{b-a}\int_a^b f(t)dt. \qquad (9)$$

This last inequality corresponds to the first inequality as indicated on the thesis.

On the other hand, for all $t\in[0,1]$, we have that

$$f(ta+(1-t)b)\le t^s f(a)+m(1-t)^s f\left(\frac{b}{m}\right)$$

Integrating the above inequality over the interval $(0,1)$, we obtain

$$\int_0^1 f(ta+(1-t)b)dt \le f(a)\int_0^1 t^s dt + mf\left(\frac{b}{m}\right)\int_0^1 (1-t)^s dt = \frac{1}{s+1}\left[f(a)+mf\left(\frac{b}{m}\right)\right].$$

Consequently,

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{s+1}\left[f(a)+mf\left(\frac{b}{m}\right)\right].$$

Using a similar procedure, we obtain that

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{s+1}\left[f(b)+mf\left(\frac{a}{m}\right)\right].$$

Therefore we get the inequality on the right side at (7), this completes the proof of the theorem.

Theorem 12. *Let $f : [0,+\infty) \to [0,+\infty)$ be is $(s,m)-$convex in second sense function, $a,b \in [0,+\infty)$ with $a<b$, $s,m \in L_1([0,1])$. Then*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(t)dt + \frac{1}{b-ma}\int_{ma}^{b} f(t)dt\right] \le \frac{f(a)+f(b)}{s+1}. \tag{10}$$

Proof. Since f is $(s,m)-$convex in second sense on $[0,+\infty)$, we get

$$\begin{aligned}
f(ta+m(1-t)b) &\le t^s f(a) + m(1-t)^s f(b)\\
f((1-t)a+mtb) &\le (1-t)^s f(a) + mt^s f(b)\\
f(mta+(1-t)b) &\le mt^s f(a) + (1-t)^s f(b)\\
f(m(1-t)a+tb) &\le m(1-t)^s f(a) + t^s f(b)
\end{aligned}$$

Adding the above four inequalities, we get

$$\begin{aligned}
&f(ta+m(1-t)b) + f((1-t)a+mtb)\\
&+f(mta+(1-t)b) + f(m(1-t)a+tb)\\
&\le ((1-t)^s+t^s)(mf(b)+f(a)+f(b)+mf(a)).
\end{aligned}$$

Now, integrating over the interval $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 [f(ta+m(1-t)b) + f((1-t)a+mtb)dt\\
+&\int_0^1 f(mta+(1-t)b) + f(m(1-t)a+tb)]dt\\
&\le \int_0^1 (m+1)((1-t)^s+t^s)(f(a)+f(b))dt.
\end{aligned}$$

Where,

$$\frac{2}{mb-a}\int_a^{mb} f(t)dt + \frac{2}{b-ma}\int_{ma}^{b} f(t)dt$$

$$\le \int_0^1 (m+1)((1-t)^s+t^s)(f(a)+f(b))dt.$$

Now,

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(t)dt + \frac{1}{b-ma}\int_{ma}^{b} f(t)dt\right] \le \frac{f(a)+f(b)}{s+1}.$$

Remark. Note that if $m=1$ in Theorem 12 we obtain the right side of the Hermite Hadamard inequality for s-convex functions.

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{s+1}. \tag{11}$$

We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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References

[1] M. Bessenyei and Zs. Páles, Characterization of convexity via Hadamard's inequality, Math. Inequal. Appl. 9/1 (2006), 53-62.

[2] W. W. Breckner, Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, Publ. Inst. Math. (Beograd), 23, (1978), 13-20.

[3] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2002. (http://rgmia.vu.edu.au/monographs/).

[4] S. S. Dragomir, J. Pečarić and L. E.Persson , Some Inequalities for Hadamard type, Soochow J. Math. 21(1995),335-341.

[5] S. S. Dragomir and S. Fitzpatrik, The Hadamard inequality for s-convex functions in the second sense, Demonstration Math. 32(4), (1999),687-696.

[6] S. S. Dragomir and G. Toader, Some inequalities for m-convex functions, Studia Univ. Babes-Bolyai Math., 38(1), (1993), 21-28.

[7] N. Eftekhari, Some remarks on $(s,m)-$convexity in the second sense, Journal of mathematical inequalities, Vol 8, N 2 (2014), 489-495.

[8] L. Fejér. Uber die Fourierreinhen,II. Math. Naturwiss. Anz. Ungar. Akadd. Wiss. 24 (1906) 369 - 390.

[9] J.S Hadamard. Etude sur les propiètés des fonctions entieres et en particulier d' une fontion considerer per Riemann, J. Math. Pure and Appl. 58 (1893) 171- 215

[10] Ch. Hermite, Sur deux limites d'une intégrale défine, Mathesis 3, (1883), 82.

[11] M. Klaričić Bacula, J. Pečarić and M. Ribičić,Companion Inequalities to Jensen's inequality for m-convex and (α, m) convex functions, J. Inequal. Pure and Appl. Math, 7(5), (2006), Art. 194,http//jipam.vu.edu.au/article.php?sid=811.

[12] M. Klaričić Bacula, M. E. Özdemir and J. Pečarić ,Hadamard type inequalities for m-convex and (α,m)-convex functions, J. Inequal. Pure and Appl. Math, vol 9 (2008), Art. 96.

[13] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality, PWN - Uniwersytet Slski, Warszawa-Krakow-Katowice, 1985. Second Edition: Birkhauser, Basel-Boston-Berlin, 2009.

[14] T. Lara, E. Rosales and J. Sánchez, New Properties of m-Convex Functions, International Journal of Mathematical Analysis Vol. 9, 2015, no. 15, 735 - 742. http://dx.doi.org/10.12988/ijma.2015.412389

[15] N. Merentes y S. Rivas, El desarrollo del concepto de función convexa. XXVI Escuela Venezolana de Matemáticas. EMALCA-Venezuela 2013.

[16] V. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania, 1993.

[17] C. P. Niculescu and L. E. Persson, Convex Functions and their Applications. A Contemporary Approach, CMS Books in Mathematics vol. 23, Springer-Verlag, New York, 2006.

[18] J. E. Pečarić, F. Proschan, Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Acad. Press,Inc., Boston, 1992.

[19] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions, Acta Math. Univ Comenianae, Vol LXXIX, 2 (2010),pp. 265-272.

[20] G. Toader, Some generalizations of the convexity, Proc. Colloq. Approx. Optim., (1984), 329-338.

[21] G. Toader, On a generalizations of the convexity, Mathematica,30 (53), (1988), 83-87.

[22] S. Toader, The order of star-convex function, Bull Applied and Comp. Math. ,85-B (1998). BAM-1473,347-350.

[23] C. Zalinescu, Convex Analysis in General Vector Spaces, World Scientíc, New Jersey, 2002.

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