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An Efficient Numerical Technique for Solving the Inverse Gravity Problem of Finding a Lateral Density

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Abstract: The main goal of our paper is to construct a technique for the gravity inversion problem of finding a variable density in a horizontal layer on the basis of gravitational data. This technique consists of two steps: extracting the gravitational field and solving the linear integral equation of the density. After discretization and approximation of integral operator, this problem is reduced to solving large systems of linear algebraic equations. To solve these systems, we proposed a memory-efficient algorithm based on the iterative method of minimal residuals. The idea of memory optimization is based on exploiting the block-Toeplitz structure of coefficients matrix. The algorithms were parallelized and implemented using the Uran and UrFU supercomputers. A model problem with synthetic gravitational data was solved.

Keywords: gravity data inversion, potential fields, gradient method, parallel algorithm, multicore processor

1 Introduction

The problem of localizing the gravity field sources in a horizontal layer between given depths and finding a variable density in this layer is very important. This problem arises during the Earth's crust model construction and in the prognostic geological explorations. The solution process consists of two separate steps. The first one is to extract anomalous effect of the considered layer from the observed gravity field. The extracted anomalous field is the right part of an integral equation of the desired density. This problem was formulated and studied in [1]. Some approaches to its solving are presented in [2,3].

In this work, we propose an effective numerical technique for solving this problem and test it on synthetic gravitational data. Note that both steps include solving the Fredholm integral equation of the first kind, which belongs to a class of ill-posed problems.

The algorithms were implemented using the multicore processors and Intel Xeon Phi coprocessors of the Uran supercomputer installed at the Institute of Mathematics and Mechanics UrB RAS and UrFU supercomputer installed at the Ural Federal University.

2 Anomalous field extraction

As a preliminary processing of gravity observations, we should recalculate the measured gravitational field to the horizontal level $z = 0$, see [4,5].

The next step is to use the technique from [2]. It allows one to suppress the effects of shallow and deeper objects and to extract approximately the gravity signal $\Delta g(x,y)$ of sources located in a horizontal layer between given depths H_1 and H_2.

Let's describe this technique under assumption that there are no horizontally elongated sources above the considered layer. The technique is based on the upward and downward analytic continuation.

After the continuation upward to level $z = -H$ (assuming that the z axis is directed downwards), the influence of subsurface sources (located above the level $z = H$) significantly abates. Therefore, the field $g(x,y,z)\big|_{z=0}$ should be continued to the level $z = -H$.

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This continuation is implemented using the Poisson integral

$$g(x,y,-H) = \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H}{[(x-x')^2+(y-y')^2+H^2]^{3/2}}\times \\ \times g(x',y',0)dx'dy'. \quad (1)$$

The distortions generated by this procedure are most substantial near the boundary of the area D due to integration on the limited area. Assuming that the field of the desired object is adequately traced, the residual anomaly at the boundary of the area D should be close to zero. To reduce the distortions, we should preliminary subtract a function $f(x,y)$ from the measured field $g(x,y,0)$.

This function is a solution of a plane Dirichlet problem for the Laplace equation $\Delta f(x,y) = 0$, $f(x,y)|_{\delta D} = \phi(x,y)$. $\phi(x,y) = g(x,y,0)$, *i. e.*, the desired function has the same values at the boundary of the area D as the observed gravitational field. To solve the Dirichlet problem, we use the successive over-relaxation method [6].

To eliminate influence of the sources in a horizontal layer between the Earth surface and depth $z = H$, the field $g(x,y,-H)$ should be continued downward to depth $z = H$. To find a function $g(x,y,H)$ describing this field, we should solve the following equation:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2H}{[(x-x')^2+(y-y')^2+(2H)^2]^{3/2}}\times \\ \times g(x',y',H)dx'dy' = g(x,y,-H). \quad (2)$$

According to the definiton of the analytic continuation, the singularities of this function are located below the level $z = H$. It is harmonic above this level, therefore, it can be used as a field of the sources located below the level $z = H$.

Now to extract approximately a gravitational effect $\Delta g(x,y)$ of sources located in a horizontal layer $\Pi = \{(x,y,z) : (x,y) \in D, H_1 \leqslant z \leqslant H_2\}$, we should perform the procedure described above for two values H_1, H_2 and take the difference of two fields.

Let K be the integral operator from (2). Then we can rewrite this equation as

$$Kg(x',y',H) = g(x,y,-H). \quad (2')$$

Problem (2-2') is a linear two-dimensional Fredholm integral equation of the first kind. After discretization of the area Π into $n = M \times N$ grid and approximation of the integral operator using quadrature rules it takes the form

$$\frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{2H}{[(x_v-x_i)^2+(y_u-y_j)^2+(2H)^2]^{3/2}}\times \\ \times g(x_i,y_j,H)\Delta x\Delta y = g(x_v,y_u,-H), \\ u = \overline{1,M}, v = \overline{1,N}. \quad (3)$$

This is a system of linear algebraic equations, which can be rewritten in the form

$$\overline{K}g = d, \quad (3')$$

where $\overline{K}$ is a matrix of $n \times n = MN \times MN$ dimension, g and d are vectors of n dimension.

3 Density reconstruction

The next problem is to find a variable density $\sigma(x,y)$ in a horizontal layer Π using aquired gravitational data $\Delta g(x,y)$. It is assumed that the density distribution $\sigma(x,y)$ doesn't depend on z coordinate (Fig. 1).

The gravitational effect is described by the following equation [2]:

$$\gamma\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{(x-x')^2+(y-y')^2+H_1^2}}- \right. \\ \left. -\frac{1}{\sqrt{(x-x')^2+(y-y')^2+H_2^2}}\right\}\times \\ \times \sigma(x',y')dx'dy' = \Delta g(x,y), \quad (4)$$

where γ is the gravitational constant. Let A be the integral

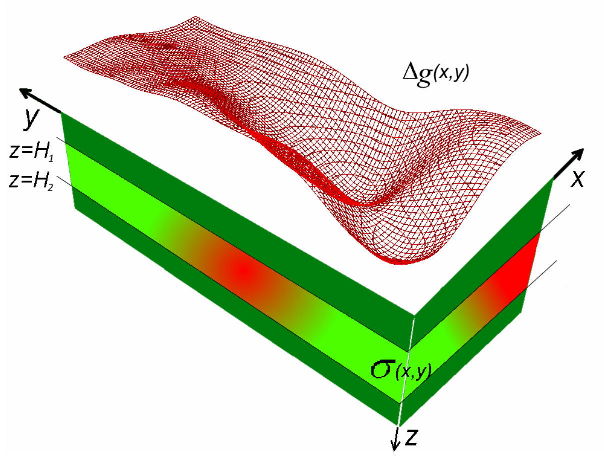

Fig. 1: Model of a horizontal layer.

operator from (4). We can rewrite this equation as

$$A\sigma(x',y') = \Delta g(x,y). \quad (4')$$

This problem is also a linear two-dimensional Fredholm integral equation of the first kind. After discretization of the area Π into $n = M \times N$ grid and approximation of the integral operator it takes the form

$$\gamma \sum_{i=1}^{N} \sum_{j=1}^{M} \left\{ \frac{1}{\sqrt{(x_v - x_i)^2 + (y_u - y_j)^2 + H_1^2}} - \frac{1}{\sqrt{(x_v - x_i)^2 + (y_u - y_j)^2 + H_2^2}} \right\} \times \times \sigma(x_i, y_j) \Delta x \Delta y = \Delta g(x_v, y_u), \quad u = \overline{1,M}, v = \overline{1,N}. \tag{5}$$

This is a system of linear algebraic equations which can also be rewritten in the form

$$\overline{A}\sigma = c, \tag{5'}$$

where $\overline{A}$ is a matrix of $n \times n = MN \times MN$ dimension, σ and c are vectors of n dimension.

4 Methods for solving SLAE

Problems (2) and (4) belong to the class of ill-posed problems. The roof depth H_1 and grid size $M \times N$ greatly affect the stability of the solution: the larger is H_1, the greater is the steps of the grid. Therefore, the resulting SLAEs (3') and (5') are ill-conditioned. The matrices $\overline{K}$ and $\overline{A}$ are symmetric positive defined [7]. So, we can use the Lavrentyev regularization scheme [8]. They both will take the form

$$Az = b, \tag{6}$$

where $A = \overline{A} + \alpha E$, $z = \sigma$, $b = c$ for the system (5') or $A = \overline{K} + \alpha E$, $z = g$, $b = d$ for the system (3'), and α is a regularization parameter.

To solve system (6), we used the following iterative method of minimal residuals [9]:

$$z^{k+1} = z^k - \frac{\langle A(Az^k - b), Az^k - b \rangle}{\|A(Az^k - b)\|^2}(Az^k - b), \tag{7}$$

where z^k is a solution estimate at the k-th iteration. This method requires less arithmetic operations than other gradient type methods described in [9]. The initial estimate is $z^0 \equiv 0$. The condition $\frac{\|Az^k - b\|}{\|b\|} < \varepsilon$ for some sufficiently small ε is taken as a termination criterion.

5 Matrix structure investigation and storage method construction

Storing the matrix A can be very memory consuming for large grids; thus, it is worthwhile to investigate its structure to optimize the storage method.

Let's consider A as a block matrix. Then the elements can be defined as

$$a_{k,p,l,q} = a_{(k-1)M+p,(l-1)M+q} = = \gamma \Delta x \Delta y \left(\frac{1}{\sqrt{(x_k - x_l)^2 + (y_p - y_q)^2 + H_1^2}} - - \frac{1}{\sqrt{(x_k - x_l)^2 + (y_p - y_q)^2 + H_2^2}} \right),$$

or

$$a_{k,p,l,q} = a_{(k-1)M+p,(l-1)M+q} = = \frac{1}{2\pi} \Delta x \Delta y \left(\frac{2H}{[(x_k - x_l)^2 + (y_p - y_q)^2 + (2H)^2]^{3/2}} \right),$$

where $k, l = \overline{1,M}$ are the block indices and $p, q = \overline{1,N}$ are the indices of elements inside each block.

Apparently, the matrix elements depend only on the terms $(x_k - x_l)^2 + (y_p - y_q)^2$.

Note that

$$(y_{p+1} - y_{q+1})^2 = (y_p + \Delta y - y_q - \Delta y)^2 = (y_p - y_q)^2,$$
$$(x_{k+1} - x_{l+1})^2 = (x_k + \Delta x - x_l - \Delta y)^2 = (x_k - x_l)^2.$$

The first equation means that $p = q \Rightarrow \Rightarrow a_{k,p,l,q} = a_{k,p+1,l,q+1}$, *i.e.*, in each block, each descending diagonal from left to right is constant. The second one means that $k = l \Rightarrow a_{k,p,l,q} = a_{k+1,p,l+1,q}$, *i.e.*, each block diagonal is constant as well. In other words, the matrix A is symmetric Toeplitz-block-Toeplitz. The scheme of its structure is shown in Fig. 2.

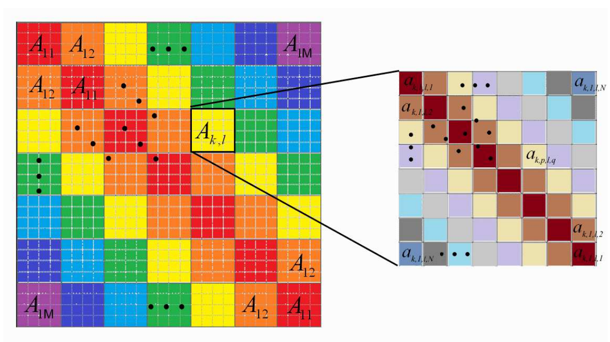

Fig. 2: Matrix structure.

The obvious way of storing this matrix is to store the first row only. Each subsequent row is obtained by the following operations:

1) shifting the element rows inside each block rightwise by one element and complementing each row from the left by symmetrically positioned element;

2) shifting the entire block row rightwise by one element and complementing it by symmetrically positioned block.

Although this method requires only $O(NM)$ memory, it needs too many index recalculations to obtain an element.

More effective way of storing this matrix is to store the symmetrically complemented first row of blocks. Each subsequent row is obtained by moving the M-wide "window" to the left by one block. The storage and access scheme is shown in Fig. 3.

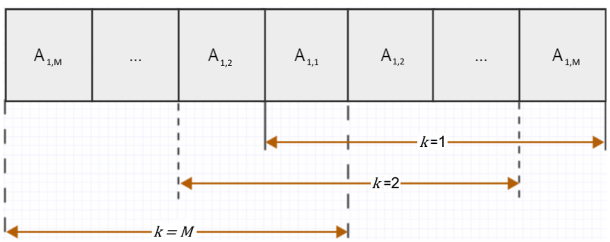

Fig. 3: Optimized matrix blocks storing method and access scheme.

6 Test problem

The test problem of finding the density in the layer between the depths of 10 and 11 km for the area of 128×128 km^2 was considered. The model gravitational field $g_{\text{model}} = g_{\text{original}} + g_{\text{noise}}$ is shown in Fig. 4.

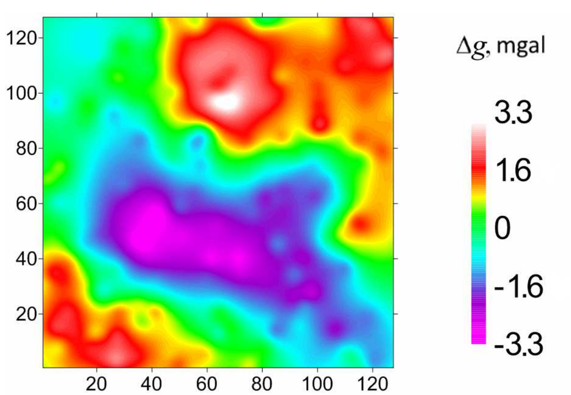

Fig. 4: Model (synthetic) gravitational field g_{model}.

The field g_{original} shown in Fig. 5 was obtained by solving the direct gravity problem using the original synthetic density distribution σ_{original} shown in Fig. 8 and adding the noise g_{noise} with 80% peak value. This noise was generated by 200 point sources located randomly above the 10 km level. The noise field is shown in Fig. 6

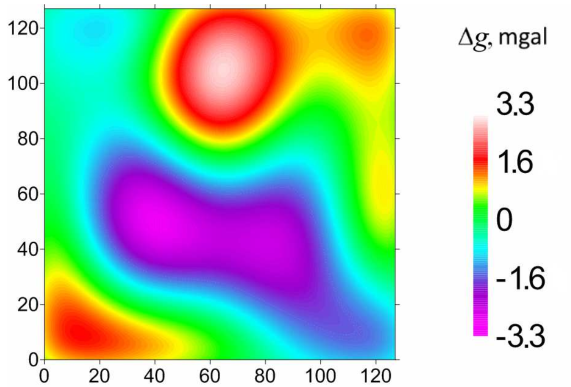

Fig. 5: Original gravitational field g_{original}.

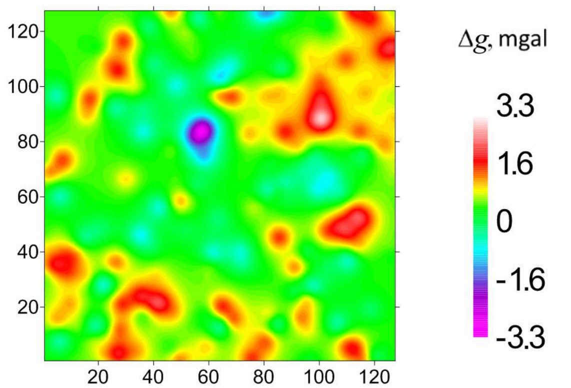

Fig. 6: Noise gravitational field g_{noise}.

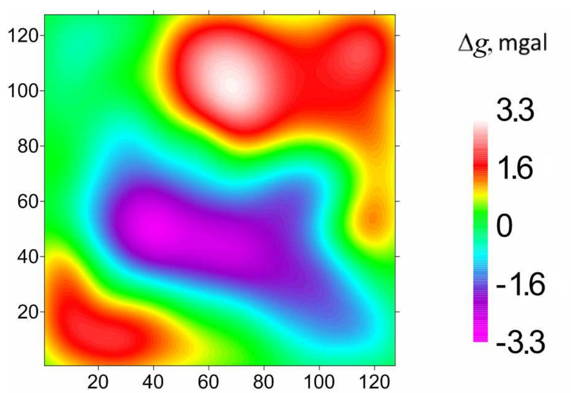

Fig. 7: Extracted gravitational field Δg.

To reduce the boundary effects we used the technique described in [10]. The main idea of this technique is to use the mean density as background density. The background gravitational field generated by this density has constant value, therefore, the anomalous field is generated by difference between the actual density and background density. Thus, the "step" of the field values on the boundary of the area can be reduced.

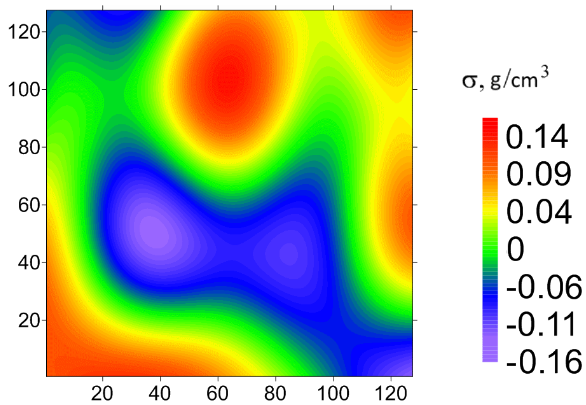

Fig. 8: The original density distributions σ_{original}.

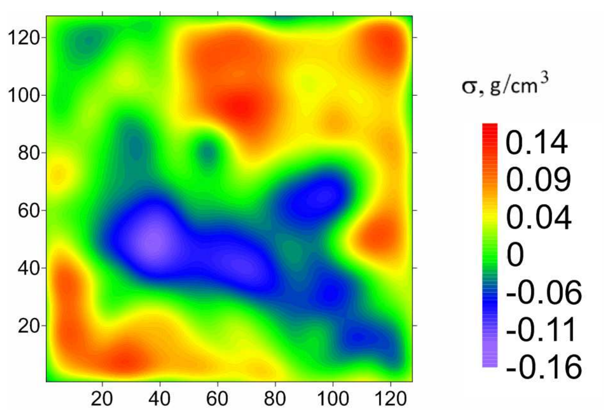

Fig. 9: The reconstructed density distributions $\sigma_{\text{reconstructed}}$.

Then two problems were solved. The first one is to remove the noise from the field using the extraction technique described above. The field Δg of anomalous masses below the level $H = 10$ km was extracted. It is shown in Fig. 7. This problem was solved by the minimal residuals method (7). The termination criterion was $\varepsilon = 0.005$, and the solving process took 20 iterations.

The second problem was to find the density distribution using the aquired denoised field. The reconstructed density $\sigma_{\text{reconstructed}}$ is shown in Fig. 9. The problem of finding a variable density was solved by the method (7). The termination criterion was $\varepsilon = 10^{-5}$, and the solving process took 15 iterations. The resulting relative error is $\frac{\||\sigma_{\text{original}} - \sigma_{\text{reconstructed}}\||}{\|\sigma_{\text{original}}\|} < 0.2$.

7 Adjusting the regularization parameter

Proper choice of the regularization parameter α for the given data is very important problem. If this parameter is too big, then the solution z will not fit the given data b and the residual $\|Az - b\|$ will be too big. On the other hand, if α is too small, then the fit will be good but the solution will be dominated by the contributions from the data errors and, hence, $\|z - z_{\text{exact}}\|$ will be too big.

In this work, the regularization parameter was adjusted using the L-curve method [11]. It is a convenient graphical tool for displaying the trade-off between the error of a regularized solution and its fit to the right part, as the regularization parameter α varies.

The L-curve for the model field extraction problem is shown in Fig. 10. The error is $\frac{\|g_{\text{original}} - \Delta g(\alpha)\|}{\|g_{\text{original}}\|}$ and the residual is $\frac{\|A\Delta g(\alpha) - b\|}{\|b\|}$. The optimal regularization parameter is located in the "corner" of the L-curve and is about $\alpha = 0.025$.

The L-curve for the model density reconstruction problem is shown in Fig. 11. The error is $\frac{\|\sigma_{\text{original}} - \sigma_{\text{reconstructed}}(\alpha)\|}{\|\sigma_{\text{original}}\|}$ and the residual is $\frac{\|A\sigma_{\text{reconstructed}}(\alpha) - b\|}{\|b\|}$. Apparently, the optimal regularization parameter is about $\alpha = 2$.

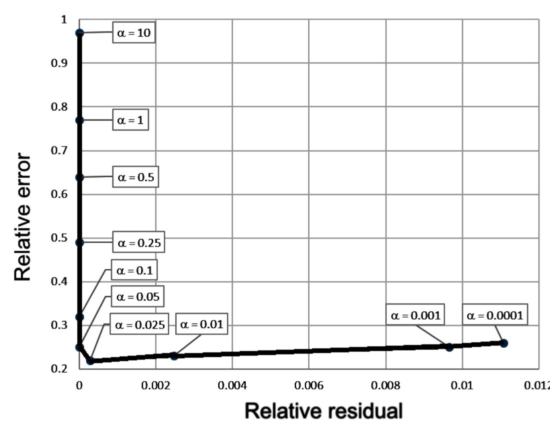

Fig. 10: L-curve for field extraction problem.

8 Parallelization and numerical experiments

Note that storing the SLAE matrix for a $2^9 \times 2^9$ grid takes about 2 GB, *i.e.*, one node is sufficient for solving the problem. The full storage takes 525 GB, which exceeds the one node memory limit. Thus, several nodes are needed for solving the same problem.

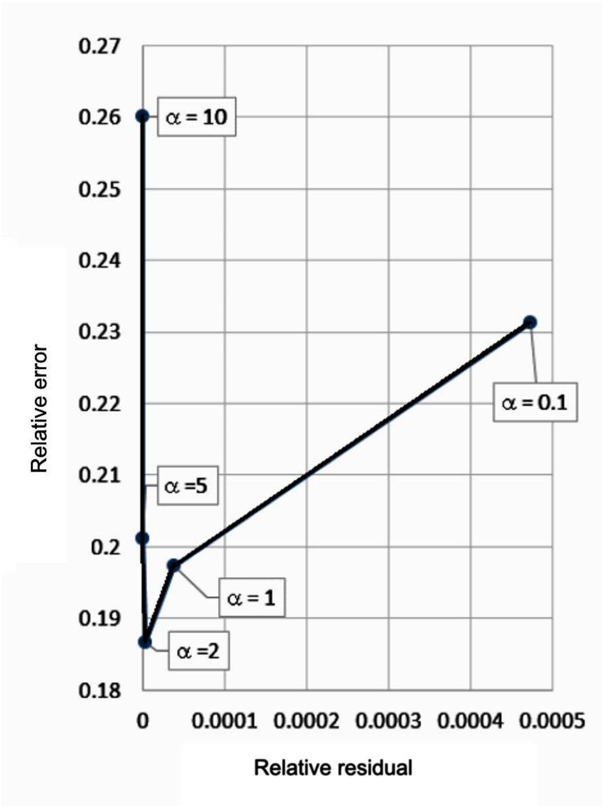

Fig. 11: L-curve for density reconstruction problem.

The parallelization was implemented by dividing the data into a number of fragments with respect to the number of used nodes and processor cores. In the case of full storage, the matrix is divided into horizontal bars, and each fragment is stored on its own node. In the case of optimized storage, the complemented first row of blocks is stored on each node. Moreover, additional parallelization was performed using the OpenMP technology.

The problem was solved using the Uran supercomputer nodes with eight-core Intel Xeon E5-2660 (2.2 GHz) CPUs, six-core Intel Xeon X5675 (3.07 GHz), four-core Intel Xeon E5450 (3.0 GHz), and UrFU supercomputer nodes with six-core Intel Xeon E5-2620 (2.1 GHz). Each node has 32 GB of RAM. Two storage methods was tested: the first one with the full storage and the second one with the optimized storage. The MPI+OpenMP hybrid technology was utilized.

The problem was also solved using the Intel Xeon Phi coprocessors [12]. The Intel Xeon Phi coprocessors are based on the Intel MIC architecture, run a full service Linux operating system, and support x86 memory order model and IEEE 754 arithmetic. The Intel Xeon Phi coprocessor provides high performance, and performance per watt for highly parallel HPC workloads, while not requiring a new programming model, API, nor language or restrictive memory model. It is able to do this with an array of general purpose cores with multiple thread contexts, wide vector units, caches, and high bandwidth on die and memory interconnect.

The computation times for the Uran nodes and memory requirements for a $2^9 \times 2^9$ grid are shown in Table 1. The dash denotes lack of memory to run.

The main advantage of optimized storage method is its ability to run on one node. The computation times for the same problem using only one Uran node are shown in Table 2.

The computation times for the same problem using Xeon Phi 5110P coprocessor (60 cores, 1.053 GHz, 8GB of RAM) with the Intel Xeon host processor of the UrFU supercomputer with several thread configurations are shown in Table 3. Each Xeon Phi physical core has 4 hardware threads, therefore, the optimal thread number is $4n$ where n is the number of available cores $(60-1)$. Thus, the optimal thread number is 236.

Table 1: Results of numerical experiments (several nodes of the Uran supercomputer).

Number of nodes	Total number of cores	Optimized storage (requires 2 GB)		Full storage (requires 2 GB)	
		Computation time, mins	Memory usage on each node, GB	Computation time, mins	Memory usage on each node, GB
1	8	53.9	0.25	–	66
2	16	13.4	0.12	–	33
4	32	6.8	0.06	7.9	16
8	64	3.5	0.03	4.8	8

Table 2: Results of numerical experiments (optimized storage, one Uran node).

Node	Number of cores	Computation time, minutes
2× Intel Xeon E5450 (3.0 GHz, 4 cores)	8	14.7
2× Intel Xeon X5675 (3.07GHz, 6 cores)	12	3.6
2× Intel Xeon E5-2650 (2.6 GHz, 8 cores)	16	1

Table 3: Results of numerical experiments using the Intel Xeon Phi coprocessor.

Node	Threads number	Computation time,mins
2× Intel Xeon E5-2620 (2.1GHz, 12 cores)	12	2.6
Intel Xeon Phi 5110P (1.05GHz, 60 cores)	59	1.5
	118	1
	236	0.8
	472	0.9

The experiments performed show that the proposed optimized storage method is very promising for several reasons. Firstly, it has much lesser memory requirements: $O(MN^2)$ rather than $O(M^2N^2)$. It means that application of several nodes is not needed. Secondly, the data have a convenient structure, therefore, some speedup is achieved. The experiments for the Xeon Phi coprocessor show major speedup in comparison with several multicore nodes. Therefore, the optimized storage method allows one to solve a problem using only one high performance node with many cores.

The parallel algorithm was incorporated into the remote computational system "Specialized Web-Portal for Solving Geophysical Problems on Multiprocessor Computers" [13].

9 Conclusion

The effective parallel algorithm for solving the linear inverse gravity problem was constructed. This algorithm is based on exploiting the structure of discretized integral equations matrices. The proposed algorithm significantly reduces the memory usage, as well as the computation time. The parallel algorithm was constucted and numerically implemented using the Uran and UrFU supercomputers. The comparison in terms of computation time was carried out. The experiments show that application of the Intel Xeon Phi coprocessor gives a major speedup in comparison with using CPUs of several nodes. The test problem of finding the lateral density using synthetic data was solved. Adjusting the regularization parameter was implemented using the L-curve method.

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