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The Min-Pareto Power Series Distributions of Lifetime

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Abstract: The paper sets out to study some properties of the new mathematical probability power series distributions called a Min Pareto power series (MinParPS). Several models in special cases used in the study of the lifetime are presented. Expressions for certain numerical and reliability characteristics (e.g. expectation, variance, survival function, hazard rate) are obtained. In the final part of the paper the problem of the statistical estimation using the EM algorithm is treated.

Keywords: lifetime; Pareto distribution; distribution of the minimum; power series distributions; maximum likelihood estimation

1 Introduction

Alongside the most common reliability distributions (exponential, Erlang, Weibull) there is also the Pareto distribution which has the applications not only in economy (incomes from a population), but also in the study of the lifetime of the k out of n type systems and the estimation of reliability power stress (”stress-strength reliability”, [8]). This is the reason why this new class of distribution MinParPS in order to study the reliability behavior of the complicated systems is introduced.

The working methodology and techniques are presented and analyzed in the paper [6], which enables the study of the distribution of the minimum value of the sample of the random size Z from the statistical population with Pareto distribution.

The random variable Z has a distribution that belongs to the power series distributions class (PSD ”power series distribution” [5]).

The general problem of determining the maximum and minimum distribution of a random sequence of a random variable has been solved by Louzada et al. in the work [7], using as a working tool the composing generating function of the number of the random variable of the sequence with survival function of random variable components of the sequence.

Instead, the present paper approaches in a unitary manner the distribution of a minimum number of independent and identically distributed random variables (i.i.d.r.v.) Pareto distributed in terms of PSD family, distribution characterised by the number of the random variable of the sequence.

Let’s consider random variable Z such that $\mathbb{P}(Z \in \{1,2,\ldots\}) = 1$.

Definition 1.1.([5]) We say that random variable Z has a power series distribution if:

$$\mathbb{P}(Z = z) = \frac{a_z \Theta^z}{A(\Theta)},\ z = 1,2,\ldots;\ \Theta \in (0,\tau), \qquad (1)$$

where $a_1, a_2, \ldots$ are nonnegative real numbers, τ is a positive number bounded by the convergence radius of power series (*series function*) $A(\Theta) = \sum_{z\geq 1} a_z \Theta^z$, $\forall\, \Theta \in (0,\tau)$ and Θ is *power parameter* of the distribution (Table 1).

PSD denotes the power series distribution functions families. If the random variable Z has the distribution from relationship (1), then we write that $Z \in PSD$.

2 The Min Pareto power series distributions

We consider that the random variable $X_i \sim Par(\mu,\alpha)$, $\mu,\alpha > 0$, where $(X_i)_{i\geq 1}$ are i.i.d.r.v. with the cumulative distribution function (cdf) $F_{X_i}(x) = F_{Par}(x) = 1 - \left(\frac{\mu}{x}\right)^{\alpha}$, $x \geq \mu$ and the probability density function (pdf) $f_{X_i}(x) = f_{Par}(x) = \frac{\alpha\mu^{\alpha}}{x^{\alpha+1}}$, $x \geq \mu$. Also, we denote by $V_{Par} = \min\{X_1,\ldots,X_Z\}$, where random variable $Z \in PSD$.

Cdf, pdf, as well as some reliability characteristics (survival function, hazard rate) are given as follows.

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Table 1: The representative elements of the PSD families for various truncated (*) distributions

Distribution	a_z	Θ	$A(\Theta)$	τ
$Binom^*(n,p)$	$\binom{n}{z}$	$\frac{p}{1-p}$	$(1+\Theta)^n-1$	∞
$Poisson^*(\lambda)$	$\frac{1}{z!}$	λ	$e^{\Theta}-1$	∞
$Log(p)$	$\frac{1}{z}$	p	$-ln(1-\Theta)$	1
$Geom^*(p)$	1	$1-p$	$\frac{\Theta}{1-\Theta}$	1
$Pascal(k,p)$	$\binom{z-1}{k-1}$	$1-p$	$\left(\frac{\Theta}{1-\Theta}\right)^k$	1
$Bineg^*(k,p)$	$\binom{z+k-1}{z}$	p	$(1-\Theta)^{-k}-1$	1

Proposition 2.1. *If the random variable $V_{Par} = \min\{X_1, X_2, \ldots, X_Z\}$, where $(X_i)_{i\geq 1}$ are independent and identically distributed (i.i.d.) random variables, $X_i \sim Par(\mu,\alpha)$, $\mu,\alpha > 0$ and $Z \in PSD$ with $\mathbb{P}(Z=z) = \frac{a_z\Theta^z}{A(\Theta)}$, $z = 1,2,\ldots;$ $\Theta \in (0,\tau)$, $\tau > 0$, the random variable $(X_i)_{i\geq 1}$ and Z independent, then cdf of the random variable V_{Par} is the following:*

$$V_{Par}(x) = 1 - \frac{A\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A(\Theta)},\ x \geq \mu. \tag{2}$$

Proof. Given the relationship that characterizes cdf of a minimum of a random sequence of i.i.d.r.v. (see [6], Proposition 2.2), where cdf $F(x) \equiv F_{Par}(x), \forall\, x > 0$, we obtain (2).□

Consequence 2.1. *The survival function of the random variable V_{Par} is the following:*

$$S_{V_{Par}}(x) = \frac{A\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A(\Theta)},\ x \geq \mu. \tag{3}$$

Proof. Taking into account the survival function definition, $S_{V_{Par}}(x) = 1 - V_{Par}(x)$, $x \geq \mu$ and the relationship (2), we obtain the relationship (3).□

Consequence 2.2. *Pdf of the random variable V_{Par} is characterized by:*

$$v_{Par}(x) = \frac{\alpha\Theta\mu^{\alpha}\frac{d}{dx}\left\{A\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]\right\}}{x^{\alpha+1}A(\Theta)},\ x \geq \mu. \tag{4}$$

Proof. According to the definition of the pdf in the case of the minimum (see [6], Consequence 2.3) we obtain the relationship (4).□

Definition 2.1. We say that the random variable V_{Par} has a Min-Pareto power series distributions of parameters μ, α and Θ (is denoted $V_{Par} \sim MinParPS(\mu,\alpha,\Theta)$), if it has the cdf defined by the relationship (2) and the pdf defined by the relationship (4).

Proposition 2.2. *The hazard rate for the random variable V_{Par} is given by:*

$$h_{V_{Par}}(x) = \frac{\alpha\Theta\mu^{\alpha}\frac{d}{dx}\left\{A\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]\right\}}{x^{\alpha+1}A\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]},\ x \geq \mu. \tag{5}$$

Proof. Given the Proposition 2.3 from [6] which characterizes the hazard rate as well as Consequences 2.1 and 2.2, follows the relationship (5). □

The next limit result aimed at MinParPS distribution:

Proposition 2.3. *If $(X_i)_{i\geq 1}$ is a sequence of independent, identically Pareto distributed random variables, with the cdf F_{Par} and $Z \in PSD$ with $\mathbb{P}(Z=z) = \frac{a_z\Theta^z}{A(\Theta)}$, where $(a_z)_{z\geq 1}$ is a sequence of nonnegative real numbers, $A(\Theta) = \sum_{z\geq 1} a_z\Theta^z$, $\forall\, \Theta \in (0,\tau)$, then:*

$$\lim_{\Theta\to 0^+} V_{Par}(x) = 1 - \left[1 - F_{Par}(x)\right]^l,\ x \geq \mu,$$

where $l = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$.

Proof. By repeatedly applying the l'Hopital rule, we have:

$$\lim_{\Theta\to 0^+} V_{Par}(x) = 1 - \lim_{\Theta\to 0^+}\frac{A\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A(\Theta)}$$
$$= \underbrace{\ldots}_{l-times} = 1 - \lim_{\Theta\to 0^+}\frac{\left(\frac{\mu}{x}\right)^{\alpha l}A^{(l)}\left[\Theta\left(\frac{\mu}{x}\right)^{\alpha}\right]}{A^{(l)}(\Theta)}$$
$$= 1 - \frac{l!a_l\left(\frac{\mu}{x}\right)^{\alpha l}}{l!a_l} = 1 - \left(\frac{\mu}{x}\right)^{\alpha l},\ x \geq \mu.\square$$

Consequence 2.3. *The r^{th} moments, $r \in \mathbb{N}$, $r \geq 1$ of the random variable $V_{Par} \sim MinParPS(\mu,\alpha,\Theta)$ are given by the relationship:*

$$\mathbb{E}V_{Par}^r = \sum_{z\geq 1}\frac{a_z\Theta^z}{A(\Theta)}\mathbb{E}\left[\min\{X_1,\ X_2,\ldots,X_z\}\right]^r, \tag{6}$$

where pdf of the random variable $\min\{X_1,\ X_2,\ldots,X_z\}$ is characterized by the relationship:

$$f_{\min\{X_1,\,X_2,\ldots,X_z\}}(x) = z f_{Par}(x)\left[1 - F_{Par}(x)\right]^{z-1}.$$

Proof. It is known that the distribution function of the minimum of the sample of size $Z = z$ which has the cdf F_{Par} is $V_z(x) = 1 - \left[1 - F_{Par}(x)\right]^z$. With the total probability formula, the cdf of the minimum of a sequence of i.i.d.r.v. a random number Z has the expression:

$$V_{Par}(x) = \sum_{z\geq 1} V_z(x)\cdot\mathbb{P}(Z=z)$$
$$= \sum_{z\geq 1} 1 - \left[1 - F_{Par}(x)\right]^z\cdot\mathbb{P}(Z=z).$$

Deriving previous relationship relative to variable x, we obtain :

$$v_{Par}(x) = \sum_{z\geq 1} z f_{Par}(x)\left[1-F_{Par}(x)\right]^{z-1}\cdot \mathbb{P}(Z=z), \quad (7)$$

where $z f_{Par}(x)\left[1-F_{Par}(x)\right]^{z-1}$ is pdf of the random variable $\min\{X_1, X_2, \ldots, X_z\}$. Applying the means relationship (7), we obtain (6).□

In order to formulate a special case of the Poisson Limit Theorem, we need to consider two particular cases of Min Pareto distributions called Min Pareto Binomial zero truncated (MinParB) and Min Pareto Poisson zero truncated (MinParP) distributions.

The Min-Pareto-Binomial (MinParB) power series distributions is defined by the function (2), where $Z \sim Binom^{\star}(n,p) \in PSD$, $k,n \in \{1,2,\ldots\}$, $A(\Theta) = (\Theta+1)^n - 1$, $\Theta = \frac{p}{1-p}$, $p \in (0,1)$ (Table 1), namely:

$$V_{ParB}(x) = \frac{1-\left(1-p+p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n}}{1-(1-p)^n}, \; x \geq \mu. \quad (8)$$

Pdf of the random variable V_{ParB} is defined according to the relationship (4), namely:

$$v_{ParB}(x) = \frac{np\alpha\mu^{\alpha}\left(1-p+p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n-1}}{x^{\alpha+1}\left(1-(1-p)^n\right)}, \; x \geq \mu. \quad (9)$$

Definition 2.2. We say that the random variable V_{ParB} has a Min-Pareto-Binomial power series distributions with parameters μ, α, n and p (is denoted $V_{ParB} \sim MinParB(\mu,\alpha,n,p)$), where $\mu,\alpha > 0$, $n \in \{1,2,\ldots\}$ and $p \in (0,1)$, if it has the cdf defined by the relationship (8) and pdf defined by the relationship (9).

Consequence 2.4. *The survival function and the hazard rate of the random variable V_{ParB} are defined by the following relationships:*

$$S_{V_{ParB}}(x) = \frac{\left(1-p+p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n} - (1-p)^n}{1-(1-p)^n}, \; x \geq \mu \quad (10)$$

and

$$h_{V_{ParB}}(x) = \frac{np\alpha\mu^{\alpha}\left(1-p+p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n-1}}{x^{\alpha+1}\left[\left(1-p+p\left(\frac{\mu}{x}\right)^{\alpha}\right)^{n} - (1-p)^n\right]}. \quad (11)$$

Proof. Particularizing the results of Proposition 2.2 and Consequence 2.1 for $A(\Theta) = (\Theta+1)^n - 1$ with $\Theta = \frac{p}{1-p}$, $p \in (0,1)$, we obtain the relationships (10) and (11). □

Consequence 2.5. *The means and variance of the random variable $V_{ParB} \sim MinParB(1,\alpha,n,p)$, $\alpha > 0$, $n \in \{1,2,\ldots\}$ and $p \in (0,1)$ are characterized by the following relationships:*

$$\mathbb{E}V_{ParB} = \frac{np\alpha}{1-(1-p)^n} \cdot \sum_{z=1}^{n} \frac{(-1)^{z-1}\binom{n}{z-1}(z-1)!(\alpha p)^{z-1}}{(\alpha-1)(2\alpha-1)\ldots(z\alpha-1)}, \quad (12)$$

$\forall\, \alpha > 1$ and

$$\mathbb{V}ar V_{ParB} = \frac{np\alpha}{1-(1-p)^n}\left[\sum_{z=1}^{n} \frac{(-1)^{z-1}\binom{n}{z-1}(z-1)!(\alpha p)^{z-1}}{(\alpha-2)(2\alpha-2)\ldots(z\alpha-2)} - \frac{np\alpha}{1-(1-p)^n}\left(\sum_{z=1}^{n} \frac{(-1)^{z-1}\binom{n}{z-1}(z-1)!(\alpha p)^{z-1}}{(\alpha-1)(2\alpha-1)\ldots(z\alpha-1)}\right)^2\right], \quad (13)$$

$\forall\, \alpha > 2$.

Proof. Taking into account the definition of the means and (9), we can write:

$$\mathbb{E}V_{ParB} = \frac{np\alpha}{1-(1-p)^n}\int_1^{\infty} \frac{\left[1-p\left(1-\frac{1}{x^{\alpha}}\right)\right]^{n-1}}{x^{\alpha}}\,\mathrm{d}x,$$

the existence of the means is ensured by the condition $\alpha > 1$. By developing the counter integrants by binomial formula, we obtain a sum of n integrals than can be solved with elementary methods (method of integration by parts), which leads to the equation (12). With the second order moment $\mathbb{E}V_{ParB}^2 = \frac{np\alpha}{1-(1-p)^n}\int_1^{\infty} \frac{\left[1-p\left(1-\frac{1}{x^{\alpha}}\right)\right]^{n-1}}{x^{\alpha-1}}\,\mathrm{d}x$, where it is finite to $\alpha > 2$, we can write the variance of the random variable V_{ParB} which in the end is described by the relationship (13). □

The Min-Pareto-Poisson (MinParP) power series distributions is characterized by the cdf defined by the relationships (2), where $Z \sim Poisson^{\star}(\lambda) \in PSD$, $\lambda > 0$ and $A(\Theta^{\star}) = e^{\Theta^{\star}} - 1$, $\Theta^{\star} = \lambda$ (Table 1), namely:

$$V_{ParP}(x) = \frac{1-e^{-\lambda\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)}}{1-e^{-\lambda}}, \; x \geq \mu \quad (14)$$

and pdf according to the relationship (4):

$$v_{ParP}(x) = \frac{\alpha\lambda\mu^{\alpha}e^{-\lambda\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)}}{x^{\alpha+1}\left(1-e^{-\lambda}\right)}, \; x \geq \mu. \quad (15)$$

Definition 2.3. We say that the random variable V_{ParP} has a Min-Pareto-Poisson power series distributions with parameters μ, α and λ (is denoted $V_{ParP} \sim MinParP(\mu,\alpha,\lambda)$), where $\mu,\alpha,\lambda > 0$, if it has the cdf defined by the relationship (14) and pdf defined by the relationship (15).

Consequence 2.6. *The survival function and the hazard rate of the random variable V_{ParP} are defined by the following relationships:*

$$S_{V_{ParP}}(x) = \frac{e^{-\lambda\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)} - e^{-\lambda}}{1-e^{-\lambda}}, \; x \geq \mu \quad (16)$$

and

$$h_{V_{ParP}}(x) = \frac{\alpha\lambda\mu^{\alpha}e^{-\lambda\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)}}{x^{\alpha+1}\left[e^{-\lambda\left(1-\left(\frac{\mu}{x}\right)^{\alpha}\right)}-e^{-\lambda}\right]}. \qquad (17)$$

Proof. Particularizing the results of Proposition 2.2 and Consequence 2.1 for $A(\Theta) = e^{\Theta} - 1$ with $\Theta = \lambda,\ \lambda > 0$, we obtain the relationships (16) and (17).□

As a particular case of the Poisson Limit Theorem proved in the paper [6] for Min PSD distributions, we have the following:

Proposition 2.4.*(Poisson Limit Theorem for MinParPS)*. *If the random variable $V_{ParB} \sim MinParB(\mu,\alpha,n,p)$ with $n \to \infty$ and $p \to 0^+$ such that $n \cdot p \to \lambda,\ \lambda > 0$, then:*

$$\lim_{\substack{n\to\infty \\ p\to 0^+}} V_{ParB}(x) = V_{ParP}(x),\ \forall\, x \geq \mu,$$

where $V_{ParB}(x)$, respectively $V_{ParP}(x)$, $x \geq \mu$ are distribution functions $V_{ParB} \sim MinParB(\mu,\alpha,n,p)$, respectively $V_{ParP} \sim MinParP(\mu,\alpha,\lambda)$, defined by relationships (8) and (14).

Remark 2.1. The Poisson Limit Theorem for MinParPS distribution is confirmed visually in the graphical representation of Figures 1 and 2, where the probability densities functions and cumulative distributions function of the MinParB and MinParP distributions for the following parameters $\mu = 1$, $\alpha \in \left\{\frac{1}{2},1,3\right\}$, $n = 10$, $\lambda = 1$ and $p = 1/10$ are presented.

3 Statistical simulation for the MinParB distribution

We consider the simulation algorithm for the MinParB distribution using the uniform pseudorandom number generator with its implemenation in the Eclipse SDK 4.2.0 programming environment.

The description of the statistical simulation algorithm for MinParB distribution can be achieved in the following conditions: the random variable $V_{ParB} \sim \text{MinParB}(\mu,\alpha,n,p)$, $\mu,\alpha > 0$, $n \in \{1,2\ldots\}$, $p \in (0,1)$ has the same distribution as the random variable $\min_{1\leq i\leq Z} X_i$, where $(X_i)_{i\geq 1}$ are i.i.d.r.v., $X_i \sim Par(\mu,\alpha)$, $\mu,\alpha > 0$, and the value of random variable $Z \sim Binom^{\star}(n,p)$, $p \in (0,1)$, $n \in \{1,2,\ldots\}$ coincides with the value of the random variable zero truncated binomial, distributed with the same parameters.

Statistical simulation algorithm for the MinParB distribution:

Step 1:We generate a value $z^{\star}$ of the random variable $Z^{\star} \sim Binom(n,p)$, $p \in (0,1)$, $n \in \{1,2,\ldots\}$;

Step 2:If $z^{\star} = 0$ then GO TO pas 1, otherwise $z = z^{\star}$;

Step 3:For the value z of the random variable Z (generated in steps 1 and 2), simulate the values x_i, $i = 1,2,\ldots$ as a values of z-i.i.d.r.v. with distribution $Par(\mu,\alpha)$, $\mu,\alpha > 0$;

Step 4:It is considered $v_{ParB} = \min_{1\leq i\leq z} x_i$, STOP.

Having obtained the values from the simulation, we can apply the **Chi-square test concordance**, where the sample is generated: $\left(v^1_{ParB}, v^2_{ParB}, \ldots, v^m_{ParB}\right)$. The basic and alternative hypotheses are taken into consideration:

H_0: sample values $\left(v^1_{ParB}, v^2_{ParB}, \ldots, v^m_{ParB}\right)$ are values of the random variable distributed MinParB$(1,\alpha,n,p)$;

H_1: sample values $\left(v^1_{ParB}, v^2_{ParB}, \ldots, v^m_{ParB}\right)$ not the values of the random variable distributed MinParB$(1,\alpha,n,p)$.

The test is considered valid if the empirical value of χ^2_c is less than the upper critical value of the Chi-square $(r-1)-L = (12-1)-4 = 7$ freedom degrees ($\chi^2_{0.05;7} = 14.067$).

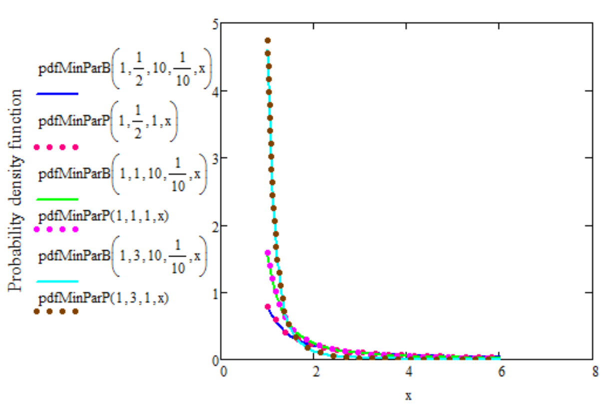

Fig. 1: The pdf's for the MinParB and MinParP distributions - graphical illustration of the Poisson Limit Theorem

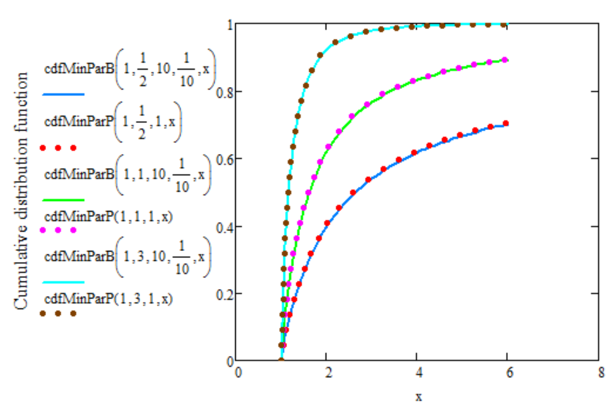

Fig. 2: The cdf's for the MinParB and MinParP distributions - graphical illustration of the Poisson Limit Theorem

The statistical of the Pearson's test is calculated using the relationship:

$$\chi_c^2 = \sum_{j=1}^{r} \frac{(n_j - \mathbf{n}_0 p_j)^2}{\mathbf{n}_0 p_j},$$

where $n_j,\ j = \overline{1,r}$ represents the number of observed values in the interval $[t_{j-1}, t_j)$, $\mathbf{n}_0 = \sum_{j=1}^{r} n_j$.

The probabilities p_j that the random variable V_{ParB} takes values in the interval $[t_{j-1}, t_j)$ are calculated using the equation:

$$p_j = V_{ParB}(t_j) - V_{ParB}(t_{j-1}) \stackrel{(8)}{=} \frac{1}{1-(1-p)^n}\left[\left(1-p+p\left(\frac{\mu}{t_{j-1}}\right)^{\alpha}\right)^n - \left(1-p+p\left(\frac{\mu}{t_j}\right)^{\alpha}\right)^n\right],$$

where $t_j,\ j = \overline{0, r-1}$ represent the ends of the intervals after merging.

The results of the statistical simulation for the MinParB$(1,\alpha,n,p)$ distribution, $\alpha > 0$, $n \in \{1,2\ldots\}$, $p \in (0,1)$ and the Pearson's test are centralized in Table 2, where the mean and variance (theoretical value) is computed with the relationships (12) and (13).

Table 2: The validation of the simulation results of the MinParB distribution with the application of the Chi-square test

Sample size	Mean		Variance		Chi-square
	Theoretical	Empirical	Theoretical	Empirical	
100	1,0470	1,1207	0,0674	0,0132	5,2685
1000		1,1073		0,0147	5,0485
10000		1,1036		0,0145	1,7359
100000		1,1024		0,0142	12,6967
1000000		1,1020		0,0141	7,7002
10000000		1,1021		0,0140	5,6087

4 EM algorithm

The description of the EM algorithm, [4], for *MinParB*$(1,\alpha,n,p)$ distribution is related to the existence of a sample $(x_1,x_2,\ldots,x_m)$ of size m a statistical population governed by a MinParB distribution with pdf $v_{ParB}(x,\boldsymbol{\Omega})$, $x > 0$ which depends on the parameter vector $\boldsymbol{\Omega} = (\alpha, p)$, given that the parameter n of the zero truncated binomial distribution is given. Its based on the relationship (9), the maximum likelihood function is the following:

$$L(x_1,x_2,\ldots,x_m;\boldsymbol{\Omega}) = \frac{(np\alpha\mu^{\alpha})^m}{(1-(1-p)^n)^m} \prod_{j=1}^{m} \frac{\left(1-p+p\left(\frac{\mu}{x_j}\right)^{\alpha}\right)^{n-1}}{x_j^{\alpha+1}}. \quad (18)$$

Write the function:

$$\ln L(x_1,x_2,\ldots,x_m;\alpha,p) = m(\ln n + \ln p + \ln\alpha + \alpha\ln\mu) - m\ln(1-(1-p)^n) + \sum_{j=1}^{m}\left[(n-1)\ln\left(1-p+p\left(\frac{\mu}{x_j}\right)^{\alpha}\right) - (\alpha+1)\ln x_j\right]$$

allows us to get the system $\boldsymbol{S}(\boldsymbol{\Omega}) = \left(\frac{\partial \ln L}{\partial \alpha}, \frac{\partial \ln L}{\partial p}\right) = \mathbf{0}$, we provide the maximum likelihood equations. Because the system cannot be easly solved, the implementation of the EM algorithm is required. Therefore, we assume that random variable Z is considered a random variable latency and we consider the sample $((x_1,z_1),(x_2,z_2),\ldots,(x_m,z_m))$ by m observations of random variable (V_{ParB}, Z). This shows that $((x_1,z_1),(x_2,z_2),\ldots,(x_m,z_m))$ can be interpreted as a complete set of statistics, being in this case, a sample of incomplete data.

The description of EM algorithm supposes the knowledge of the conditional mean $\mathbb{E}(Z|V_{ParB};\boldsymbol{\Omega})$, where $\boldsymbol{\Omega} = (\alpha, p)$.

The random variable V_{ParB} which corresponds to an incomplete set of data has pdf $v_{ParB}(x,\boldsymbol{\Omega})$, $x > 0$ defined by the relationship (4).

In these conditions, pdf $v_{ParB}(x,z)$ of the random variable (V_{ParB}, Z) which corresponds to a complete set of data, is given by:

$$v_{ParB}(x,z;\boldsymbol{\Omega}) = z f_{Par}(x)(1-F_{Par}(x))^{z-1}\mathbb{P}(Z=z) = \frac{z\alpha a_z \Theta^z}{x^{\alpha z+1}A(\Theta)},$$

where $A(\Theta) = (1+\Theta)^n - 1$, $\Theta = \frac{p}{1-p}$, $p \in (0,1)$, $a_z = \binom{n}{z}$, $z \le n$, while $f_{Par}(x)$ and $F_{Par}(x)$, $x > 0$ are pdf, respectively cdf of a random variable which has the distribution $Par(1,\alpha)$, $\alpha > 0$.

Then pdf of random variable Z conditioned by random variable V_{ParB} has the expression:

$$v_{ParB}(z|x) = \frac{v_{ParB}(x,z)}{v_{ParB}(x)} = \frac{z a_z \Theta^{z-1}}{x^{\alpha(z-1)}A'\left[\Theta\left(\frac{1}{x^{\alpha}}\right)\right]},$$

from where, the conditional mean:

$$\mathbb{E}(Z|V_{ParB};\boldsymbol{\Omega}) = \sum_{z=1}^{n} z \cdot v_{ParB}(z|x;\boldsymbol{\Omega})$$

$$= \frac{1}{\frac{d}{dx}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right]}\sum_{z=1}^{n}\frac{z^2 a_z \Theta^{z-1}}{x^{\alpha(z-1)}} = \frac{\Theta}{x^{\alpha}\frac{d}{dx}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right]}\sum_{z=1}^{n}\frac{z^2 a_z\Theta^{z-2}}{x^{\alpha(z-2)}}$$

$$= \frac{\Theta}{x^{\alpha}\frac{d}{dx}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right]}\cdot\left[\frac{d^2}{dx^2}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right] + \frac{x^{\alpha}}{\Theta}\underbrace{\sum_{z=1}^{n}\frac{z a_z \Theta^{z-1}}{x^{\alpha(z-1)}}}_{\frac{d}{dx}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right]}\right]$$

$$= \frac{\Theta\frac{d^2}{dx^2}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right]}{x^{\alpha}\frac{d}{dx}\left[A\left(\frac{\Theta}{x^{\alpha}}\right)\right]} + 1$$

or

$$\mathbb{E}(Z|V_{ParB};\boldsymbol{\Omega}) = \frac{np+(1-p)x^{\alpha}}{p+(1-p)x^{\alpha}}, \tag{19}$$

where we took into account that $Z \sim Binom^{\star}(n,p) \in PSD$, $k,n \in \{1,2,\ldots\}$ with $A(\Theta) = (1+\Theta)^n - 1$, $\Theta \in (0,+\infty)$, $\Theta = \frac{p}{1-p}$, $p \in (0,1)$.

The condition under which the EM algorithm works for $MaxParB(1,\alpha,n,p)$ distribution in order to estimate the unknown parameter $\boldsymbol{\Omega} = (\alpha,p)$ by $\boldsymbol{\Omega}^{(h)} = (\alpha^{(h)},p^{(h)})$, calculated for a few steps $h \geq 1$, is the following:

$$\max\left(\left|\alpha^{(h)} - \alpha^{(h-1)}\right|, \left|p^{(h)} - p^{(h-1)}\right|\right) \leq \varepsilon \tag{20}$$

or $h = K$ be accomplished when $\varepsilon > 0$ and K represents the number of preset iterations.

The steps **EM algorithm for MinParB distribution** are the following:

Step 1:We take $\alpha = \alpha^{(0)}$, $p = p^{(0)}$, $\alpha^{(0)} > 0$, $p^{(0)} \in (0,1)$;

Step 2:(*Expectation*) For iterating h, $h \geq 1$, we calculate the mean value of $z_j^{(h-1)}$, $j = \overline{1,m}$ according to the relationship (19):

$$z_j^{(h-1)} = \frac{np^{(h-1)} + (1-p^{(h-1)})x_j^{\alpha^{(h-1)}}}{p^{(h-1)} + (1-p^{(h-1)})x_j^{\alpha^{(h-1)}}},$$

where x_j, $j = \overline{1,m}$ are the values of z-i.i.d.r.v. with distribution $Par(1,\alpha)$, $\alpha > 0$ (see simulation algorithm for MinParB distribution);

Step 3:(*Maximization*) Through the maximum likelihood estimation (MLE) method, we take into consideration the sample

$$\left(\left(x_1, z_1^{(h-1)}\right), \left(x_2, z_2^{(h-1)}\right), \ldots, \left(x_m, z_m^{(h-1)}\right)\right),$$

with the maximum likelihood function:

$$L\left(x_1,x_2,\ldots,x_m,z_1^{(h-1)},z_2^{(h-1)},\ldots,z_m^{(h-1)};\boldsymbol{\Omega}^{(h-1)}\right) =$$
$$\prod_{j=1}^{m} v_{ParB}\left(x_j,z_j^{(h-1)};\boldsymbol{\Omega}^{(h-1)}\right) =$$
$$\prod_{j=1}^{m} \frac{\binom{n}{z_j^{(h-1)}} z_j^{(h-1)} \alpha^{(h-1)} \left(p^{(h-1)}\right)^{z_j^{(h-1)}} \left(1-p^{(h-1)}\right)^{n-z_j^{(h-1)}}}{x_j^{\alpha^{(h-1)} z_j^{(h-1)}+1} \left[1-\left(1-p^{(h-1)}\right)^n\right]} =$$
$$\frac{\left(\alpha^{(h-1)}\right)^m}{\left[1-\left(1-p^{(h-1)}\right)^n\right]^m} \cdot$$
$$\cdot \prod_{j=1}^{m} \frac{\binom{n}{z_j^{(h-1)}} z_j^{(h-1)} \left(p^{(h-1)}\right)^{z_j^{(h-1)}} \left(1-p^{(h-1)}\right)^{n-z_j^{(h-1)}}}{x_j^{\alpha^{(h-1)} z_j^{(h-1)}+1}}.$$

thus you can find iteration $\boldsymbol{\Omega}^{(h)} = (\alpha^{(h)},p^{(h)})$ which estimates the parameter $\boldsymbol{\Omega} = (\alpha,p)$.

Step 4:We examine (20). If NOT, then GO TO Step 2, otherwise, $\boldsymbol{\Omega} := \boldsymbol{\Omega}^{(h)}$, STOP.

Given the function:

$$\ln L\left(x_1,x_2,\ldots,x_m,z_1^{(h-1)},z_2^{(h-1)},\ldots,z_m^{(h-1)};\boldsymbol{\Omega}^{(h-1)}\right) =$$
$$m\ln\alpha^{(h-1)} - m\ln\left[1-\left(1-p^{(h-1)}\right)^n\right] +$$
$$\sum_{j=1}^{m}\left[\ln\binom{n}{z_j^{(h-1)}} + \ln z_j^{(h-1)} + \left(n - z_j^{(h-1)}\right)\ln\left(1-p^{(h-1)}\right) +\right.$$
$$\left. z_j^{(h-1)}\ln p^{(h-1)} - \left(\alpha^{(h-1)} z_j^{(h-1)} + 1\right)\ln x_j\right].$$

the maximum likelihood equations are characterized by nonlinear system $\boldsymbol{S}(\boldsymbol{\Omega}^{(h-1)}) = \left(\frac{\partial \ln L}{\partial \alpha^{(h-1)}}, \frac{\partial \ln L}{\partial p^{(h-1)}}\right) = \mathbf{0}$, namely:

$$\begin{cases} \frac{m}{\alpha^{(h-1)}} - \sum_{j=1}^{m} z_j^{(h-1)} \ln x_j = 0 \\ -\frac{mn\left(1-p^{(h-1)}\right)^{n-1}}{1-\left(1-p^{(h-1)}\right)^n} - \frac{mn}{1-p^{(h-1)}} + \frac{1}{p^{(h-1)}\left(1-p^{(h-1)}\right)} \sum_{j=1}^{m} z_j^{(h-1)} = 0 \end{cases}.$$

The estimation of the parameters α and p of the MinParB distribution was achieved through the MLE method. This is implemented in Octave 1.5.4 GUI programming environment, and the results after the execution of the program are shown in Table 3 for different size m of the sample and the known values of the parameters $\mu = 1$ and $n = 4$.

Table 3: The estimate of the parameter vector $\boldsymbol{\Omega} = (\alpha,p)$ of $MinParB(1,\alpha,4,p)$ distribution by $\hat{\boldsymbol{\Omega}} = (\hat{\alpha},\hat{p})$

Sample size	(α,p)	$\hat{\alpha}$	$\hat{p}$	h
100	(10;0,1)	9,1538	0,0000	207
1000		10,2906	0,0048	2556
10000		9,9098	0,0935	796
100000		9,9775	0,0988	767
1000000		9,9597	0,1043	736

5 Conclusions

The conclusions revealed by from this paper are related to the study of power series distributions type of a minimum of a sequence of i.i.d.r.v. which are found in a random number.

The basic results of this paper are aimed at extending the results that had as a starting point the studies of Adamidis and Loukas [1] later generalized by Chahkandi and Ganjali [3] or Baretto-Souza and Cribari [2].

Also, it was presented in a unitary approach, the distribution of a minimum number of i.i.d.r.v. through the PSD family, distribution characterised by the number of the random variable in the sequence.

The Poisson Limit Theorem has been formulated for the situations when the random variable number of the sum is a zero truncated binomial distribution and the limit distribution is Poisson type distribution.

For this purpose there have been developed programs for the statistical simulation of the MinParB power series distributions type. The validity of the minimum distributions was performed using the Pearson's test of consistency and is reflected in Table 2. Describing the EM algorithm implemented in the GUI Octave 1.5.4 programming environment to estimate the parameters (Table 3) of the MinParPS distribution.

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