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A Wilf Class Composed of 7 Symmetry Classes of Triples of 4-Letter Patterns

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Abstract: In this paper, we make a contribution to the enumeration of permutations avoiding a triple of 4-letter patterns by establishing a Wilf class composed of 7 symmetry classes.

Keywords: pattern avoidance, Catalan number, kernel method, indecomposable permutation

1 Introduction

We say a permutation is *standard* if its support set is an initial segment of the positive integers, and for a permutation π whose support is any set of positive integers, $\mathrm{St}(\pi)$ is the standard permutation obtained by replacing the smallest entry of π by 1, next smallest by 2, and so on. As usual, a standard permutation π *avoids* a standard permutation τ if there is no subsequence ρ of π for which $\mathrm{St}(\rho)=\tau$. In this context, τ is a pattern, and for a list T of patterns, $S_n(T)$ denotes the set of permutations of $[n]=\{1,2,\ldots,n\}$ that avoid all the patterns in T.

In recent decades pattern avoidance has received a lot of attention. It has its formal origins in Knuth [5] and Simion and Schmidt [7] who considered the problem on permutations and enumerated the number of members of S_n avoiding a particular element or subset, respectively, of 3-letter patterns. Since then the problem has been addressed on several other discrete structures, such as compositions, k-ary words, and set partitions; see, e.g., the texts [3, 6] and references contained therein. Here, we provide further enumerative results concerning the classical avoidance problem on permutations.

Members of S_n avoiding a single 4-letter pattern have been well studied (see, e.g., [8–10]). There are 56 symmetry classes of pairs of 4-letter patterns, for all but 8 of which the avoiders have been enumerated [1]. Less is known about the 317 symmetry classes of triples of 4-letter patterns. Here, we show that for precisely 7 symmetry classes (as defined in [1]) of triples of 4-letter patterns, their avoiders are counted by the sequence $(u_n)_{n\geq 0}=(1,1,2,6,21,77,287,1079,\ldots)$ defined by the recurrence $u_0=1,\ u_n=u_{n-1}+\binom{2n-2}{n-2}$ for $n\geq 1$, whence $u_n=1+\sum_{k=2}^{n}\binom{2k-2}{k-2}$. These 7 symmetry classes thus form a Wilf class [1].

Our approach mostly uses the generating function, $\sum_{n\geq 0}u_nx^n$, which is, as a routine computation shows,

$$1+\frac{1-2x}{2(1-x)}\left(\frac{1}{\sqrt{1-4x}}-1\right)=1+\frac{x}{1-x}\big(1+xC'(x)\big).$$

Throughout, $C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n-1}$ denotes the Catalan number, and $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ the generating function $\sum_{n\geq 0}C_nx^n$; $C'(x)$ denotes the derivative of $C(x)$. For the first class, we give a bijective enumeration that explains the $\binom{2n-2}{n-2}$ summand in u_n.

Numerical computations show that at most 7 of the 317 symmetry classes of triples of 4-letter patterns have avoiders counted by this sequence. Our result is the following.

Theorem 1. *Let*

$$T_1=\{2413,3142,1324\},\quad T_2=\{2143,2413,1324\},$$
$$T_3=\{2143,1324,1342\},\quad T_4=\{3142,4132,1243\},$$
$$T_5=\{3142,4123,1423\},\quad T_6=\{4132,1432,1243\},$$
$$T_7=\{4132,1342,1324\}.$$

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Then

$$F_j(x) = \sum_{n\geq 0} |S_n(T_j)|x^n = 1 + \frac{x}{1-x}\big(1 + xC'(x)\big),$$

for all $j = 1, 2, \dots, 7$.

2 Proofs

For cases $T = T_1, T_2$, we note that all patterns involved are reverse indecomposable, also known as skew indecomposable. (See [2] for terminology. As a reminder, the permutation 21534 is decomposable with components 21 and 534, but 21534 is reverse indecomposable and the reverse components, also known as skew components, of 68721534 are 687 and 21534.) So a permutation avoids T_1 (resp. T_2) if and only if each of its reverse components does so. This observation reduces the problem to finding the number v_n of reverse indecomposable avoiders of length n, for then with $V(x) := \sum_{n\geq 1} v_n x^n$, the desired generating function $\sum_{n\geq 0} |S_n(T)|x^n$ is $1/(1-V(x))$ by the combinatorial interpetation of the Invert transform.

2.1 $T_1 = \{2413, 3142, 1324\}$

Suppose π is a reverse indecomposable T_1-avoider. Recall that a separable permutation is one that avoids the first two patterns in T_1 and a separable permutation of length ≥ 2 is either decomposable or reverse decomposable [2]. Hence, for $n \geq 2$, π is decomposable and so π is uniquely expressible as $\pi_1 \oplus \pi_2$ with π_1, π_2 nonempty T_1-avoiders and π_1 indecomposable. (Here, $\oplus$ is the direct sum of standard permutations, thus $213 \oplus 1243 = 2134576$.) But now, avoiding the last pattern, 1324, in T_1 implies that π_1 avoids 132 and π_2 avoids 213. Conversely, if π_1 is an indecomposable 132-avoider and π_2 is a nonempty 213-avoider, then $\pi_1 \oplus \pi_2$ avoids T_1. The generating function for indecomposable 132-avoiders is $x(1 + xC(x)^3)$ and for nonempty 213-avoiders is $C(x) - 1$. Hence, the generating function for reverse indecomposable T_1-avoiders is $V(x) = x(1 + xC(x)^3)(C(x) - 1)$, and one checks that $F(x) = 1/(1-V(x))$ coincides with $1 + \frac{x}{1-x}\big(1 + xC'(x)\big)$.

Alternatively, we can give a direct bijective count for case T_1. A permutation π whose last entry is 1 avoids T_1 iff $\mathrm{St}(\pi\backslash\{1\})$ avoids T_1. Hence, with $u_n = |S_n(T_1)|$, $u_n - u_{n-1}$ counts $\mathscr{A}_n := \{\pi \in S_n(T_1) : \text{last entry of } \pi \text{ is not } 1\}$. On the other hand, $\binom{2n-2}{n-2}$ counts lattice paths of $n-2$ upsteps $(1,1)$ and n downsteps $(1,-1)$, or, by prepending and appending an upstep, $\binom{2n-2}{n-2}$ counts the set $\mathscr{B}_n$ of lattice paths of n upsteps and n downsteps that start and end with an upstep. Here is a bijection from $\mathscr{A}_n$ to $\mathscr{B}_n$.

Given $\pi \in \mathscr{A}_n$, the staircase with corners at the right-left maxima (heavy line in Figure 1) identifies the reverse components of π by its contacts with the diagonal.

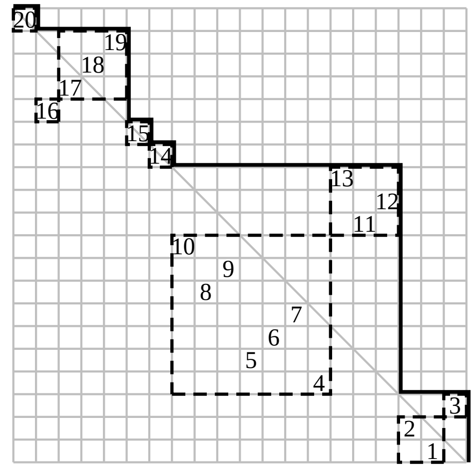

Fig. 1: A T_1-avoider, last entry $\neq 1$, with its reverse components decomposed

Among the reverse components one can now identify the singletons, all on the diagonal, and for each non-singleton its first component and tail (all entries following the first component). These are the dashed square enclosures in Figure 1. Since each first component is a 132-avoider, it is determined by its left-right minima which in turn are specified by the lattice path for which the left-right min serve as corners (drawn heavy in Figure 2). Similarly each tail is a 213-avoider and so is determined by an analogous lattice path based on its right-left maxima (also drawn heavy in Figure 2). The

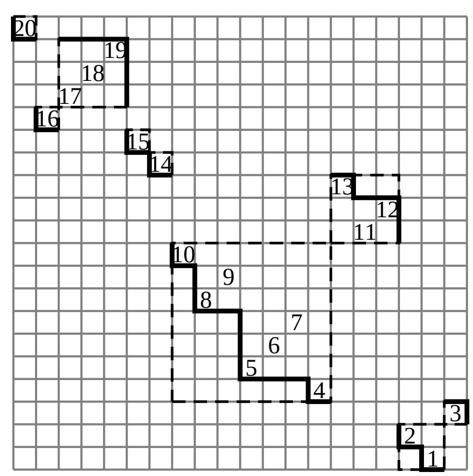

Fig. 2: The lattice paths that determine a T_1-avoider, last entry $\neq 1$

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singletons are bracketed by a south-then-east 2-step path as in Figure 2. Note that each first component is either a singleton or else, since it is indecomposable and separable, it is reverse decomposable and consequently its lattice path has an interior return to the diagonal line joining its endpoints.

The heavy-line paths, taking left-right order into account, determine π. Note that the first path always starts with a south step because it brackets either a singleton or a first component. Also, the last path always ends with a south step since it brackets the tail of a non-singleton reverse component (π does not end with a 1). Rotate each path 135° clockwise and concatenate, with bullets to mark the division into reverse components (Figure 3). The tail paths appear below the dotted line, while the first component and singleton paths appear above it.

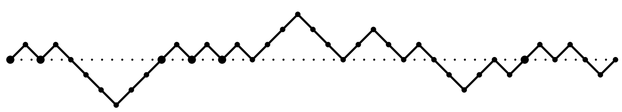

Fig. 3: A balanced path with distinguished returns

The resulting path has n upsteps U and n downsteps D and starts and ends with an upstep. To ensure invertibility, we must first tweak the subpaths corresponding to first components of non-singleton reverse components before erasing the bullets. Each such subpath is a Dyck path and has at least 2 upsteps from ground level since the corresponding permutation is indecomposable. Transfer the *second* upstep from ground level to the beginning of the subpath and then erase the bullets (Figure 4).

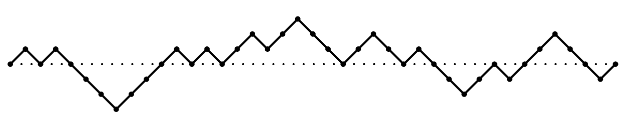

Fig. 4: A balanced path

The resulting path is in $\mathscr{B}_n$. To reverse the map, take each maximal subpath above ground level—a nonempty Dyck path—and transform it, inserting bullets, as follows: (i) $(UD)^k \to (\bullet UD)^k$ where $k \geq 1$, (ii) $(UD)^k UUPDQDR \to (\bullet UD)^k \bullet UPDUQDR$ where $k \geq 0$ and P,Q,R are possibly empty Dyck paths (every nonempty Dyck path has one of these two forms) . $\square$

2.2 $T_2 = \{2143, 2413, 1324\}$

We have the following simple characterization of reverse indecomposables among T_2-avoiders; the proof follows from the fact that if 1 appears after n in a 2413-avoider π, then π is reverse decomposable.

Lemma 1. *A T_2-avoider π of length n is reverse indecomposable if and only if 1 appears before n in π.* $\square$

Suppose π is a permutation on $[n]$ in which 1 appears before n. Write π as $A1BnC$ where A,B,C are substrings, possibly empty. If π is a T_2-avoider, then (i) B is increasing, for else 1 and n are the "1" and "4" of a 1324 pattern, and (ii) $A > C$ (meaning all entries of A exceed all entries of C), for else 1 and n are the "1" and "4" of a 2143 pattern. Consequently we may refine B to write π as

$$\pi = \begin{cases} A\,1\,B_1\,B_2\,B_3\,n\,C & \text{with } B_1 < \max(C) < B_2 < \min(A) < B_3 \\ & \quad \text{if } A \text{ and } B \text{ are both nonempty,} \\ 1\,B_1\,B_3\,n\,C & \text{with } B_1 < \max(C) < B_3 \\ & \quad \text{if } A = \emptyset,\ C \neq \emptyset, \\ A\,1\,B_1\,B_3\,n & \text{with } B_1 < \min(A) < B_3 \\ & \quad \text{if } A \neq \emptyset,\ C = \emptyset\,. \end{cases}$$

Then we also have, for a T_2-avoider, (iii) $\pi_1 := \mathrm{St}(A_1\,1\,B_3)$ is nonempty, avoids 132 for else n is the "4" of a 1324, and 1 is not immediately followed by 2 in π_1 by definition of B_3, and (iv) $\pi_2 := \mathrm{St}(B_1\,n\,C)$ is nonempty, avoids 213 for else 1 is the "1" of 1324, and $\max(\pi_2)$ is not immediately preceded by $\max(\pi_2) - 1$ in π_2 by definition of B_1. Next, note that π can be recovered from knowledge of $j := |B_2|$, π_1, and π_2: add 1 to each entry of π_2 except replace $\max(\pi_2)$ by n to get $B_1\,n\,C$, then add $|B_1| + |B_2| + |C|$ to each non-1 entry of π_1 to get $A_1\,1\,B_3$, and lastly fill in the (increasing) entries of B_2.

Conversely, for $n \geq 2$, given $j \geq 0$ and standard permutations π_1, π_2 with $j + |\pi_1| + |\pi_2| = n$ and π_1, π_2 satisfying conditions (iii) and (iv) respectively, the latter construction produces a reverse indecomposable T_2-avoider on $[n]$. Thus we let w_n denote the number of pairs (π_1, π_2) of total length n satisfying (iii) and (iv) so that, for $n \geq 2$, $v_n = w_2 + w_3 + \cdots + w_n$ gives the number of reverse indecomposable T_2-avoiders on $[n]$, while $v_1 = 1$.

To compute w_n, we have the following elementary counts.

Lemma 2. *Define $w(r,s)$ by $w(1,1) = 1$ and $w(r,s) = C_{r-s-1,s} = \frac{s+1}{2r-s-1}\binom{2r-s-1}{r-s-1}$ for $r \geq 2$, $1 \leq s \leq r$. Then, for $1 \leq s \leq r$, $w(r,s)$ is both*

(i) the number of 132-avoiding permutations of $[r]$ in which the number of entries weakly after 1 is s (equivalently, 1 is in position $r-s+1$) and 1 is not immediately followed by 2, and

(ii) the number of 213-avoiding permutations of $[r]$ in which the number of entries weakly before r is s (equivalently, r is in position s) and r is not immediately preceded by $r-1$. $\square$

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We have $w_2 = 1$, and for $n \geq 3$, by Lemma 2,

$$w_n = \sum_{r=1}^{n-1} \sum_{s=1}^{r} w(r,s) \sum_{t=1}^{n-r} w(n-r,t). \qquad (1)$$

To evaluate these sums, we use

Lemma 3. *For $r \geq 2$,*

$$\sum_{s=1}^{r} w(r,s) = C_{r-2,2}.$$

□

The $r = 1$ term in (1) contributes $\sum_{t=1}^{n-1} w(n-1,t) = C_{n-3,2}$. The $r = n-1$ term similarly contributes $\sum_{s=1}^{n-1} w(n-1,s) = C_{n-3,2}$. The remaining terms in (1) contribute (only for $n \geq 4$)

$$\sum_{r=2}^{n-2} \sum_{s=1}^{r} w(r,s) \sum_{t=1}^{n-r} w(n-r,t) = \sum_{r=2}^{n-2} C_{r-2,2} C_{n-r-2,2} = C_{n-4,5},$$

the last equality using the convolution property, $(C_{n,r})_{n\geq 0} * (C_{n,s})_{n\geq 0} = (C_{n,r+s+1})_{n\geq 0}$, of the Catalan triangle numbers. Hence, for $n \geq 3$,

$$w_n = 2C_{n-3,2} + C_{n-4,5}$$

with $C_{-1,5} := 0$.

Since $\sum_{n\geq 0} C_{n,k} x^n = C(x)^{k+1}$, it is now routine to compute the generating function $W(x) := \sum_{n\geq 2} w_n x^n = x^2 + 2x^3 C(x)^3 + x^4 C(x)^6$ and, since $v_n = w_2 + w_3 + \cdots + w_n$, we find the generating function $V(x) := \sum_{n\geq 1} v_n x^n = x + W(x)/(1-x)$. After simplification, this $V(x)$ agrees with the $V(x)$ in Case T_1.

2.3 $T_3 = \{2143, 1324, 1342\}$

Lemma 4. *Let $a_n = |S_n(T_3)|$. Then*

$$\begin{aligned} a_n &= 4a_{n-1} - 2a_{n-2} \\ &\quad + \sum_{i=2}^{n-2} \sum_{j=1}^{i-1} \sum_{i_1+\cdots+i_{n-j-1}=j-1} a_{i_{n-j-1}+1} \prod_{s=1}^{n-j-2} C_{i_s} \end{aligned}$$

with $a_0 = a_1 = 1$.

Proof. Let $\pi = i\pi'$ be a member of $S_n(T_3)$. If $i = n, n-1$ then there are a_{n-1} possible permutations. So assume $1 \leq i \leq n-2$ and $1 \leq j \leq n$, and let $\pi = ij\pi'$ be a member of $S_n(T_3)$. If $j > i$, one may verify that π avoids T_3 if and only if either $j = i+1$ or $j = n$. Clearly, $\pi = i(i+1)\pi'$ (resp. $\pi = in\pi'$) avoids T_3 if and only if $(i+1)\pi'$ (resp. $i\pi'$) avoids T_3, which implies there are a_{n-1} possible permutations. Note that the case $i = n-1$ and $j = n$ is counted twice, which has a_{n-2} possible permutations. Hence

$$a_n = 4a_{n-1} - 2a_{n-2} + \sum_{i=2}^{n-2} \sum_{j=1}^{i-1} a_n(i,j),$$

where $a_n(i,j)$ is the number of permutations $\pi = ij\pi'$ in $S_n(T_3)$.

Now suppose $n \geq i > j \geq 1$. Since π avoids 2143, we have that π contains $i, (i+1), \ldots, n$ in that order. Since π avoids 1324 and 1342, it must be that π does not contain any letter ℓ with $j+1 \leq \ell \leq i-1$ to the left of $i+1$. Thus, π contains $i, j, (j+1), \ldots, (i-1), (i+1), (i+2), \ldots, n$ in that order. Hence, we can express π as

$$\begin{aligned} \pi =& ij\pi^{(1)}(j+1)\pi^{(2)} \cdots (i-1)\pi^{(i-j)} \\ & (i+1)\pi^{(i-j+1)}(i+2)\pi^{(i-j+2)} \cdots n\pi^{(n-j)}, \end{aligned}$$

where $\pi^{(s)}$ avoids 132 for all $s = 1, 2, \ldots, n-j-2$, $\pi^{(n-j-1)} n\pi^{(n-j)}$ avoids T_3, each letter of $\pi^{(s)}$ is greater than each letter of $\pi^{(s+1)}$ for all $s = 1, 2, \ldots, n-j-3$, and each letter of $\pi^{(n-j-2)}$ is greater than each letter of $\pi^{(n-j-1)}\pi^{(n-j)}$. Hence, since $|S_n(132)| = C_n$ (see [5]), we obtain

$$a_n(i,j) = \sum_{i_1+\cdots+i_{n-j-1}=j-1} a_{i_{n-j-1}+1} \prod_{s=1}^{n-j-2} C_{i_s},$$

which completes the proof.

By Lemma 4, we have

$$a_n = 4a_{n-1} - 2a_{n-2} + \sum_{j=1}^{n-3} j \sum_{i_1+\cdots+i_{j+1}=n-3-j} a_{i_{j+1}+1} \prod_{s=1}^{j} C_{i_s}$$

with $a_0 = a_1 = 1$. Let $A(x) = \sum_{n\geq 0} a_n x^n$. Then our recurrence can be written as

$$\begin{aligned} A(x) - 1 - x &= 4x(A(x)-1) - 2x^2(A(x)-1) \\ &\quad + \sum_{j\geq 1} jx^{j+2}(A(x)-1)C(x)^j, \end{aligned}$$

leading to

Theorem 2. *The generating function for the number of permutations of $S_n(T_3)$ is given by*

$$F_3(x) = 1 + \frac{x}{1-x}\big(1 + xC'(x)\big).$$

2.4 $T_4 = \{3142, 4132, 1243\}$

Let $G_m(x)$ be the generating function for T_4-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_4(x)$. Let $H_m(x)$ be the generating function for T_4-avoiders where the left-right maxima form a list of m consecutive integers.

Lemma 5. *Let $m \geq 1$. Then*

$$H_m(x) = xH_{m-1}(x) + x\sum_{j\geq m} G_j(x),$$

where $H_1(x) = G_1(x) = xF_4(x)$.

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*Proof.*We denote the set of permutations of $S_n(T_4)$ where the left-right maxima form a list of m consecutive integers by $\mathscr{A}_{n,m}$. Let us write an equation for $H_m(x)$. If $\pi=(n+1-m)(n+2-m)\pi^{(2)}\cdots n\pi^{(m)}\in\mathscr{A}_{n,m}$, then the permutation that obtained from π by removing the first letter belongs to $\mathscr{A}_{n-1,m-1}$. Thus, the contribution of this case is given by $xH_{m-1}(x)$. Otherwise, let $\pi=(n+1-m)\pi^{(1)}(n+2-m)\pi^{(2)}\cdots n\pi^{(m)}\in\mathscr{A}_{n,m}$ such that $\pi^{(1)}$ is a nonempty sequence with exactly k left-right maxima. So $\pi'=\mathrm{St}(\pi^{(1)}(n+2-m)\cdots n\pi^{(m)})\in S_{n-1}(T_4)$ has exactly $m+k-1$ left-right maxima. Note that any permutation π' in $S_{n-1}(T_4)$ with exactly $m+k-1$, $k\geq 1$, left-right maxima, then π' can be written as $\pi'^{(1)}(n+2-m)\cdots n\pi'^{(m)}$, where $\pi'^{(1)}$ has exactly k left-right maxima (π' avoids 1243). Thus, the contribution of this case is given by $x\sum_{k\geq1}G_{m-1+k}(x)=x\sum_{j\geq m}G_j(x)$.

Hence, by combining these two cases, we have

$$H_m(x)=xH_{m-1}(x)+x\sum_{j\geq m}G_j(x),$$

where $H_1(x)=G_1(x)=xF_4(x)$ (by the definitions), which completes the proof.

Lemma 6.*For all $m\geq 2$,*

$$G_m(x)=H_m(x)+\frac{x^m}{1-x}(C^{m-1}(x)-1).$$

*Proof.*Let $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ be any permutation of $S_n(T_4)$ with exactly m left-right maxima. Since π avoids 1243, we can write π as

$$\pi=i_1\pi^{(1)}(n+2-m)\pi^{(2)}\cdots n\pi^{(m)},$$

where each letter of $\pi^{(1)}$ is at most i_1-1, and each letter of $\alpha=\pi^{(2)}\cdots\pi^{(m)}$ is at most $n+1-m$. Now let us write an equation for $G_m(x)$. If α is empty then the contribution is $H_m(x)$ (see Lemma 5). Otherwise, since π avoids 2413 and 2431, we see that each letter of α is at least i_1+1. Moreover, by the fact π avoids 1243, then $\pi^{(1)}=(i_1-1)\cdots21$, and $i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ avoids 132, that is, each letter of $\pi^{(j)}$ is greater than each letter of $\pi^{(j+1)}$, $j=2,3,\ldots,m-1$. Thus, by the fact that the generating function for the number of permutations in $S_n(132)$ is $C(x)$ (see [5]), we obtain that the contribution is given by

$$\frac{x^m}{1-x}\left(C^{m-1}(x)-1\right).$$

By combining the two contributions, we complete the proof.

Now, we are ready to find an explicit formula for $F_4(x)$. Let $G(x,u)=\sum_{j\geq1}G_j(x)u^{j-1}$. Note that $F_4(x)=1+G(x,1)$. By Lemma 5 and Lemma 6, we have

$$G_m(x)=x^mF_4(x)+\frac{x^m}{1-x}\left(C^{m-1}(x)-1\right)+\sum_{\ell\geq2}x^{\ell-1}\sum_{j\geq m-\ell+2}G_j(x),$$

which implies

$$G_m(x)-xG_{m-1}(x)=\frac{x^m}{1-x}\left(C^{m-1}(x)-C^{m-2}(x)\right)+x\sum_{j\geq m}G_j(x)$$

with $G_1(x)=xF_4(x)=x(G(x,1)+1)$. By multiplying the recurrence by u^{m-1} and summing over $m\geq2$, we obtain

$$G(x,u)-x(G(x,1)+1)-xuG(x,u)=\frac{xu}{1-u}(G(x,1)-G(u,x))+\frac{x^2u(C(x)-1)}{(1-x)(1-uxC(x))},$$

which is equivalent to

$$\left(1+\frac{xu^2}{1-u}\right)G(x,u)=x+\frac{x}{1-u}G(x,1)+\frac{x^2u(C(x)-1)}{(1-x)(1-uxC(x))}.$$

This functional equation can be solved by the kernel method (see [4] and references therein) using $u=C(x)$, leading to

Theorem 3.*The generating function for the number of permutations of $S_n(T_4)$ is given by*

$$F_4(x)=1+G(x,1)=\frac{x}{1-x}\left(1+xC'(x)\right).$$

2.5 $T_5=\{3142,4123,1423\}$

Let $G_m(x)$ be the generating function for the number of permutations in $S_n(T_5)$ with exactly m left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_5(x)$. In the next lemma we deal with the hardest case, namely $m=2$.

Lemma 7.*We have*

$$G_2(x)=\frac{x^2\left(C(x)-1-xC(x)\right)F_5(x)+x(F_5(x)-1)}{1-x}.$$

*Proof.*Let $\pi=i\alpha'n\alpha''$ be any permutation with exactly two left-right maxima. Let us write an equation for $G_2(x)$. In case $i=n-1$, the contribution is $x(F_5(x)-1)$. If $i<n-1$, then α'' contains $n-1$ and we consider two cases:

– π contains a letter ℓ between n and $n-1$ such that $\ell>i$. In this case each letter of α'' is greater than i. Thus, the contribution of this case is given by $x^2\left(C(x)-1-xC(x)\right)F_5(x)$, where $C(x)-1-xC(x)$ counts the number of nonempty permutation of $S_n(231)$ such that the first letter is not n (see [5]).
– π does not contain any letter ℓ between n and $n-1$ such that $\ell>i$. So $\pi=i\alpha'n(n-1)\alpha'''$, which implies that the contribution is given by $xG_2(x)$.

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By combining all three cases, we obtain $G_2(x)=x(F_5(x)-1)+x^2\big(C(x)-1-xC(x)\big)F_5(x)+xG_2(x)$, which completes the proof.

Now we treat the case $m\geq 3$.

Lemma 8. *For all $m\geq 3$, $G_m(x)=\big(xC(x)\big)^{m-2}G_2(x)$.*

Proof. Let $\pi\in S_n(T_5)$ be any permutation with exactly m left-right maxima. Then π can be written as

$$\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$$

with $i_m=n$. Note that π avoids T_5 if and only if (1) $\pi^{(j)}$ avoids 231 for all $j=3,4,\ldots,m$, (2) $i_1\pi^{(1)}i_2\pi^{(2)}$ avoids T_5, (3) each letter of $\pi^{(j+1)}$ is greater than each letter of $\pi^{(j)}$ for $j=3,4,\ldots,m-1$, (4) each letter of $\pi^{(3)}$ is greater than each letter of $i_1\pi^{(1)}i_2\pi^{(2)}$. Hence, by the fact that $\sum_{n\geq 0}|S_n(231)|x^n = C(x)$ (see [5]), we get $G_m(x)=\big(xC(x)\big)^{m-2}G_2(x)$, as claimed.

Using the expressions above for $G_0(x)$ and $G_1(x)$ and Lemma 8, we have

$$F_5(x)=\sum_{j\geq 0}G_j(x)=1+xF_5(x)+G_2(x)C(x),$$

which, by Lemma 7, implies the following result.

Theorem 4. *The generating function for the number of permutations of $S_n(T_5)$ is given by*

$$F_5(x)=1+\frac{x}{1-x}\big(1+xC'(x)\big).$$

2.6 $T_6=\{4132,1432,1243\}$

Let $G_m(x)$ be the generating function for the number of permutations of $S_n(T_6)$ with exactly m left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_6(x)$. By using similar arguments as in the proof of Lemma 5, we obtain the following relation.

Lemma 9. *Let $m\geq 1$ and let $H_m(x)$ be the generating function for the number of permutations π in $S_n(T_6)$ where the left-right maxima of π are exactly $n+1-m,n+2-m,\ldots,n$. Then*

$$H_m(x)=xH_{m-1}+x\sum_{j\geq m}G_j(x)$$

with $H_1(x)=xF_6(x)$.

Lemma 10. *For all $m\geq 2$,*

$$G_m(x)=H_m(x)+\frac{x^{m+1}}{1-x}F_6(x)\sum_{j=1}^{m-1}\left(\frac{1-x}{1-2x}\right)^j.$$

Proof. Let $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ be any permutation of $S_n(T_6)$ with exactly m left-right maxima. Since π avoids 1243, we can write π as

$$\pi=i_1\pi^{(1)}(n+2-m)\pi^{(2)}(n+3-m)\pi^{(3)}\cdots n\pi^{(m)},$$

where each letter of $\pi^{(1)}$ is at most i_1-1, and each letter of $\alpha=\pi^{(2)}\cdots\pi^{(m)}$ is at most $n+1-m$. Now let us write an equation for $G_m(x)$. If α is empty then the contribution is $H_m(x)$ (see Lemma 9). Otherwise, since π avoids T_6, we see that there exactly unique j, $2\leq j\leq m$, such that $\pi^{(j)}$ is not empty, which leads to $\pi^{(j)}=(i_1+1)\cdots(n+1-m)$. Thus,

– Since π avoids 1243, we have that
$$\pi^{(1)}\pi^{(2)}\cdots\pi^{(j-1)}=(i_1-1)\cdots 21.$$

– Since π avoids 2431, we have that $\pi^{(j)}$ can be decomposed as
$$\pi^{(j,0)}\pi^{(j,1)}\cdots\pi^{(j,i_1-1)},$$
where all letters of $\pi^{(j)}$ on the left side of the letter i_1+1, $\pi^{(j,0)}$ avoids T_6, and $\pi^{(j,k)}$ avoids 21 (since π avoids 1432), for $k=1,2,\ldots,i_1-1$.

Hence, the contribution in this case is

$$\frac{x^{m+1}}{1-x}F_6(x)\sum_{j=1}^{m}\left(\frac{1}{1-\frac{x}{1-x}}\right)^j.$$

By combining the two contributions, we complete the proof.

Now, we ready to find an explicit formula for $F_6(x)$. Let $G(x,u)=\sum_{j\geq 1}G_j(x)u^{j-1}$. Note that $F_6(x)=1+G(x,1)$. By Lemma 9 and Lemma 10, we have

$$\begin{aligned}&G_m(x)-xG_{m-1}(x)\\&=\frac{x^{m+1}}{1-x}(G(x,1)+1)\frac{(1-x)^{m-1}}{(1-2x)^{m-1}}+x\sum_{j\geq m}G_j(x)\end{aligned}$$

with $G_1(x)=xF_6(x)=x(G(x,1)+1)$. By multiplying the recurrence by u^{m-1} and summing over $m\geq 2$, we obtain

$$\begin{aligned}&G(x,u)-x(G(x,1)+1)-xuG(x,u)\\&=\frac{xu}{1-u}(G(x,1)-G(u,x))+\frac{x^3(G(x,1)+1)}{1-2x-x(1-x)u},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\left(1+\frac{xu^2}{1-u}\right)G(x,u)\\&=x+\frac{x^3}{1-2x-x(1-x)u}\\&\qquad+\left(\frac{x}{1-u}+\frac{x^3}{1-2x-x(1-x)u}\right)G(x,1).\end{aligned}$$

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This functional equation can be solved by the kernel method (see [4] and references therein) using $u = C(x)$, leading to

Theorem 5. *The generating function for the number of permutations of $S_n(T_6)$ is given by*

$$F_6(x) = 1 + G(x,1) = 1 + \frac{x}{1-x}\big(1 + xC'(x)\big).$$

2.7 $T_7 = \{4132, 1342, 1324\}$

Let $G_m(x)$ be the generating function for the number of permutations in $S_n(T_7)$ with exactly m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = x\sum_{n\geq 0}|S_n(132)|x^n = xC(x)$ (see [5]). In the next lemma we deal with the hardest case, namely $m = 2$.

Lemma 11. *We have*

$$G_2(x) = \frac{x^2(3-4x-\sqrt{1-4x})}{(1-x)\sqrt{1-4x}(1+\sqrt{1-4x})}.$$

Proof. Let $G_{2,k}(x)$ denote the generating function for $S_{n,2,k} := \{\pi \in S_n(T_7) : \pi$ has exactly two left-right maxima and the leftmost letter of π is $n-1-k\}$. Thus $G_2(x) = \sum_{k=0}^{n-2} G_{2,k}(x)$.

Now let $\pi \in S_{n,2,k}$ and consider the cases $k = 0$, $k = 1$, and $2 \leq k \leq n-2$ separately:

– $k = 0$. Here π can be presented as $\pi = (n-1)\alpha' n\alpha''$. If π has at least three letters, then either $\pi = (n-1)\beta'(n-2)\beta'' n\alpha''$ where each letter of β' is greater than each letter of $\beta''\alpha''$, or $\pi = (n-1)\alpha' n\beta'(n-2)\beta''$ where each letter of $\alpha'\beta'$ is greater than each letter of β''. Thus, by the fact that $\sum_{n\geq 0}|S_n(132)|x^n = C(x)$, we obtain

$$G_{2,0}(x) = x^2 + xC(x)G_{2,0}(x) + xC(x)G_{2,0}(x),$$

which implies

$$G_{2,0}(x) = \frac{x^2}{\sqrt{1-4x}}.$$

– $k = 1$. Here π can be presented as $\pi = (n-2)\alpha' n\alpha''(n-1)\alpha'''$. Similar to the case $k = 0$, by considering the position of $n-3$, we obtain that

$$\begin{aligned} G_{2,1}(x) &= x^3 + xC(x)G_{2,1}(x) \\ &\quad + (xC(x))^2 G_{2,0}(x) + x^2C(x)G_{2,0}(x). \end{aligned}$$

By Case $k = 0$, we obtain

$$G_{2,1}(x) = \frac{x^2(1-x)(1-\sqrt{1-4x})}{\sqrt{1-4x}(1+\sqrt{1-4x})}.$$

– $2 \leq k \leq n-2$. Here π can be presented as $\pi = (n-1-k)\alpha' n\alpha''$. If $n-1$ is the leftmost letter $\ell > n-1-k$ in α'' then since π avoids 4132, we have that $n-1$ is the leftmost letter of α'. Otherwise, since π avoids T_7, we have that $n-1$ is the rightmost letter $\ell > n-1-k$ of α''. Since π avoids 1324, we can write π as

$$\pi = (n-1-k)\alpha' n\beta^{(0)}(n-k)\beta^{(k)}\cdots(n-1)\beta^{(1)}.$$

Since π avoids T_7, we see that each letter of $\beta^{(j)}$ is greater than each letter of $\beta^{(j-1)}$, for $j = k, k-1, \ldots, 2$. Moreover, each letter of $\alpha'\beta^{(0)}$ is greater than each letter of $\beta^{(k)}$. It is not hard to see that $\beta^{(j)}$ avoids 132 for all $j = 1, 2, \ldots, k$ and $(n-1-k)\alpha'(n-k)\beta^{(0)}$ is in $S_{n-k,2,0}$. Hence,

$$G_{2,k}(x) = xG_{2,k-1}(x) + (xC(x))^k G_{2,0}(x).$$

By summing over $k \geq 2$, we obtain

$$\begin{aligned} &G_2(x) - G_{2,0}(x) - G_{2,1}(x) \\ &= x(G_2(x) - G_{2,0}(x)) + \frac{(xC(x))^2}{1-xC(x)}G_{2,0}(x). \end{aligned}$$

Using the evaluations above for $G_{2,0}$ and $G_{2,0}$, we complete the proof.

Now we treat the case $m \geq 3$.

Lemma 12. *For all $m \geq 3$, $G_m(x) = (xC(x))^{m-2}G_2(x)$.*

Proof. Let $\pi \in S_n(T_7)$ be any permutation with exactly m left-right maxima. Then π can be written as

$$\pi = i\pi^{(0)}(i+1)\pi^{(1)}\cdots(i+m-2)\pi^{(m-2)}n\pi^{(m-1)}.$$

Note that π avoids T_7 if and only if (1) $\pi^{(j)}$ avoids 132 for all $j = 0, 1, \ldots, m-3$, (2) $i\pi^{(m-2)}n\pi^{(m-1)}$ avoids T_7, (3) each letter of $\pi^{(j)}$ is greater than each letter of $\pi^{(j+1)}$ for $j = 0, 1, \ldots, m-4$, (4) there is no letter ℓ in $\pi^{(m-2)}\pi^{(m-1)}$ such that ℓ between the minimal letter of $\pi^{(m-3)}$ and $i + m - 2$. Hence, $G_m(x) = (xC(x))^{m-2}G_2(x)$, as claimed.

Using the expressions above for $G_0(x)$ and $G_1(x)$ and Lemma 12, we have

$$F_7(x) = \sum_{j\geq 0} G_j(x) = 1 + xC(x) + G_2(x)C(x),$$

which, by Lemma 11, implies the following result.

Theorem 6. *The generating function for the number of permutations of $S_n(T_7)$ is given by*

$$F_7(x) = 1 + \frac{x}{1-x}\big(1 + xC'(x)\big).$$

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3 Conclusion

In this paper, we have used generating functions to determine all symmetry classes of permutations avoiding a triple of 4-letter patterns with counting sequence $u_n = 1 + \sum_{k=2}^{n} \binom{2k-2}{k-2}$. A bijective argument helps to explain the $\binom{2k-2}{k-2}$ summand in u_n.

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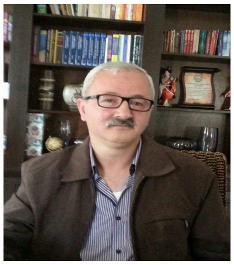

Toufik Mansour obtained his PhD degree in mathematics from the University of Haifa in 2001. He spent one year as a postdoctoral researcher at the University of Bordeaux (France) supported by a Bourse Chateaubriand scholarship, and a second year at the Chalmers Institute in Gothenburg (Sweden) supported by a European Research Training Network grant. Toufik has been a permanent faculty member at the University of Haifa since 2003 and was promoted to associate professor in 2008, and to full professor in 2014. He spends his summers as a visitor at institutions around the globe, for example, at the Center for Combinatorics at Nankai University (China) where he was a faculty member from 2004 to 2007, and at The John Knopfmacher Center for Applicable Analysis and Number Theory, University of the Witwatersrand (South Africa). Toufik's area of specialty is enumerative combinatorics and more generally, discrete mathematics and its applications. Originally focusing on pattern avoidance in permutations, he has extended his interest to set partitions, words, compositions, and normal ordering. Toufik has authored or co-authored many papers in these areas. He serves as a reviewer for many journals.

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