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Common Fixed Point Theorems for Generalized Quadratic (ψ_1, ψ_2, ϕ)-Weak Contraction in Complete Metric Spaces

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Abstract: In this note, we establish some unique common fixed point theorems in complete metric spaces by using quadratic(ψ_1, ψ_2, ϕ)-weak contractive condition and as an application of our result we have supported this by giving an example.

Keywords: Common Fixed Point, (ψ_1, ψ_2, ϕ)- Weak Contraction Condition, Complete Metric Space

1 Introduction and Preliminaries

Banach ([8]) proved a unique fixed point for a self mapping T defined on a complete metric space (X,d) satisfying the following condition for all $x,y \in X$

$$d(Tx,Ty) \leq kd(x,y), \tag{1}$$

where $k \in (0,1)$.

It is remarkable that a self map T satisfying condition (1) implies that T is continuous. Banach Fixed Point Theorem was generalized by many authors by using different types of *control functions*. Mishra et al. ([10]-[12]) have discussed some results of Fixed point theorems in partial metric spaces and other spaces with different type of contraction conditions. Applications of these type of common fixed point theorems were discussed in ([13]-[23]). The weak contraction condition in Hilbert Space was introduced by Alber and Gurerre - Delabriere ([9]). Later Rhoades ([2]) has shown that the result of Alber and Gurerre - Delabriere ([9]) in Hilbert Spaces is also true in a complete metric space. Rhoades [2] established a fixed point theorem in a complete metric space by using the following contraction condition:

A weakly contractive mapping $T : X \to X$ which satisfies the condition for all x, y $\in X$

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)), \tag{2}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark: In the above result if $\varphi(t) = (1-k)t$ where $k \in (0,1)$, then we have the condition (1) due to Banach. So, in view of (1), the condition (2) is weaker condition and we call this condition as a Weakly Contraction Condition.

Recently, Dutta and Choudhury ([6]) proved the following theorem by employing two control functions. The theorem follows:

Theorem 1.1 Let (X,d) be a complete metric space and let $T : X \to X$ satisfying the inequality

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for all x, y$\in X$

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y)) \quad (3)$$

where $\phi,\psi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\phi(t) = \psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Zhang and Song([7]) used generalized $\phi-$ weak contraction for a pair of two maps and proved a common fixed point theorem which follows.

Theorem 1.2 Let (X,d) be a complete metric space and $T,S : X \to X$ are two mappings such that for all $x,y \in X$

$$d(Tx,Sy) \leq M(x,y) - \psi(M(x,y)), \quad (4)$$

where $\psi : [0,\infty) \to [0,\infty)$ is lower semi-continuous function with $\psi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0$,

$$M(x,y) = max\{d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[d(y,Tx)+d(x,Sy)]\}.$$

Then there exists the unique point $u \in X$ such that $u = Tu = Su$.

Doric ([4]) generalized the above Theorem 1.2 by using more than one control function to establish the following theorem:

Theorem 1.3 Let (X,d) be a complete metric space and let $T,S : X \to X$ be the two self-mappings such that for all x, y $\in X$

$$\psi(d(Tx,Sy)) \leq \psi(M(x,y)) - \phi(M(x,y)), \quad (5)$$

where

1. $M(x,y) = max\{d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[d(y,Tx)+d(x,Sy)]\}$,
2. $\psi : [0,\infty) \to [0,\infty)$ is continuous monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$,
3. $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$,

Then there exists unique fixed point $u \in X$ such that $u = Tu = Su$.

In 2012 Choudhury and Kundu ([3]) proved some coincidence point and common fixed point theorems in a partial order metric spaces using (ψ_1,ψ_2,ϕ)-weak contraction condition. Akbar and Choudhury ([1]) also proved some results using (ψ_1,ψ_2,ϕ)-weak contraction in partial order metric spaces.
In this paper, we prove some theorems using quadratic (ψ_1,ψ_2,ϕ)- weak contractive condition and produce an example to support our results.

2 Main Results

Theorem 2.1. Let (X,d) be a complete metric space. Let $T,S : X \to X$ be two self-mappings such that for all $x,y \in X$

$$\psi_1(d^2(Tx,Sy)) \leq \psi_2(M(x,y)) - \phi(M(x,y)) \quad (6)$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Sy), d(x,Tx).d(y,Sy), d(x,Sy).d(y,Tx), d(x,Tx).d(x,Sy), \frac{d(y,Sy).d(y,Tx)}{2}\}$,
2. $\psi_1,\psi_2 : [0,\infty) \to [0,\infty)$ is continuous monotone non-decreasing functions,
3. $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$, $\phi(t) > 0$ for all $t \in (0,\infty)$,
4. satisfying $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for $t > 0$.

Then there exists unique fixed point $u \in X$ such that $u = Tu = Su$.

Proof. For an arbitrary $x_0 \in X$ such that $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$, $x_4 = Tx_3$... and so on. In general, for all $n \in N\cup\{0\}$ we can construct a sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}.$$

At first we shall assume $x_n \neq x_{n+1}$, for all $n \geq 0$.
Putting that $x = x_n$ and $y = x_{n-1}$ in (6), we have

$$\psi_1(d^2(x_{n+1},x_n)) \leq \psi_2(M(x_n,x_{n-1})) - \phi(M(x_n,x_{n-1})), \quad (7)$$

where
$M(x,y) = max\{d^2(x_n,x_{n-1}), d^2(x_n,x_{n+1}), d^2(x_{n-1},x_n),$

$d(x_n,x_{n+1}).d(x_{n-1},x_n), d(x_n,x_n).d(x_{n-1},x_{n+1}),$

$d(x_n,x_{n+1}).d(x_n,x_n), \frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{2}\}.$
If

$$d(x_n,x_{n+1}) > d(x_n,x_{n-1}) \quad (8)$$

then

$$M(x_n,x_{n-1}) = d^2(x_n,x_{n+1}). \quad (9)$$

By using (7) and (9), we have

$$\psi(d^2(x_{n+1},x_n)) \leq \psi(d^2(x_n,x_{n+1})) - \phi(d^2(x_n,x_{n+1})),$$

a contradiction. Hence for all $n \geq 0$,

$$d(x_n,x_{n+1}) \leq d(x_n,x_{n-1}). \quad (10)$$

Using (10), we get

$$M(x_n,x_{n-1}) = d^2(x_n,x_{n-1}). \quad (11)$$

Therefore for all $n \geq 0$. The sequence $\{d(x_n,x_{n+1}\}$ is monotone decreasing and bounded. So, we have

$$lim_{n\to\infty} d(x_n,x_{n+1}) = r \geq 0.$$

Using (7) and (11) , we then obtain

$$\psi_1(d^2(x_n,x_{n+1})) \le \psi_2(d^2(x_n,x_{n-1})) - \phi(d^2(x_n,x_{n-1})).$$

On letting $n \to \infty$, we have

$$\psi_1(r^2) \le \psi_2(r^2) - \phi(r^2),$$

a contradiction by the property of the ϕ function. This implies that $r = 0$.
Hence

$$lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{12}$$

Next, we shall prove that $\{x_n\}$ is a cauchy sequence. For this, it is enough to prove that subsequence $\{x_{2n}\}$ is a Cauchy sequence. Then $\exists$ an $\varepsilon > 0$ for which we can find subsequence $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ such that $n(k)$ is smallest positive integer for which $n(k) > m(k) > k$,

$$d(x_{2m(k)},x_{2n(k)}) \ge \varepsilon \tag{13}$$

is satisfied. Then we have

$$d(x_{2m(k)},x_{2n(k)-1}) < \varepsilon. \tag{14}$$

By triangle inequality, we have

$$d(x_{2m(k)},x_{2n(k)}) \le d(x_{2m(k)},x_{2n(k)-1}) + d(x_{2n(k)-1},x_{2n(k)}).$$

Taking $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2m(k)},x_{2n(k)}) = \varepsilon. \tag{15}$$

Again

$$d(x_{2n(k)-1},x_{2m(k)}) \le d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)},x_{2m(k)}),$$

$$d(x_{2n(k)},x_{2m(k)}) \le d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)-1},x_{2m(k)}).$$

By taking the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2n(k)-1},x_{2m(k)}) = \varepsilon. \tag{16}$$

Again

$$\begin{aligned} d(x_{2n(k)-1},x_{2m(k)-1}) \le d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)},\\ x_{2m(k)}) + d(x_{2m(k)},x_{2m(k)-1}),\end{aligned}$$

$$\begin{aligned} d(x_{2n(k)},x_{2m(k)}) \le d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)-1},\\ x_{2m(k)-1}) + d(x_{2m(k)-1},x_{2m(k)}).\end{aligned}$$

Again taking the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2n(k)-1},x_{2m(k)-1}) = \varepsilon. \tag{17}$$

Again for all positive integer k $\in$ N,

$$d(x_{2m(k)-1},x_{2n(k)}) \le d(x_{2m(k)-1},x_{2m(k)}) + d(x_{2n(k)},x_{2m(k)}),$$

$$d(x_{2n(k)},x_{2m(k)}) \le d(x_{2m(k)},x_{2m(k)-1}) + d(x_{2m(k)-1},x_{2n(k)}).$$

On letting $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2m(k)-1},x_{2n(k)}) = \varepsilon. \tag{18}$$

By putting $x = x_{2n(k)-1}$ and $y = x_{2m(k)-1}$ in (6), we get

$$\begin{aligned}\psi_1(d^2(x_{2n(k)},x_{2m(k)})) \le\ & \psi_2(M(x_{2n(k)-1},x_{2m(k)-1})) \\ & - \phi(M(x_{2n(k)-1},x_{2m(k)-1}))\end{aligned} \tag{19}$$

where
$M(x_{2n(k)-1},x_{2m(k)-1}) = max\{d^2(x_{2n(k)-1},x_{2m(k)-1}),$

$d^2(x_{2n(k)-1},x_{2n(k)}), d^2(x_{2m(k)-1},x_{2m(k)}), d(x_{2n(k)-1},x_{2n(k)}).$

$d(x_{2m(k)-1},x_{2m(k)}), d(x_{2n(k)-1},x_{2m(k)}).d(x_{2m(k)-1},x_{2n(k)}),$
$d(x_{2n(k)-1},x_{2n(k)}).d(x_{2n(k)-1},x_{2m(k)}),$
$\frac{d(x_{2m(k)-1},x_{2m(k)}).d(x_{2m(k)-1},x_{2n(k)})}{2}\}.$

On letting $k \to \infty$, we get

$$\begin{aligned} lim_{n\to\infty} M(x_{2n(k)-1},x_{2m(k)-1}) &= max\{\varepsilon^2,0,0,0,\varepsilon^2,0,0\} \\ &= \varepsilon^2.\end{aligned} \tag{20}$$

Using (20) and (19), we shall obtain

$$\psi_1(\varepsilon^2) \le \psi_2(\varepsilon^2) - \phi(\varepsilon^2),$$

which is a contradiction. Hence we have shown that $\{x_n\}$ be a Cauchy sequence in X. Since X is a complete metric space so that $\exists$ $u \in X$ such that $x_n \to u$ as $n \to \infty$ consequently the subsequences, Sx_{2n} and $Tx_{2n+1} \to u$ as $n \to \infty$.

To prove that z is the fixed point of T and S.
First we shall assume that, let $d(u,Su) \ne 0$. By putting $x = x_{2n+1}$ and $y = u$ in (6), we get

$$\psi_1(d^2(Tx_{2n+1},Su)) \le \psi_2(M(x_{2n+1},u)) - \phi(M(x_{2n+1},u)),$$

On letting $n \to \infty$, we get

$$\begin{aligned}\psi_1(d^2(u,Su)) \le\ & \psi_2(lim_{n\to\infty} M(x_{2n+1},u)) \\ & -\phi(lim_{n\to\infty} M(x_{2n+1},u)),\end{aligned} \tag{21}$$

where
$M(x_{2n+1},u) = max\{d^2(x_{2n+1},u), d^2(x_{2n+1},x_{2n+2}),$

$d^2(u,Su), d(x_{2n+1},x_{2n+2}).d(u,Su), d(x_{2n+1},Su).d(u,x_{2n+2}),$
$d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su), \frac{d(u,Su).d(u.Tx_{2n+1})}{2}\}.$

By taking the limit $n \to \infty$, we get

$$M(u,u) = max\{0,0,d^2(u,Su),0,0,0\} = d^2(u,Su). \tag{22}$$

So, we have

$$\psi_1(d^2(u,Su)) \le \psi_2(d^2(u,Su)) - \phi(d^2(u,Su)),$$

which is a contradiction. Hence $d^2(u,Su) = 0 \Rightarrow u = Su$. Similarly, by putting $x = u$, $y = x_{2n}$ in (6), we get $d^2(Tu,u) = 0 \Rightarrow Tu = u$. Hence u is common fixed point of S and T.

To prove the uniqueness: Assume w is the second common fixed point of S and T i.e. $Tw = w$ and $Sw = w$, we let $d(u,w) \neq 0$. By putting $x = u$ and $y = w$ in (6), we get

$$\psi_1(d^2(Tu,Sw)) \leq \psi_2(M(u,w)) - \phi(M(u,w)), \tag{23}$$

where

$$M(u,w) = d^2(u,w). \tag{24}$$

We have

$$\psi_1(d^2(u,w)) \leq \psi_2(d^2(u,w)) - \phi(d^2(u,w)),$$

a contradiction in terms of the function ϕ . Therefore $d^2(u,w) = 0 \Rightarrow u = w$. Hence T and S has a unique common fixed point in X.

If we put $S = T$ inTheorem 2.1 we shall obtain the following theorem:

Theorem 2.2. Let (X,d) be a complete metric space. Let $T : X \to X$ be a self-mapping such that for all $x,y \in X$

$$\psi_1(d^2(Tx,Ty)) \leq \psi_2(M(x,y)) - \phi(M(x,y)), \tag{25}$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Ty),$
$d(x,Tx).d(y,Ty), d(x,Ty).d(y,Tx), d(x,Tx).d(x,Ty), \frac{d(y,Ty).d(y,Tx)}{2}\}$,
2. $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ is continuous monotone non-decreasing functions,
3. $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$, $\phi(t) > 0$ for all $t \in (0,\infty)$,
4. satisfying $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for $t > 0$.

Then there exists unique fixed point $u \in X$ such that $u = Tu$.

Again if we put $\psi_1(t_1) = t_1$ and $\psi_2(t_2) = t_2$ in above Theorem 2.1 and Theorem 2.2 respectively, we have the followings corollaries:

Corollary 2.3. Let (X,d) be a complete metric space. Let $T,S : X \to X$ are two self-mappings such that for all $x,y \in X$

$$d^2(Tx,Sy) \leq M(x,y) - \phi(M(x,y)), \tag{26}$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Sy),$
$d(x,Tx).d(y,Sy), d(x,Sy).d(y,Tx), d(x,Tx).d(x,Sy), \frac{d(y,Sy).d(y,Tx)}{2}\}$,
2. $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$, $\phi(t) > 0$ for all $t \in (0,\infty)$.

Then there exists unique fixed point $u \in X$ such that $u = Tu = Su$.

Corollary 2.4. Let (X,d) be a complete metric space . Let $T : X \to X$ be a self-mapping such that for all $x,y \in X$

$$d^2(Tx,Sy) \leq M(x,y) - \phi(M(x,y)) \tag{27}$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Ty),$
$d(x,Tx).d(y,Ty), d(x,Ty).d(y,Tx), d(x,Tx).d(x,Ty), \frac{d(y,Ty).d(y,Tx)}{2}\}$, and
2. $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$ for all $t \in (0,\infty)$.

Then there exists unique fixed point $u \in X$ such that $u = Tu$.

Theorem 2.5. Let (X,d) be a complete metric space Let T, S : $X \to X$ be mappings satisfying

$$\psi_1(d^2(Tx,Sy)) \leq \psi_2(M(x,y)) - \phi(N(x,y)), \tag{28}$$

for all x, y $\in X$, where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Sy),$
$d(x,Tx).d(y,Sy), d(x,Sy).d(y,Tx), d(x,Tx).d(x,Sy), \frac{d(y,Sy).d(y,Tx)}{2}\}$
and
$N(x,y) = min\{d^2(x,y), d^2(x,Tx), d^2(y,Sy),$
$d(x,Tx).d(y,Sy), \frac{d(y,Sy).d(y,Tx)}{2}\}$,
2. $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ are continuous monotone non-decreasing functions,
3. $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,
4. satisfying $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for $t > 0$.

Then there exists unique fixed point $u \in X$ such that $u = Tu = Su$.

*Proof.*Let $x_0 \in X$ be an arbitrary point of X and define $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$, $x_4 = Tx_3$........ and so on. In general, for all $n \geq 0$ we can construct a sequence in the following manner

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}. \tag{29}$$

First we shall assume that $n \in N \cup \{0\}$, and for

$$x_n \neq x_{n+1}. \tag{30}$$

Put $x = x_n$ and $y = x_{n-1}$ in (28), we have

$$\psi_1(d^2(x_{n+1},x_n)) \leq \psi_2(M(x_n,x_{n-1})) - \phi(N(x_n,x_{n-1})), \tag{31}$$

where
$M(x,y) = max\{d^2(x_n,x_{n-1}), d^2(x_n,x_{n+1}),$

$d^2(x_{n-1},x_n), d(x_n,x_{n+1}).d(x_{n-1},x_n), d(x_n,x_n).d(x_{n-1},x_{n+1}),$
$d(x_n,x_{n+1}).d(x_n,x_n),$
$\frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{2}\}$
and
$N(x,y) = min\{d^2(x_n,x_{n-1}), d^2(x_n,x_{n+1}), d^2(x_{n-1},x_n),$
$d(x_n,x_{n+1}).d(x_{n-1},x_n), \frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{2}\}$.
If

$$d(x_n,x_{n+1}) > d(x_n,x_{n-1}) \tag{32}$$

then

$$M(x_n,x_{n-1}) = d^2(x_n,x_{n+1}). \tag{33}$$

By the virtue of (30), we have $N(x_n,x_{n-1}) > 0$.By using (28), and (33) , we have

$$\psi_1(d^2(x_n,x_{n+1})) \leq \psi_2(d^2(x_n,x_{n+1})) - \phi(N(x_n,x_{n-1})),$$

a contradiction. Hence for all $n \in N \cup \{0\}$,

$$d(x_n,x_{n+1}) \leq d(x_n,x_{n-1}). \tag{34}$$

Using (28) and (34), we get

$$M(x_n,x_{n-1}) = d^2(x_n,x_{n-1}) \tag{35}$$

$$N(x_n,x_{n-1}) = d^2(x_n,x_{n+1}). \tag{36}$$

Thus, the sequence $\{d(x_n,x_{n+1})\}$ is monotone decreasing sequence of non negative real numbers, there exists a number$r \geq 0$ such that

$$lim_{n\to\infty} d(x_n,x_{n+1}) = r \geq 0.$$

By using (28), (35) and (36), we can write

$$\psi_1(d^2(x_n,x_{n+1})) \leq \psi_2(d^2(x_n,x_{n-1})) - \phi(d^2(x_n,x_{n-1})).$$

Taking the limit $n \to \infty$ in the above inequality , we get

$$\psi_1(r^2) \leq \psi_2(r^2) - \phi(r^2),$$

which is a contradiction. Then $\forall n \in N \cup \{0\}$ we have

$$lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{37}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient if we prove that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. Then $\exists\ \varepsilon > 0$ for which we can find subsequence $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ such that $n(k)$ is smallest positive integer for which $n(k) > m(k) > k$,

$$d(x_{2m(k)},x_{2n(k)}) \geq \varepsilon \tag{38}$$

is satisfied. Then we have

$$d(x_{2m(k)},x_{2n(k)-1}) < \varepsilon. \tag{39}$$

Using triangle inequality, we have

$$d(x_{2m(k)},x_{2n(k)}) \leq d(x_{2m(k)},x_{2n(k)-1}) + d(x_{2n(k)-1},x_{2n(k)}).$$

Letting the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2m(k)},x_{2n(k)}) = \varepsilon. \tag{40}$$

Again for all $k \in N$,

$$d(x_{2n(k)-1},x_{2m(k)}) \leq d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)},x_{2m(k)}),$$

and

$$d(x_{2n(k)},x_{2m(k)}) \leq d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)-1},x_{2m(k)}).$$

Letting the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2n(k)-1},x_{2m(k)}) = \varepsilon. \tag{41}$$

Now again,

$$d(x_{2n(k)-1},x_{2m(k)-1}) \leq d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)},x_{2m(k)-1}),$$

$$d(x_{2n(k)},x_{2m(k)}) \leq d(x_{2n(k)-1},x_{2n(k)}) + d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-1},x_{2m(k)}).$$

Letting $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2n(k)-1},x_{2m(k)-1}) = \varepsilon. \tag{42}$$

Again for all positive integer $k \in N$, we have

$$d(x_{2m(k)-1},x_{2n(k)}) \leq d(x_{2m(k)-1},x_{2m(k)}) + d(x_{2n(k)},x_{2m(k)}),$$

and

$$d(x_{2n(k)},x_{2m(k)}) \leq d(x_{2m(k)},x_{2m(k)-1}) + d(x_{2m(k)-1},x_{2n(k)}).$$

Letting the limit $k \to \infty$, we get

$$lim_{k\to\infty} d(x_{2m(k)-1},x_{2n(k)}) = \varepsilon. \tag{43}$$

Putting $x = x_{2n(k)-1}$ and $y = x_{2m(k)-1}$ in (28) for all $k \in N$, we have

$$\psi_1(d^2(x_{2n(k)},x_{2m(k)})) \leq \psi_2(M(x_{2n(k)-1},x_{2m(k)-1})) - \phi(M(x_{2n(k)-1},x_{2m(k)-1})) \tag{44}$$

where
$M(x_{2n(k)-1},x_{2m(k)-1}) = max\{d^2(x_{2n(k)-1},x_{2m(k)-1}),$

$d^2(x_{2n(k)-1},x_{2n(k)}), d^2(x_{2m(k)-1},x_{2m(k)}) d(x_{2n(k)-1},x_{2n(k)}).$

$d(x_{2m(k)-1},x_{2m(k)}), d(x_{2n(k)-1},x_{2m(k)}).d(x_{2m(k)-1},x_{2n(k)}),$
$d(x_{2n(k)-1},x_{2n(k)}).d(x_{2n(k)-1},x_{2m(k)}),$
$\frac{d(x_{2m(k)-1},x_{2m(k)}).d(x_{2m(k)-1},x_{2n(k)})}{2}\}$ and
$N(x_{2n(k)-1},x_{2m(k)-1}) = min\{d^2(x_{2n(k)-1},x_{2m(k)-1}),$

$d^2(x_{2n(k)-1},x_{2n(k)}), d^2(x_{2m(k)-1},x_{2m(k)}) d(x_{2n(k)-1},x_{2n(k)}).$

$d(x_{2m(k)-1},x_{2m(k)}), \frac{d(x_{2m(k)-1},x_{2m(k)}).d(x_{2m(k)-1},x_{2n(k)})}{2}\}.$

Taking the limit $k \to \infty$, we get

$$lim_{k\to\infty} M(x_{2n(k)-1},x_{2m(k)-1}) = max\{\varepsilon^2,0,0,0,\varepsilon^2,0,0\} = \varepsilon^2, \tag{45}$$

$$lim_{k\to\infty} N(x_{2n(k)-1},x_{2m(k)-1}) = min\{\varepsilon^2,0,0,0,0\} = 0. \tag{46}$$

On taking limit $k \to \infty$in (43) and using (44) and (45) we shall obtain

$$\psi_1(\varepsilon^2) \leq \psi_2(\varepsilon^2) - lim_{k\to\infty}\phi(N(x_{m(k)-1}, x_{n(k)-1})).$$

The fact that ϕ has a discontinuous at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term of the right hand side of the above inequality is non zero, so we arrive at a contradiction. Hence $\{x_n\}$ is Cauchy sequence in complete metric space X, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$ we can also write,

$$Sx_{2n} \to u \text{ and } Tx_{2n+1} \to u \text{ as } n \to \infty$$

To prove that z is the fixed point of T and S.

For this, let $d(u,Su) \neq 0$ and putting $x = x_{2n+1}$ and $y = u$ in (28), we get

$$\psi_1(d^2(Tx_{2n}, Su)) \leq \psi_2(M(x_{2n+1}, u)) - \phi(N(x_{2n+1}, u)), \tag{47}$$

where
$M(x_{2n+1},u) = max\{d^2(x_{2n+1},u), d^2(x_{2n+1},x_{2n+2}),$

$d^2(u,Su), d(x_{2n+1},x_{2n+2}).d(u,Su), d(x_{2n+1},Su).d(u,x_{2n+2}),$
$d(x_{2n+1},x_{2n+2}).d(x_{2n+1},Su), \frac{d(u,Su).d(u.Tx_{2n+1})}{2}\}$
and
$N(x_{2n+1},u) = min\{d^2(x_{2n+1},u), d^2(x_{2n+1},x_{2n+2}),$

$d^2(u,Su), d(x_{2n+1},x_{2n+2}).d(u,Su), \frac{d(u,Su).d(u.Tx_{2n+1})}{2}\}.$

On applying limit as $n \to \infty$ in the above conditions, we shall obtain

$$lim_{n\to\infty}M(x_{2n+1},u) = max\{0,0,d^2(u,Su),0,0,0\} = d^2(u,Su)$$

and

$$lim_{n\to\infty}N(x_{2n+1},u) = min\{0,0,d^2(u,Su),0\} = d^2(u,Su).$$

Letting limit $n \to \infty$ in (47), we have

$$\psi_1(d^2(u,Su)) \leq \psi_2(d^2(u,Su)) - lim_{k\to\infty}\phi(N(x_{2n+1},u)),$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term of the right hand side of the above inequality in non zero, Therefore we obtain,

$$\psi_1(d^2(u,Su)) \leq \psi_2(d^2(u,Su)) - \phi(d^2(u,Su)),$$

a contradiction. Hence $d^2(u,Su) = 0 \Rightarrow u = Su$. Similarly by putting $x = u$, $y = x_{2n}$ in (28), and arguing as above, we get, $d^2(Tu,u) = 0 \Rightarrow Tu = u$. Hence u is common fixed point of S and T.

For uniqueness:- Let w is the another fixed point of S and T such that $Tw = w$ and $Sw = w$ and let $d(u,w) \neq 0$, putting $x = u$ and $y = w$ in (28), we get

$$\psi(d^2(u,w)) \leq \psi(d^2(u,w))\phi(d^2(u,w)),$$

a contradiction of the assumption, so, we have $d^2(u,w) = 0 \Rightarrow u = w$.

If we put $S = T$ in Theorem 2.5, we have the following theorem:

Theorem 2.6. Let (X,d) be a complete metric space. Let $T : X \to X$ be a map such that for all $x,y \in X$

$$\psi_1(d^2(Tx,Ty)) \leq \psi_2(M(x,y)) - \phi(N(x,y)), \tag{48}$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Ty),$

   $d(x,Tx).d(y,Ty), d(x,Ty).d(y,Tx), d(x,Tx).d(x,Ty), \frac{d(y,Ty).d(y,Tx)}{2}\}$
   and
   $N(x,y) = min\{d^2(x,y), d^2(x,Tx), d^2(y,Ty),$
   $d(x,Tx).d(y,Ty), \frac{d(y,Ty).d(y,Tx)}{2}\},$
2. $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ are continuous monotone non-decreasing functions,
3. $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,
4. satisfying $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for $t > 0$.

Then there exists a unique fixed point $u \in X$ such that $u = Tu$.

If we put $\psi_1(t) = t_1$ and $\psi_2(t) = t_2$ in above Theorem 2.8 and Theorem 2.9 we get the followings corollaries:

Corollary 2.7. Let (X,d) be a complete metric space and let T, S : $X \to X$ be the two self-mappings such that for all x, y $\in X$

$$d^2(Tx,Sy) \leq M(x,y) - \phi(N(x,y)), \tag{49}$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Sy),$

   $d(x,Tx).d(y,Sy), d(x,Sy).d(y,Tx), d(x,Tx).d(x,Sy), \frac{d(y,Sy).d(y,Tx)}{2}\}$
   and
   $N(x,y) = min\{d^2(x,y), d^2(x,Tx), d^2(y,Sy),$
   $d(x,Tx).d(y,Sy), \frac{d(y,Sy).d(y,Tx)}{2}\},$

2. $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$,

Then there exists a unique fixed point $u \in X$ such that $u = Tu = Su$.

Corollary 2.8. Let (X,d) be a complete metric space. Let $T : X \to X$ be a map such that for all x, y $\in X$

$$d^2(Tx,Ty) \leq M(x,y) - \phi(N(x,y)), \tag{50}$$

where

1. $M(x,y) = max\{d^2(x,y), d^2(x,Tx), d^2(y,Ty),$
$d(x,Tx).d(y,Ty), d(x,Ty).d(y,Tx), d(x,Tx).d(x,Ty), \frac{d(y,Ty).d(y,Tx)}{2}\}$
and
$N(x,y) = min\{d^2(x,y), d^2(x,Tx), d^2(y,Ty),$
$d(x,Tx).d(y,Ty), \frac{d(y,Ty).d(y,Tx)}{2}\}$.
2. $\phi : [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ and lower semi-continuous for all $t > 0$, ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$.

Then there exists a unique fixed point $u \in X$ such that $u = Tu$.

Example 2.9. Let $X = \{0,1,2\}$ and define

$$d(x,y) = \begin{cases} x+y : \text{if } x \neq y \\ 0 \quad : \text{if } x = y \end{cases}$$

is a complete metric space. Define $T,S : X \to X$ such that

$$T0 = 0,\ T1 = 0,\ T2 = 1,$$

$$S0 = 0,\ S1 = 2,\ S2 = 0$$

Let $\psi_1(t) = t$ for all $t \geq 0$ and

$$\psi_2(t) = \begin{cases} 2t & : \text{if } 0 \leq t \leq 1 \\ t + \frac{1}{t} & : \text{if } t > 1 \end{cases} \quad and \quad \phi(t) = \begin{cases} 1 : \text{if } t > 0 \\ 0 : \text{if t =0} \end{cases}$$

We can see $\psi_1(t) - \psi_2(t) + \phi(t) > 0$ for $t > 0$. Now we have to verify that the following inequality of Theorem 2.5 for the following cases.

Case 1: if $x = 0$ and $y = 0$ then $\psi_1(d^2(Tx,Sy)) = 0$, $M(x,y) = 0$ and $N(x,y) = 0$. So, $\psi(M(x,y) - \phi(N(x,y) = \psi_1(d^2(Tx,Sy))$.

Case 2: if $x = 2$ and $y = 1$ then $\psi_1(d^2(Tx,Sy)) = 9$, $M(x,y) = 12$ and $N(x,y) = 3$. So, $\psi_2(M(x,y) - \phi(N(x,y) = 133/12 > 9 = \psi_1(d^2(Tx,Sy))$.

Case 3: if $x = 1$ and $y = 2$ then $\psi_1(d^2(Tx,Sy)) = 0$, $M(x,y) = 9$ and $N(x,y) = 1$. So, $\psi_2(M(x,y) - \phi(N(x,y) = 73/9 > 0 = \psi_1(d^2(Tx,Sy))$.

Case 4: if $x = 2$ and $y = 0$ then $\psi_1(d^2(Tx,Sy)) = 1$, $M(x,y) = 9$ and $N(x,y) = 0$. So, $\psi_2(M(x,y) - \phi(N(x,y) = 82/9 > 1 = \psi_1(d^2(Tx,Sy))$.

Case 5: if $x = 0$ and $y = 2$ then $\psi_1(d^2(Tx,Sy)) = 0$, $M(x,y) = 4$ and $N(x,y) = 0$. So, $\psi_2(M(x,y) - \phi(N(x,y) = 17/4 > 0 = \psi_1(d^2(Tx,Sy))$.

Case 6: if $x = 1$ and $y = 0$ then $\psi_1(d^2(Tx,Sy)) = 0$, $M(x,y) = 1$ and $N(x,y) = 0$. So, $\psi_2(M(x,y) - \phi(M(x,y) = 2 > 0 = \psi_1(d^2(Tx,Sy))$.

Case 7: if $x = 2$ and $y = 2$ then $\psi_1(d^2(Tx,Sy)) = 1$, $M(x,y) = 16$ and $N(x,y) = 3$. So, $\psi_2(M(x,y) - \phi(M(x,y) = 241/16 > 1 = \psi_1(d^2(Tx,Sy))$.

Case 8: if $x = 1$ and $y = 1$ then $\psi_1(d^2(Tx,Sy)) = 4$, $M(x,y) = 9$ and $N(x,y) = 1$. So, $\psi_2(M(x,y) - \phi(M(x,y) = 73/9 > 4 = \psi_1(d^2(Tx,Sy))$.

Case 9: if $x = 0$ and $y = 1$ then $\psi_1(d^2(Tx,Sy)) = 4$, $M(x,y) = 9$ and $N(x,y) = 0$. So, $\psi_2(M(x,y) - \phi(M(x,y) = 82/9 > 4 = \psi_1(d^2(Tx,Sy))$. Inequality holds for all cases. Hence T and S has a unique common fixed point $x = 0$ in X.

3 Conclusion

The authors have tried to established some unique common fixed point theorems in a complete metric space using (ψ_1, ψ_2, ϕ)- Weak Contraction Conditions. Proved theorems are interesting as they strengthen the results due to : (i) D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc, 20(1969), 458-468.(ii) B. E. Rhoades, Some theorms on weakly contractive maps, Nonlinears Analysis 47 (2001), 2683-2693, (iii) Penumarthy Parvateesam Murthy, Kenan Tas and Uma Devi Patel, Common Fixed Point Theorems for Generalized (ϕ,ψ)-weak contraction condition in Complete Metric Spaces, Journal of Inequalities and Applications (2015) 2015:139. Three control functions were used in place of real non-negative constants used in many fixed point theorems before 2000 in general.

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