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χ-Rough Approximation Spaces

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Abstract: In this paper, we generalized the notions of Pawlak's rough set theory to a topological model where the set approximations are defined using the topological notion χ-open sets. Also, we study some of their basic properties of Pawlak's rough set model. Moreover, several important measures, related to the new model, such as accuracy measure and quality of approximation are presented.

Keywords: Rough set, lower and upper approximations, Pawlak's rough set model, χ-open sets, accuracy measure.

1 Introduction

Rough set theory, introduced by Pawlak [1], is a mathematical approach to deal with vagueness and uncertainty of imprecise data. In this approach, vagueness is expressed by a boundary region of a set. Pawlak rough set theory is an extension of the set theory for study and analyzes various types of data [1–5]. There are many applications of rough set theory especially in artificial intelligence fields such as machine learning, pattern recognition, decision analysis, cognitive sciences, intelligent decision making and process control [6–13]. Some of rough set applications are to approximate an arbitrary an universe by two definable subsets called lower and upper approximations, and to reduce the number of the set of attributes in data sets. Suppose, we are given an information system $S = (U, A)$, where U is a nonempty, finite set of objects and is called the universe and A is a nonempty, finite set of attributes. Set A will contain two disjoint sets of attributes, called condition and decision attributes and the system is denoted by $S = (U, C, D)$ where C is called condition attribute and D is called decision attribute. With every attribute $a \in A$ we associate a set Va, of its values, called the domain of a.

Classical rough set philosophy is based on an assumption that every object in the universe of discourse is associated with some information. Objects characterized by the same information are indiscernible with the available information about them. The indiscernibility relation generated in this way is the mathematical basis for the rough set theory. Classical rough set theory has used successfully in the analysis of data in complete information systems. The indiscernibility relation is reflexive, symmetric and transitive. The set of all indiscernible objects is called an elementary set or equivalent class. Any set of objects, being an union of some elementary sets is referred to as crisp set, otherwise is called rough set. A rough set can be described by a pair of crisp sets, called the lower and upper approximations.

Rough set theory is a recent approach for reasoning about data. This theory depends basically on certain topological structure and has achieved great success in many fields of real life applications. The concept of topological rough set by Wiweger [14] is one of the most important topological generalizations of rough sets. Abu-Donia and Salama [15] discussed generalization of Pawlak's rough approximation spaces by using $\delta\beta$-open sets. M.A. Abd Allah and A.S. Nawar [16] ntroduced the concept of ψ*-open sets. H.M. Abu-Donia, M.A. Abd Allah and A.S. Nawar [17] generalized the notions of Pawlak's rough set theory to a topological model where the set approximations are defined using the topological notion ψ^*-open sets. O. Tantawy, M.A. Abd Allah and A.S. Nawar [18] introduced the concept of χ-open sets.

In rough set theory, the accuracy measure is an important numerical characterization that quantifies the imprecision of a rough set caused by its boundary region. In our study, we reduce the boundary region of a set A in Pawlak's approximation space by χ-boundary of A. Also, we extend exterior of A which contains the elements that don't belong to A by χ-exterior of A.

In this paper, we investigate some important and basic issues of generalized rough sets induced by χ-open sets

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and topological preliminaries. The rest of the paper is organized as follows; Section 2 shows the basic concepts of χ-open sets and rough set theory preliminaries. The main aim of χ-open generalizations is given in Section 3. Conclusions are presented in Section 4.

2 Preliminaries

We shall recall some concepts about some near open sets which are essential for our present study.

Definition 2.1 A subset A of a topological space (X, τ) is called semi-open [19] if $A \subseteq cl(int(A))$ and semi-closed if $int(cl(A)) \subseteq A$. The class of semi-open subsets of (X,τ) is denoted by $SO(X)$. For a subset A of a topological space (X, τ), the semi-closure of A, denoted by $scl(A)$ is the intersection of all semi-closed subsets of X containing A. Dually, the semi-interior of A, denoted by $sint(A)$ is the union of all semi-open subsets of X contained in A.

Definition 2.2 A subset A of a topological space (X, τ) is called χ-open [18] if $A \supseteq U$, $U \in SGC(X) \Rightarrow sint(A) \supseteq U$. The class of χ-open subsets of (X, τ) is denoted by $\chi O(X)$.

Definition 2.3 [18] Let A be a subset of topological space (X, τ), then we have:

(i) The union of all χ-open sets contained in A is called the χ-interior of A and is denoted by $\chi int(A)$.

(ii) The intersection of all χ-closed sets containing A is called the χ-closure of A and is denoted by $\chi cl(A)$.

Motivation for rough set theory has come from the need to represent subsets of an universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X, R)$, where X is a set called the universe and R is an equivalence relation [4]. The equivalence classes of R are also known as the granules, atoms, elementary sets or blocks. We will use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$. In the approximation space $K = (X, R)$, we consider two operators $\overline{R}(A) = \{x \in X : R_x \cap A \neq \phi\}$ and $\underline{R}(A) = \{x \in X : R_x \subseteq A\}$, called the upper approximation and the lower approximation of $A \subseteq X$ respectively. Also let $POS_R(A) = \underline{R}(A)$ denote the positive region of A, $NEG_R(A) = X - \overline{R}(A)$ denote the negative region of A and $BN_R(A) = \overline{R}(A) - \underline{R}(A)$ denote the borderline (boundary) region of A.

The degree of completeness can also be characterized by the accuracy measure, in which $|A|$represents the cardinality of a subset $A \subseteq X$ as follows:

$$\eta_R(A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|}, \text{ where } A \neq \phi.$$

The accuracy measure $\eta_R(A)$ is tried to express the degree of completeness of knowledge. Obviously $0 \leq \eta_R(A) \leq 1$, for every R and $A \subseteq X$; if $\eta_R(A) = 1$, A is crisp with respect to R; if $\eta_R(A) < 1$, A is rough with respect to R.

If the lower and upper approximation are identical (i.e., $\overline{R}(A) = \underline{R}(A)$), then set A is definable, otherwise, set A is undefinable in X. There are four types of undefinable sets in X:

(1) If $\underline{R}(A) \neq \phi$ and $\overline{R}(A) \neq X$, then A is called roughly R-definable,

(2) If $\underline{R}(A) = \phi$ and $\overline{R}(A) \neq X$, then A is called internally R-undefinable,

(3) If $\underline{R}(A) \neq \phi$ and $\overline{R}(A) = X$, then A is called externally R-undefinable,

(4) If $\underline{R}(A) = \phi$ and $\overline{R}(A) = X$, then A is called totally R-undefinable.

We denote the set of all roughly R-definable (resp. internally R-undefinable, externally R-undefinable and totally R-undefinable) sets by $RD(X)$ (resp. $IUD(X)$, $EUD(X)$ and $TUD(X)$).

3 Generalizations of χ-open sets to χ-rough sets

We introduce the following definitions:

Definition 3.1 Let X be a finite non-empty universe. The pair (X, R_{semi}) is called a *semi*-approximation space where R_{semi} is a general binary relation used to get a subbase for a topology τ on X.

Definition 3.2 Let (X, R_{semi}) be a *semi*-approximation space then *semi*-lower (resp *semi*-upper) approximation of any non-empty subset A of X is defined as:

(i) $\underline{R}_{semi}(A) = \cup\{G \in SO(X) : G \subseteq A\}$,

(ii) $\overline{R}_{semi}(A) = \cap\{F \in SC(X) : F \supseteq A\}$.

Definition 3.3 Let (X, R_{semi}) be a *semi*-approximation space and $A \subseteq X$. Then there are memberships $\underline{\in}$, $\overline{\in}$, $\underline{\in}_{semi}$ and $\overline{\in}_{semi}$, say, strong, weak, *semi*-strong and *semi*-weak memberships respectively which are defined as follows:

(1) $x \underline{\in} A$ iff $x \in \underline{R}(A)$,

(2) $x \overline{\in} A$ iff $x \in \overline{R}(A)$,

(3) $x \underline{\in}_{semi} A$ iff $x \in \underline{R}_{semi}(A)$,

(4) $x \overline{\in}_{semi} A$ iff $x \in \overline{R}_{semi}(A)$.

Definition 3.4 Let (X, R_{semi}) be a *semi*-approximation space and $A \subseteq X$. The *semi*-accuracy measure of A defined as follows:

$$\eta_{R_{semi}}(A) = \frac{|\underline{R}_{semi}(A)|}{|\overline{R}_{semi}(A)|} \text{ where } A \neq \phi.$$

Definition 3.5 Let (X, R_{semi}) be a *semi*-approximation space, the subset $A \subseteq X$ is called:

(1) Roughly R_{semi}-definable, if $\underline{R}_{semi}(A) \neq \phi$ and $\overline{R}_{semi}(A) \neq X$,

(2) Internally R_{semi}-undefinable, if $\underline{R}_{semi}(A) = \phi$ and $\overline{R}_{semi}(A) \neq X$,

(3) Externally R_{semi}-undefinable, if $\underline{R}_{semi}(A) \neq \phi$ and $\overline{R}_{semi}(A) = X$,

(4) Totally R_{semi}-undefinable, if $\underline{R}_{semi}(A) = \phi$ and $\overline{R}_{semi}(A) = X$.

We denote the set of all roughly R_{semi}-definable (resp. internally R_{semi}-undefinable, externally R_{semi}-undefinable and totally R_{semi}-undefinable) sets by $SRD(X)$ (resp. $SIUD(X)$, $SEUD(X)$ and $STUD(X)$).

In this section, we generalize and investigate the concept of *semi*-approximation space to χ-approximation space. Also, we introduce the concepts of χ-lower approximation and χ-upper approximation and study their properties.

Definition 3.6 Let X be a finite non-empty universe. The pair (X, R_χ) is called a χ-approximation space where R_χ is a general binary relation used to get a subbase for a topology τ on X which generates the class $\chi O(X)$ of all χ-open sets.

Example 3.1 Let $X = \{a,b,c,d\}$ be an universe and $R = \{(a,c),(a,d),(b,c),(b,d),(c,c),(c,d),(d,c),(d,d)\}$ is a binary relation defined on X thus aR = bR = cR = dR = {c, d}. Then the topology associated with this relation is τ={X, ϕ,{c,d}} and $\chi O(X)$={X,ϕ,{c},{d},{c,d},{a,c,d},{b,c,d}}. So (X, R_χ) is a χ-approximation space.

Definition 3.7 Let (X, R_χ) be a χ-approximation space then χ-lower (resp χ-upper) approximation of any non-empty subset A of X is defined as:

(i) $\underline{R}_\chi(A) = \cup\{G \in \chi O(X) : G \subseteq A\}$,

(ii) $\overline{R}_\chi(A) = \cap\{F \in \chi C(X) : F \supseteq A\}$.

Theorem 3.1 For any topological space (X, τ) generated by a binary relation R on X, we have, $\underline{R}(A) \subseteq \underline{R}_{semi}(A) \subseteq \underline{R}_\chi(A) \subseteq A \subseteq \overline{R}_\chi(A) \subseteq \overline{R}_{semi}(A) \subseteq \overline{R}(A)$.

Proof

$$\underline{R}(A) = \cup\{G \in \tau : G \subseteq A\} \subseteq \cup\{G \in SO(X) : G \subseteq A\}$$
$$= \underline{R}_{semi}(A) \subseteq \cup\{G \in \chi O(X) : G \subseteq A\} = \underline{R}_\chi(A) \subseteq A,$$
$i.e.$,$\underline{R}(A) \subseteq \underline{R}_{semi}(A) \subseteq \underline{R}_\chi(A) \subseteq A$. Also,
$$\overline{R}(A) = \cap\{F \in \tau^c : F \supseteq A\} \supseteq \cap\{F \in SC(X) : F \supseteq A\}$$
$$= \overline{R}_{semi}(A) \supseteq \cap\{F \in \chi C(X) : F \supseteq A\} = \overline{R}_\chi(A) \supseteq A,$$
$i.e.$,$\overline{R}(A) \supseteq \overline{R}_{semi}(A) \supseteq \overline{R}_\chi(A) \supseteq A$. Consequently,
$$\underline{R}(A) \subseteq \underline{R}_{semi}(A) \subseteq \underline{R}_\chi(A) \subseteq A \subseteq \overline{R}_\chi(A) \subseteq \overline{R}_{semi}(A) \subseteq \overline{R}(A). \blacksquare$$

Definition 3.8 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. According to the relation $int(A) \subseteq sint(A) \subseteq \chi int(A) \subseteq A \subseteq \chi cl(A) \subseteq scl(A) \subseteq cl(A)$, the universe X can be divided into 24 regions with respect to any $A \subseteq X$ as follows:

(1) The internal edge of A, $\underline{\mathrm{Edg}}(A) = A - \underline{R}(A)$,

(2) The *semi*-internal edge of A, $S\underline{\mathrm{Edg}}(A) = A - \underline{R}_{semi}(A)$,

(3) The χ-internal edge of A, $\chi\underline{\mathrm{Edg}}(A) = A - \underline{R}_\chi(A)$,

(4) The external edge of A, $\overline{\mathrm{Edg}}(A) = \overline{R}(A) - A$,

(5) The *semi*-external edge of A, $S\overline{\mathrm{Edg}}(A) = \overline{R}_{semi}(A) - A$,

(6) The χ-external edge of A, $\chi\overline{\mathrm{Edg}}(A) = \overline{R}_\chi(A) - A$,

(7) The boundary of A, $b(A) = \overline{R}(A) - \underline{R}(A)$,

(8) The *semi*-boundary of A, $Sb(A) = \overline{R}_{semi}(A) - \underline{R}_{semi}(A)$,

(9) The χ-boundary of A, $\chi b(A) = \overline{R}_\chi(A) - \underline{R}_\chi(A)$,

(10) The exterior of A, $ext(A) = X - \overline{R}(A)$,

(11) The *semi*-exterior of A, $Sext(A) = X - \overline{R}_{semi}(A)$,

(12) The χ-exterior of A, $\chi ext(A) = X - \overline{R}_\chi(A)$,

(13) $\overline{R}(A) - \underline{R}_{semi}(A)$,

(14) $\overline{R}(A) - \underline{R}_\chi(A)$,

(15) $\overline{R}(A) - \overline{R}_{semi}(A)$,

(16) $\overline{R}(A) - \overline{R}_\chi(A)$,

(17) $\overline{R}_{semi}(A) - \underline{R}(A)$,

(18) $\overline{R}_{semi}(A) - \underline{R}_\chi(A)$,

(19) $\overline{R}_{semi}(A) - \overline{R}_\chi(A)$,

(20) $\overline{R}_\chi(A) - \underline{R}(A)$,

(21) $\overline{R}_\chi(A) - \underline{R}_{semi}(A)$,

(22) $\underline{R}_{semi}(A) - \underline{R}(A)$,

(23) $\underline{R}_\chi(A) - \underline{R}(A)$,

(24) $\underline{R}_\chi(A) - \underline{R}_{semi}(A)$.

Remark 3.1 The study of χ-approximation space is a generalization for the study of approximation spaces (Graph 3.1).

The elements of the regions $[\underline{R}_\chi(A) - \underline{R}(A)]$ will be defined well in A, while those elements were undefinable in Pawlak's approximation spaces. Also, the elements of the region $[\overline{R}(A) - \overline{R}_\chi(A)]$ do not belong to A, while these elements were not well defined in Pawlak's approximation spaces.

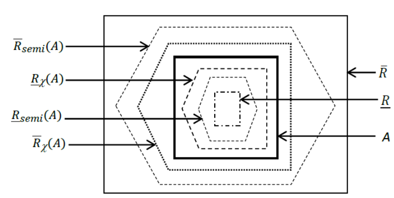

Fig. 1: Graph 3.1

In our study, we reduce the boundary region of A in Pawlak's approximation space by χ-boundary of A. Also, we extend exterior of A which contains the elements that don't belong to A by χ-exterior of A.

Proposition 3.1 For any χ-approximation space (X, R_χ) the following hold for any $A \subseteq X$:

(1) $b(A) = \overline{\text{Edg}}(A) \cup \underline{\text{Edg}}(A)$,

(2) $\chi b(A) = \chi\overline{\text{Edg}}(A) \cup \chi\underline{\text{Edg}}(A)$.

Proof. (2) It follows from

$$\chi b(A) = \overline{R}_\chi(A) - \underline{R}_\chi(A) = (\overline{R}_\chi(A) - A) \cup (A - \underline{R}_\chi(A)) = \chi\overline{\text{Edg}}(A) \cup \chi\underline{\text{Edg}}(A). \blacksquare$$

Proposition 3.2 For any χ-approximation space (X, R_χ) the following hold for any $A \subseteq X$:

(1) $\overline{R}(A) - \underline{R}_\chi(A) = \overline{\text{Edg}}(A) \cup \chi\underline{\text{Edg}}(A)$,

(2) $\overline{R}_\chi(A) - \underline{R}(A) = \chi\overline{\text{Edg}}(A) \cup \underline{\text{Edg}}(A)$.

Proof.

(1) $\overline{R}(A) - \underline{R}_\chi(A) = (\overline{R}(A) - A) \cup (A - \underline{R}_\chi(A)) = \overline{\text{Edg}}(A) \cup \chi\underline{\text{Edg}}(A)$.

(2) $\overline{R}_\chi(A) - \underline{R}(A) = (\overline{R}_\chi(A) - A) \cup (A - \underline{R}(A)) = \chi\overline{\text{Edg}}(A) \cup \underline{\text{Edg}}(A). \blacksquare$

Proposition 3.3 For any χ-approximation space (X, R_χ) the following hold for any $A \subseteq X$:

(1) $\underline{\text{Edg}}(A) = \chi\underline{\text{Edg}}(A) \cup (\underline{R}_\chi(A) - \underline{R}(A))$,

(2) $\overline{\text{Edg}}(A) = \chi\overline{\text{Edg}}(A) \cup (\overline{R}(A) - \overline{R}_\chi(A))$.

Proof. Obvious.$\blacksquare$

Definition 3.9 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then there are memberships $\underline{\in}_\chi$ and $\overline{\in}_\chi$, say, χ-strong and χ-weak memberships respectively which are defined as follows:

(1) $x \,\underline{\in}_\chi\, A$ iff $x \in \underline{R}_\chi(A)$,

(2) $x \,\overline{\in}_\chi\, A$ iff $x \in \overline{R}_\chi(A)$.

Remark 3.2 According to Definition 3.9. χ-lower and χ-upper approximations of a subset $A \subseteq X$ can be written as:

(1) $\underline{R}_\chi(A) = \{x \in A : x \,\underline{\in}_\chi\, A\}$

(2) $\overline{R}_\chi(A) = \{x \in A : x \,\overline{\in}_\chi\, A\}$

Remark 3.3 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then we have:

(1) $x \,\underline{\in}\, A \Rightarrow x \,\underline{\in}_{semi}\, A \Rightarrow x \,\underline{\in}_\chi\, A$,

(2) $x \,\overline{\in}_\chi\, A \Rightarrow x \,\overline{\in}_{semi}\, A \Rightarrow x \,\overline{\in}\, A$.

The converse of Remark 3.3 may not be true in general as seen in the following example.

Example 3.2 In Example 3.1. Let A = {c}, we have $c \,\underline{\in}_\chi\, A$ but $c \,\underline{\notin}_{semi}\, A$. Let A = {b, c, d}, we have $b \,\underline{\in}_{semi}\, A$ but $b \,\underline{\notin}\, A$. Let A = {b}, we have $a \,\overline{\in}\, A$ but $a \,\overline{\notin}_{semi}\, A$. Let A = {a, c}, we have $d \,\overline{\in}_{semi}\, A$ but $d \,\overline{\notin}_\chi\, A$.

Definition 3.10 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. The χ-accuracy measure of A defined as follows:

$$\eta_{R_\chi}(A) = \frac{|\underline{R}_\chi(A)|}{|\overline{R}_\chi(A)|}, \text{ where } A \neq \phi.$$

Example 3.3 Let $X = \{a,b,c,d,e\}$ be an universe and $R = \{(a,a),(b,c),(b,d),(c,b),(c,c),(c,d),(c,e),(d,b),(d,c),(d,d),(d,e),(e,b),(e,c),(e,d),(e,e)\}$ is a binary relation defined on X thus aR = {a}, bR = {c, d} and cR = dR = eR = {b, c, d, e}. Then the topology associated with this relation is τ = {X, ϕ,{a},{c,d},{a,c,d},{b,c,d,e}} and $\chi O(X)$ = { X, ϕ,{a},{c},{d},{a,c},{a,d},{c,d},{a,c,d}, {b,c,d},{c,d,e},{a,b,c,d},{a,c,d,e},{b,c,d,e}}. So (X, R_χ) is a χ-approximation space. In this example, we can deduce the following table showing the degree of accuracy measure $\eta_R(A)$, *semi*-accuracy measure $\eta_{R_{semi}}(A)$ and χ-accuracy measure $\eta_{R_\chi}(A)$ for some subsets of X.

Table 3.1 Comparison between some type of accuracy measures and χ-accuracy measure.

$A \subseteq X$	$\eta_R(A)$	$\eta_{R_{semi}}(A)$	$\eta_{R_\chi}(A)$
{a, b}	1/3	1/2	1/2
{a, c}	1/5	1/5	1/2
{b, c}	0	0	1/3
{b, d}	0	0	1/3
{c, d}	1/2	1/2	1/2
{d, e}	0	0	1/3
{a, b, c}	1/5	1/5	1/2
{a, b, e}	1/3	1/3	1/3
{b, c, d}	1/2	3/4	3/4
{c, d, e}	1/2	3/4	3/4
{a, b, c, d}	3/5	4/5	4/5
{a, b, d, e}	1/5	1/5	1/2

We see from Table 3.1 that the degree of exactness of the subset A = {b, c, d} by using accuracy measure equal to 50%, by using *semi*-accuracy measure equal to 75%. Also, the subset A = {a, b, c} by using *semi*-accuracy measure equal to 20% and by using χ-accuracy measure equal to 50%. Consequently χ-accuracy measure is better than accuracy and *semi*-accuracy measures in this case.

We investigate χ-rough equality and χ-rough inclusion based on rough equality and rough inclusion which introduced by Pawlak and Novotny in [2, 3].

Definition 3.11 Let (X, R_χ) be a χ-approximation space and A, $B \subseteq X$. Then we say that A and B are:

(i) χ-roughly bottom equals $(A \,\overset{\sim}{\underset{\chi}{\smile}}\, B)$ iff $\underline{R}_\chi(A) = \underline{R}_\chi(B)$,

(ii) χ-roughly top equals $(A \,\tilde{-}_\chi\, B)$ iff $\overline{R}_\chi(A) = \overline{R}_\chi(B)$,

(iii) χ-roughly equals $(A \approx_\chi B)$ iff $(A \,\overset{\sim}{\underset{\chi}{\smile}}\, B)$ and $(A \,\tilde{-}_\chi\, B)$.

Example 3.4 In Example 3.2, the subsets {a} and {a, b} are χ-roughly bottom equal, but {a,b,d,e} and {a,c,d,e} are χ-roughly top equal.

One can easily show that $\approx_\chi$ is an equivalence relation on $P(X)$ (Power set of X), hence the pair $(P(X), \approx_\chi)$ is an approximation space. The relation $\approx_\chi$ is called a χ-rough equality of the χ-approximation space (X,R_χ).

Definition 3.12 Let (X,R_χ) be a χ-approximation space. We define the equivalence relation E_χ on the set $P(X)$ by:

$$(A,B)\in E_\chi \text{ if } \chi-\text{int}(A)=\chi-\text{int}(B) \text{ and } \chi-\text{cl}(A)=\chi-\text{cl}(B).$$

The equivalence relation E_χ is precisely the same as $\approx_\chi$, where $\underline{R}_\chi(A)=\chi-\text{int}(A)$ and $\overline{R}_\chi(A)=\chi\text{cl}(A)$

Remark 3.4 For any subset A of X, the equivalence class of the relation ($\approx_\chi$ or E_χ) containing A is denoted by $[A]_{\approx_\chi}$ or $[A]_{E_\chi}$ and is defined as follows:

$$[A]_{\approx_\chi}=\{D\subset X:\underline{R}_\chi(D)=\underline{R}_\chi(A) \text{ and } \overline{R}_\chi(D)=\overline{R}_\chi(A)\}.$$

Definition 3.13 Let (X,R_χ) be a χ-approximation space and A, $B\subseteq X$. Then we say that:

(i) A is χ-roughly bottom included in B $(A\underset{\sim\chi}{\subset}B)$ iff $\underline{R}_\chi(A)\subseteq\underline{R}_\chi(B)$,

(ii) A is χ-roughly top included in B $(A\tilde{\subset}_\chi B)$ iff $\overline{R}_\chi(A)\subseteq\overline{R}_\chi(B)$,

(iii) A is χ-roughly included in B $(A\underset{\sim\chi}{\tilde{\subset}}B)$ iff $(A\underset{\sim\chi}{\subset}B)$ and $(A\tilde{\subset}_\chi B)$.

Example 3.5 In Example 3.2. {a, c} is χ-roughly bottom included in {a, c, d}. Also {a, b, c, e} is χ-roughly top included in {a, c, d, e}.

In the following definition we introduced a new concept of χ-rough set.

Definition 3.14 For any χ-approximation space (X,R_χ), a subset A of X is called:

(1) $R_\chi-$definable (χ-exact) if $\overline{R}_\chi(A)=\underline{R}_\chi(A)$ or $\chi b(A)=\phi$,

(2) χ-rough if $\overline{R}_\chi(A)\neq\underline{R}_\chi(A)$ or $\chi b(A)\neq\phi$.

Example 3.6 Let (X,R_χ) be a χ-approximation space as in Example 3.2. The set {b,c,d,e} is χ-exact while {a,b,d,e} is χ-rough.

Definition 3.15 Let (X,R_χ) be a χ-approximation space, the subset $A\subseteq X$ is called:

(1) Roughly R_χ-definable, if $\underline{R}_\chi(A)\neq\phi$ and $\overline{R}_\chi(A)\neq X$,

(2) Internally R_χ-undefinable, if $\underline{R}_\chi(A)=\phi$ and $\overline{R}_\chi(A)\neq X$,

(3) Externally R_χ-undefinable, if $\underline{R}_\chi(A)\neq\phi$ and $\overline{R}_\chi(A)=X$,

(4) Totally R_χ-undefinable, if $\underline{R}_\chi(A)=\phi$ and $\overline{R}_\chi(A)=X$.

We denote the set of all roughly R_χ-definable (resp. internally R_χ-undefinable, externally R_χ-undefinable and totally R_χ-undefinable) sets by $\chi RD(X)$ (resp. $\chi IUD(X)$, $\chi EUD(X)$ and $\chi TUD(X)$).

Remark 3.5 For any χ-approximation space (X,R_χ). The following are hold:

(1) $\chi RD(X)\supseteq SRD(X)\supseteq RD(X)$,

(2) $\chi IUD(X)\subseteq SIUD(X)\subseteq IUD(X)$,

(3) $\chi EUD(X)\subseteq SEUD(X)\subseteq EUD(X)$,

(4) $\chi TUD(X)\subseteq STUD(X)\subseteq TUD(X)$.

Lemma 3.1 For any χ-approximation space (X,R_χ), and for all $x,y\in X$, the condition $x\in\overline{R}_\chi(\{y\})$ and $y\in\overline{R}_\chi(\{x\})$ implies $\overline{R}_\chi(\{x\})=\overline{R}_\chi(\{y\})$.

Proof. Since $\chi cl(\{y\})$ is a χ-closed set containing x while $\chi cl(\{x\})$ is the smallest χ-closed set containing x, thus $\chi cl(\{x\})\subseteq\chi cl(\{y\})$. Hence $\overline{R}_\chi(\{x\})\subseteq\overline{R}_\chi(\{y\})$. The opposite inclusion follows by symmetry $\chi cl(\{y\})\subseteq\chi cl(\{x\})$. Hence $\overline{R}_\chi(\{y\})\subseteq\overline{R}_\chi(\{x\})$, which complete the proof. ■

Lemma 3.2 Let (X,R_χ) be a χ-approximation space, which satisfied that, every χ-open subset A of X is χ-closed, then $y\in\overline{R}_\chi(\{x\})$ implies $x\in\overline{R}_\chi(\{y\})$for all $x,y\in X$.

Proof. If $x\notin\overline{R}_\chi(\{y\})$, then there exists a χ-open set G containing x such that $G\cap\{y\}=\phi$ which implies that $\{y\}\subseteq(X\setminus G)$, but $(X\setminus G)$ is a χ-closed set and also is a χ-open set does not containing x, thus $(X\setminus G)\cap\{x\}=\phi$. Hence $y\notin\overline{R}_\chi(\{x\})$.■

Proposition 3.4 Let (X,R_χ) be a χ-approximation space, and every χ-open subset A of X is χ-closed. Then the family of sets $\{\overline{R}_\chi(\{x\}):x\in A\}$ is a partition of the set X.

Proof. If $x,y,z\in A$ and $z\in\overline{R}_\chi(\{x\})\cap\overline{R}_\chi(\{y\})$, then $z\in\overline{R}_\chi(\{x\}))$ and $z\in\overline{R}_\chi(\{y\})$. Thus by Lemma 3.2, $x\in\overline{R}_\chi(\{z\})$ and $y\in\overline{R}_\chi(\{z\})$ and by Lemma 3.1, we have $\overline{R}_\chi(\{x\})=\overline{R}_\chi(\{z\})$ and $\overline{R}_\chi(\{y\})=\overline{R}_\chi(\{z\})$. Therefore $\overline{R}_\chi(\{x\})=\overline{R}_\chi(\{y\})=\overline{R}_\chi(\{z\})$. Hence either $\overline{R}_\chi(\{x\})=\overline{R}_\chi(\{y\})$ or $\overline{R}_\chi(\{x\})\cap\overline{R}_\chi(\{y\})=\phi$. The proof is complete. ■

The following proposition investigates some properties of χ-approximation spaces.

Proposition 3.5 Let (X,R_χ) be a χ-approximation space and A, $B\subseteq X$. Then we have:

(i) $\underline{R}_\chi(A)\subseteq A\subseteq\overline{R}_\chi(A)$,

(ii) $\underline{R}_\chi(\phi)=\overline{R}_\chi(\phi)=\phi$, $\underline{R}_\chi(X)=\overline{R}_\chi(X)=X$,

(iii) If $A\subseteq B$ then, $\underline{R}_\chi(A)\subseteq\underline{R}_\chi(B)$ and $\overline{R}_\chi(A)\subseteq\overline{R}_\chi(B)$.

Proof.

(i) Let $x\in\underline{R}_\chi(A)$ which means that $x\in\cup\{G\in\chi O(X),G\subseteq A\}$. Then there exists $G_0\in\chi O(X)$ such that $x\in G_0\subseteq A$. Thus $x\in A$. Hence $\underline{R}_\chi(A)\subseteq A$. Also, let $x\in A$ and by definition of $\overline{R}_\chi(A)=\cap\{F\in\chi C(X),A\subseteq F\}$, then $x\in F$ for all $F\in\chi\text{C}(X)$. Hence $A\subseteq\overline{R}_\chi(A)$.

(ii) Follows directly.

(iii) Let $x\in\underline{R}_\chi(A)$, by definition of χ-lower approximation of A, we have $x\in\cup\{G\in\chi O(X),\ G\subseteq A\}$ but $A\subseteq B$, thus $G\subseteq B$ and $x\in G$, then $x\in\underline{R}_\chi(B)$. Also, let

$x \notin \overline{R}_\chi(B)$ this means that $x \notin \cap\{F \in \chi C(X), B \subseteq F\}$ then, there exists $F \in \chi\mathrm{C}(X)$, $B \subseteq F$ and $x \notin F$ which means that, there exists $F \in \chi\mathrm{C}(X)$, $A \subseteq B \subseteq F$ and $x \notin F$ which implies $x \notin \cap\{F \in \chi C(X), A \subseteq F\}$, thus $x \notin \overline{R}_\chi(A)$. Therefore $\overline{R}_\chi(A) \subseteq \overline{R}_\chi(B)$.

Proposition 3.6 Let (X, R_χ) be a χ-approximation space and A, $B \subseteq X$. Then

(i) $\underline{R}_\chi(X \backslash A) = X \backslash \overline{R}_\chi(A)$,
(ii) $\overline{R}_\chi(X \backslash A) = X \backslash \underline{R}_\chi(A)$,
(iii) $\underline{R}_\chi(\underline{R}_\chi(A)) = \underline{R}_\chi(A)$,
(iv) $\overline{R}_\chi(\overline{R}_\chi(A)) = \overline{R}_\chi(A)$,
(v) $\underline{R}_\chi(\underline{R}_\chi(A)) \subseteq \overline{R}_\chi(\underline{R}_\chi(A))$,
(vi) $\underline{R}_\chi(\overline{R}_\chi(A)) \subseteq \overline{R}_\chi(\overline{R}_\chi(A))$.

Proof.

(i) Let $x \in \underline{R}_\chi(X \backslash A)$ which is equivalent to $x \in \cup\{G \in \chi O(X), G \subseteq X \backslash A\}$. So there exists $G_0 \in \chi O(X)$ such that $x \in G_0 \subseteq X \backslash A$. Then there exists G_0^c such that $A \subset G_0^c$ and $x \notin G_0^c, G_0^c \in \chi C(X)$. Thus, $x \notin \overline{R}_\chi(A)$. So $x \in X \backslash \overline{R}_\chi(A)$. Therefore $\underline{R}_\chi(X \backslash A) = X \backslash \overline{R}_\chi(A)$.

(ii) Similar to (i).

(iii) Since $\underline{R}_\chi(A) = \cup\{G \in \chi O(X), G \subseteq A\}$. This implies

$$\begin{aligned}\text{that } \underline{R}_\chi(\underline{R}_\chi(A)) &= \cup\{G \in \chi O(X), G \subseteq \underline{R}_\chi(A) \subseteq A\} \\ &= \cup\{G \in \chi O(X), G \subseteq A\} = \underline{R}_\chi(A).\end{aligned}$$

(iv) $\overline{R}_\chi(\overline{R}_\chi(A)) = \overline{R}_\chi(X \backslash \underline{R}_\chi(X \backslash A)) = X \backslash \underline{R}_\chi(\underline{R}_\chi(X \backslash A))$. From (i), (ii) and (iii), we get

$$\begin{aligned}\overline{R}_\chi(\overline{R}_\chi(A)) &= X \backslash \underline{R}_\chi(X \backslash A) = X \backslash (X \backslash (\overline{R}_\chi(A))) \\ &= \overline{R}_\chi(A).\end{aligned}$$

(v) Since $\underline{R}_\chi(A) \subseteq \overline{R}_\chi(\underline{R}_\chi(A))$ and by (iii) we have $\underline{R}_\chi(\underline{R}_\chi(A)) = \underline{R}_\chi(A)$, then $\underline{R}_\chi(\underline{R}_\chi(A)) \subseteq \overline{R}_\chi(\underline{R}_\chi(A))$.

(iv) Since $\underline{R}_\chi(\overline{R}_\chi(A)) \subseteq \overline{R}_\chi(A)$ and by (iv), we have $\overline{R}_\chi(\overline{R}_\chi(A)) = \overline{R}_\chi(A)$, then $\underline{R}_\chi(\overline{R}_\chi(A)) \subseteq \overline{R}_\chi(\overline{R}_\chi(A))$.

Proposition 3.7 Let (X, R_χ) be a χ-approximation space and A, $B \subseteq X$. Then

(i) $\underline{R}_\chi(A \cup B) \supseteq \underline{R}_\chi(A) \cup \underline{R}_\chi(B)$,
(ii) $\overline{R}_\chi(A \cup B) = \overline{R}_\chi(A) \cup \overline{R}_\chi(B)$,
(iii) $\underline{R}_\chi(A \cap B) = \underline{R}_\chi(A) \cap \underline{R}_\chi(B)$,
(iv) $\overline{R}_\chi(A \cap B) \subseteq \overline{R}_\chi(A) \cap \overline{R}_\chi(B)$.

Proof.

(i) Since we have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then $\underline{R}_\chi(A) \subseteq \underline{R}_\chi(A \cup B)$ and $\underline{R}_\chi(B) \subseteq \underline{R}_\chi(A \cup B)$ by (iii) in Proposition 3.5, then
$$\underline{R}_\chi(A \cup B) \supseteq \underline{R}_\chi(A) \cup \underline{R}_\chi(B).$$

(ii), (iii) and (iv) Similar to (i). ■

Theorem 3.3 Let (X, R_χ) be a χ-approximation space and A, $B \subseteq X$. Then the following are hold.

(i) $\overline{R}_\chi(\mathrm{cl}(A) \cup B) = \mathrm{cl}(A) \cup \overline{R}_\chi(B)$,
(ii) $\underline{R}_\chi(int(A) \cap B) = int(A) \cap \underline{R}_\chi(B)$.

Proof.

(i) By Proposition 3.5(i) and Proposition 3.7(ii), we have $\mathrm{cl}(A) \subset \overline{R}_\chi(\mathrm{cl}(A))$. Then $\mathrm{cl}(A) \cup \overline{R}_\chi(B) \subset \overline{R}_\chi(\mathrm{cl}(A)) \cup \overline{R}_\chi(B) \subset \overline{R}_\chi(\mathrm{cl}(A) \cup B)$. On the other hand, since $\mathrm{cl}(A) \cup B \subset \mathrm{cl}(A) \cup \overline{R}_\chi(B)$ and the union of a χ-open set and a closed set is χ-closed, then $\overline{R}_\chi(\mathrm{cl}(A) \cup B) \subset \overline{R}_\chi(\mathrm{cl}(A)) \cup \overline{R}_\chi(B) = \mathrm{cl}(A) \cup \overline{R}_\chi(B)$. Therefore, $\overline{R}_\chi(\mathrm{cl}(A) \cup B) = \mathrm{cl}(A) \cup \overline{R}_\chi(B)$.

(ii)Since the intersection of an open set $int(A)$ and a χ-open set $\underline{R}_\chi(B)$ is χ-open, $int(A) \cap \underline{R}_\chi(B) = \underline{R}_\chi(int(A) \cap \underline{R}_\chi(B)) \subset \underline{R}_\chi(int(A) \cap B)$. On the other hand, by using Proposition 3.7 (iii), $\underline{R}_\chi(int(A) \cap B) \subset \underline{R}_\chi(int(A) \cap \underline{R}_\chi(B) \subset int(A) \cap \underline{R}_\chi(B)$. Therefore, $\underline{R}_\chi(int(A) \cap B) = int(A) \cap \underline{R}_\chi(B)$.■

Lemma 3.3 Let (X, R_χ) be a χ-approximation space. Then $(\overline{R}_\chi(A))^c = \underline{R}_\chi(A^c)$ for all $A \subseteq X$.

Lemma 3.4 Let (X, R_χ) be a χ-approximation space and A, $B \subseteq X$. If A is open, then $A \cap \overline{R}_\chi(B) \subseteq \overline{R}_\chi(A \cap B)$.

Proposition 3.8 Let (X, R_χ) be a χ-approximation space and A, $B \subseteq X$. Then

(1) $\underline{R}_\chi(A - B) \subseteq \underline{R}_\chi(A) - \underline{R}_\chi(B)$,
(2) $\overline{R}_\chi(A - B) \supseteq \overline{R}_\chi(A) - \overline{R}_\chi(B)$.

Proof.

(1) Since $A - B = A \cap B^c$, then
$$\underline{R}_\chi(A - B) = \underline{R}_\chi(A \cap B^c) = \underline{R}_\chi(A) \cap \underline{R}_\chi(B^c).$$
Thus by Lemma 3.3, we have
$$\begin{aligned}\underline{R}_\chi(A - B) &= \underline{R}_\chi(A) \cap (\overline{R}_\chi(B))^c = \underline{R}_\chi(A) - \overline{R}_\chi(B) \\ &\subseteq \underline{R}_\chi(A) - \underline{R}_\chi(B).\end{aligned}$$
Therefore, $\underline{R}_\chi(A - B) \subseteq \underline{R}_\chi(A) - \underline{R}_\chi(B)$.

(2) Since $\overline{R}_\chi(A) - \overline{R}_\chi(B) = \overline{R}_\chi(A) \cap (\overline{R}_\chi(B))^c$, then by Lemma 3.3, we have $\overline{R}_\chi(A) - \overline{R}_\chi(B) = \overline{R}_\chi(A) \cap \underline{R}_\chi(B^c)$. Hence by Lemma 3.4, we have
$$\begin{aligned}\overline{R}_\chi(A) - \overline{R}_\chi(B) &= \overline{R}_\chi(A) \cap \underline{R}_\chi(B^c) \subseteq \overline{R}_\chi(A \cap \underline{R}_\chi(B^c)) \\ &= \overline{R}_\chi(A \cap (\overline{R}_\chi(B))^c) = \overline{R}_\chi(A - \overline{R}_\chi(B))\end{aligned}$$
thus, $\overline{R}_\chi(A - B) \supseteq \overline{R}_\chi(A) - \overline{R}_\chi(B)$.■

Definition 3.16 Let (X, R_χ) be a χ- approximation space, then a subset A of X is said to be χ-dense (resp. χ-Co-dense) if $\overline{R}_\chi(A) = X$ (resp.$\underline{R}_\chi(A) = \phi$).

Definition 3.17 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then A is called simply residual (resp. simply nowhere dense) if $\overline{R}_\chi(A^c) = X$ or $\underline{R}_\chi(A) = \phi$ (resp. if $\underline{R}_\chi(\overline{R}_\chi(A)) = \phi$).

Proposition 3.9 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. If A is simply nowhere dense, then $A \subseteq \overline{R}_\chi(\overline{R}_\chi(A))^c$.

Proof. Since A is simply nowhere dense, then $\underline{R}_\chi(\overline{R}_\chi(A)) = \phi$. By taken the complement for both sides. Then we have $(\underline{R}_\chi(\overline{R}_\chi(A)))^c = X \supseteq A$.Therefore, $A \subseteq \overline{R}_\chi(\overline{R}_\chi(A))^c$.■

Proposition 3.10 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then the sets $A \cap \overline{R}_\chi(A^c)$ and $A^c \cap \overline{R}_\chi(A)$ are simply residual.

Proof. Since

$$\begin{aligned}\underline{R}_\chi(A \cap \overline{R}_\chi(A^c)) &= \underline{R}_\chi(A) \cap \underline{R}_\chi(\overline{R}_\chi(A^c)) \\ &= \underline{R}_\chi(A) \cap \overline{R}_\chi(A^c) = \underline{R}_\chi(A) \cap (\underline{R}_\chi(A))^c = \phi.\end{aligned}$$

Thus $\underline{R}_\chi(A \cap \overline{R}_\chi(A^c)) = \phi$ and hence $A \cap \overline{R}_\chi(A^c)$ is simply residual. similarly,

$$\begin{aligned}\underline{R}_\chi(A^c \cap \overline{R}_\chi(A)) &= \underline{R}_\chi(A^c) \cap \underline{R}_\chi(\overline{R}_\chi(A)) \\ &= (\overline{R}_\chi(A))^c \cap \overline{R}_\chi(A) = \phi.\end{aligned}$$

Thus $\underline{R}_\chi(A^c \cap \overline{R}_\chi(A)) = \phi$ and hence $A^c \cap \overline{R}_\chi(A)$ is simply residual. ■

Proposition 3.11 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then $(A \cap \overline{R}_\chi(A^c)) \cup (A^c \cap \overline{R}_\chi(A)) = \chi b(A)$.

Proof. Since $(A \cap \overline{R}_\chi(A^c)) \cup (A^c \cap \overline{R}_\chi(A))$
$= [((A \cap \overline{R}_\chi(A^c)) \cup A^c)] \cap [(A \cap \overline{R}_\chi(A^c)) \cup \overline{R}_\chi(A)]$
$= [(A \cup A^c) \cap (A^c \cup \overline{R}_\chi(A^c))] \cap [(A \cup \overline{R}_\chi(A)) \cap (\overline{R}_\chi(A^c)$
$\cup \overline{R}_\chi(A))] = [X \cap \overline{R}_\chi(A^c)] \cap [\overline{R}_\chi(A) \cap X] = \overline{R}_\chi(A^c) \cap \overline{R}_\chi(A)$
$= \overline{R}_\chi(A) \cap (\underline{R}_\chi(A))^c = \overline{R}_\chi(A) - \underline{R}_\chi(A)$. Then $(A \cap \overline{R}_\chi(A^c))$
$\cup (A^c \cap \overline{R}_\chi(A)) = \overline{R}_\chi(A) - \underline{R}_\chi(A) = \chi b(A)$.■

Proposition 3.12 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then the boundary of simply open set is simply nowhere dense.

Proof. Let A be a simply open set. Then
$A = \overline{R}_\chi(A) = \underline{R}_\chi(A)$. Hence

$$\begin{aligned}\underline{R}_\chi(\overline{R}_\chi(\chi b(A))) &= \underline{R}_\chi[\overline{R}_\chi(\overline{R}_\chi(A) \cap \overline{R}_\chi(A^c))] \\ &= \overline{R}_\chi(\overline{R}_\chi(A) \cap \overline{R}_\chi(A^c)) \\ &\subseteq \overline{R}_\chi(\overline{R}_\chi(A)) \cap \overline{R}_\chi(\overline{R}_\chi(A^c)) \\ &= \overline{R}_\chi(A) \cap \overline{R}_\chi(A^c) = \phi.\end{aligned}$$

Let A be a simply open set. Then $A = \overline{R}_\chi(A) = \underline{R}_\chi(A)$. Hence

$$\begin{aligned}\underline{R}_\chi(\overline{R}_\chi(\chi b(A))) &= \underline{R}_\chi[\overline{R}_\chi(\overline{R}_\chi(A) - \underline{R}_\chi(A))] \\ &= \overline{R}_\chi(\overline{R}_\chi(A) - \underline{R}_\chi(A)) \\ &\supseteq \overline{R}_\chi(\overline{R}_\chi(A)) - \overline{R}_\chi(\underline{R}_\chi(A)) \\ &= \overline{R}_\chi(A) - \underline{R}_\chi(A) = \phi.\end{aligned}$$

Thus $\underline{R}_\chi(\overline{R}_\chi(\chi b(A))) = \phi$. Then the boundary of simply open set is simply nowhere dense. ■

Proposition 3.13 Let (X, R_χ) be a χ-approximation space and $A \subseteq X$. Then A is simply open set if and only if $\chi b(A) = \phi$.

Proof. Let A be a simply open set. Then $A = \overline{R}_\chi(A) = \underline{R}_\chi(A)$. Therefore $\chi b(A) = \phi$. Conversely, if $\chi b(A) = \phi$. Therefore $A = \overline{R}_\chi(A) = \underline{R}_\chi(A)$. Thus A is χ-exact set and hence A is simply open set. ■

We introduce the following example to show the importance of χ-open sets.

Example 3.7 Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be five amino acids (AAs). The (AAs) are described in terms of five attributes:

a_1 = Relative mutability, a_2 = Partition energy, a_3 = Polarity, a_4 = A periodic indices for beta-proteins, and a_5 = A parameter of charge transfer capability (cf. [20]). Table 3.2 shows all quantitative attributes of five AAs.

Table 3.2 Quantitative attributes of five amino acids.

	a_1	a_2	a_3	a_4	a_5
x_1	100	0.1	0	1.1	0
x_2	20	-1.42	1.48	1.05	0
x_3	106	0.78	49.7	1.41	1
x_4	102	0.83	49.9	1.4	1
x_5	41	-2.12	0.35	0.6	0

Table 3.3 Right neighborhood of five reflexive relations.

K	x_kR_1	x_kR_2	x_kR_3	x_kR_4	x_kR_5
x_1	$\{x_1,x_3,x_4\}$	$\{x_1,x_3,x_4\}$	X	$\{x_1,x_2,x_3,x_4\}$	X
x_2	X	$\{x_1,x_2,x_3,x_4\}$	X	$\{x_1,x_2,x_3,x_4\}$	X
x_3	$\{x_1,x_3,x_4\}$	$\{x_3,x_4\}$	$\{x_3,x_4\}$	$\{x_3,x_4\}$	$\{x_3,x_4\}$
x_4	$\{x_1,x_3,x_4\}$	$\{x_3,x_4\}$	$\{x_3,x_4\}$	$\{x_3,x_4\}$	$\{x_3,x_4\}$
x_5	$\{x_1,x_3,x_4,x_5\}$	X	X	X	X

We consider five reflexive relations on X defined as follow: $R_k = \left\{(x_i, x_j) \in X \times X : x_i(a_k) - x_j(a_k) < \frac{\sigma_k}{3}, i, j, k = 1, 2, \cdots, 5\right\}$ Where σ_k represents the standard deviation of the quantitative attributes a_k, k = 1, 2, 3, 4, 5. The right neighborhoods for all elements of $X = \{x_1, x_2, x_3, x_4, x_5\}$ with respect to the relations R_k, k = 1, 2, 3, 4, 5 are shown in Table 3.3.

We find the intersection of all right neighborhoods of all elements $k = 1, 2, 3, 4, 5$ as the following:

$x_1R = \bigcap_{k=1}^{5}(x_1R_k) = \{x_1, x_3, x_4\}$,

$x_2R = \bigcap_{k=1}^{5}(x_2R_k) = \{x_1, x_2, x_3, x_4\}$,

$x_3R = \bigcap_{k=1}^{5}(x_3R_k) = \{x_3, x_4\}$,

$x_4R = \bigcap_{k=1}^{5}(x_4R_k) = \{x_3, x_4\}$, and

$x_5R = \bigcap_{k=1}^{5}(x_5R_k) = \{x_1, x_3, x_4, x_5\}$.

Consider $\{x_1R,\ x_2R,\ x_3R,\ x_4R,\ x_5R\}$ as a base for a topology τ on X, then we have

$\tau = \{ X,\ \phi,\ \{x_3,x_4\},\ \{x_1,x_3,x_4\},\ \{x_1,x_2,x_3,x_4\},$ $\{x_1,x_3,x_4,x_5\}\}$, and $\chi O(X) = \{ X,\ \phi,\ \{x_3\},\ \{x_4\},$ $\{x_3,x_4\}, \{x_1,x_3,x_4\}, \{x_2,x_3,x_4\}, \{x_3,x_4,x_5\}, \{x_1,x_2,x_3,x_4\},$ $\{x_1,x_3,x_4,x_5\}, \{x_2,x_3,x_4,x_5\}\}$.

For any concept $A \subseteq X$ (collection of Amino Acid), this concept is determined by $int_\tau(A)$ and $cl_\tau(A)$ which defines its boundary. The accuracy increases by the decreases of the boundary region. Clearly the accuracy measure by using the suggested class of χ-open sets in general is greater than the accuracy measure by using any near open sets.

4 Conclusion

In this paper, we used the class of χ-open sets to introduce a new type of approximations named χ-approximation space. Also, by using χ-approximation we can obtain 24 dissimilar granules of the universe of discourse. The class of χ-open sets used in our approach is the largest granulation based on *semi*-open sets in topological spaces. This made the accuracy measures is higher than the use of any type of near open sets such as, *semi*-open sets. Some important properties of the classical Pawlak's rough sets are generalized. Also, we defined the concept of rough membership function using χ-open sets. It is a generalization of classical rough membership function of Pawlak rough sets. The generalized rough membership function can be used to analyze which decision should be made according to a conditional attribute in decision information system.

The difference between our approach and the original approach is the use of the classes resulted from the general relation without any conditions as a sub-base for a general topological structure which has rich results compared with the quasi discrete topology of Pawlak in which every open sets is closed and is limited in applying recent near topological concepts in the approximation process.

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