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The Cesáro Lacunary Ideal Double Sequence χ^2- of $\phi-$ Statistical Defined by a Musielak-Orlicz Function

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Abstract: In this paper we introduce some definitions which are the natural combination of the definition of asymptotic equivalence, statistical convergence, $\phi-$statistical convergence of Musielak Orlicz function and ideal. In addition, we introduce asymptotically ideal equivalent of double sequences and Musielak fuzzy real numbers and established some relations related to this concept. Finally we introduce the notion of Cesáro Orlicz asymptotically equivalent sequences of Musielak Orlicz function and establish their relationship with other classes.

Keywords: Analytic sequence, Museialk-Orlicz function, double sequences, chi sequence, Lambda, Riesz space, strongly, statistical convergent

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy et al., [6]- [17] , Turkmenoglu [18], Raj [19]- [25] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, \ldots) .$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, \ldots \right\}, \quad (1.1)$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m, n \in \mathbb{N}$,

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$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . \\ . \\ . \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.

2 Definitions and Preliminaries

Definition 2.1. [see [26,28]]An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy Δ_2- condition for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u)\,, u \ge 0$.

Lemma 2.2. [see [26,28]] Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \ge \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

Definition 2.3. [see [27]] Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \le m$. A real valued function $d_p(x_1,\ldots,x_n) = \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,
(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1,0),\ldots,\alpha d_n(x_n,0))\|_p = |\alpha|\,\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p,\ \alpha \in \mathbb{R}$
(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)) = (d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p)^{1/p}$ for $1 \le p < \infty$; (or)
(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)) := \sup\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\}$,
for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn},0))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0) \\ . \\ . \\ . \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Definition 2.4. Two non-negative sequences $x = (x_{mn})$ and $y = (y_{mn})$ are asymptotically equivalent 0 if

$$lim_{mn}\frac{x_{mn}}{y_{mn}} = 0$$

and it is denoted by $x \equiv 0$.

Definition 2.5. Let K be the subset of $\mathbb{N}\times\mathbb{N}$, the set of natural numbers. Then the asymptotically density of K, denoted by $\delta(K)$, is defined as

$$\delta(K) = lim_{k,\ell}\frac{1}{k\ell}\left|\{m,n \le k,\ell : m,n \in K\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set.

Definition 2.6. A number sequence $x = (x_{mn})$ is said to be statistically convergent to the number 0 if for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{m \le k, n \le \ell : (m+n)!\,|x_{mn} - 0|^{1/m+n} \ge \varepsilon\right\}$$

has asympototic density zero

$$lim_{k\ell}\frac{1}{k\ell}\left|\left\{m \le k, n \le \ell : ((m+n)!\,|x_{mn} - 0|)^{1/m+n} \ge \varepsilon\right\}\right| = 0.$$

In this case we write $St - limx = 0$.

Definition 2.7. The two non-negative double sequences $x = (x_{mn})$ and $y = (y_{mn})$ are said to be asymptotically double equivalent of multiple L provided that for every $\varepsilon > 0$,

$$lim_{k,\ell}\frac{1}{k,\ell}\left|\left\{(m,n) : m \le k, n \le \ell, \left|\frac{x_{mn}}{y_{mn}} - L\right| \ge \varepsilon\right\}\right| = 0.$$

and simply asymptotically double statistical equivalent if $L = 1$. Furthermore, let $S^L_{\theta_{rs}}$ denote the set of all sequences $x = (x_{mn})$ and $y = (y_{mn})$ such that x *is asymptotically double equivalent to* y.

Definition 2.8. Let $\theta_{rs} = \{(m_r,n_s)\}$ be a double lacunary sequence; the two double sequences $x = (x_{mn})$ and $y = (y_{mn})$ are said to be asymptotically double lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(m,n) \in I_{r,s} : \left|\frac{x_{mn}}{y_{mn}} - L\right| \ge \varepsilon\right\}\right| = 0$$

and simply asymptotically double lacunary statistical equivalent if $L = 1$.Furthermore, let $S^L_{\theta_{rs}}$ denote the set of all sequences $x = (x_{mn})$ and $y = (y_{mn})$ such that
x *is asymptotically double lacunary equivalent to* y.

Definition 2.9. Let $\theta_{rs} = \{(m_r,n_s)\}$ be a double lacunary sequence; the two double sequences $x = (x_{mn})$ and $y = (y_{mn})$ are said to be strong asymptotically double lacunary equivalent of multiple L provided that

$$lim_{r,s}\frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}}\left|\frac{x_{mn}}{y_{mn}}-L\right|=0,$$

that is x is equivalent to y and it is denoted by $N_{\theta_{rs}}^{L}$ and simply strong asymptotically double lacunary equivalent if $L=1$. In addition, let $N_{\theta_{rs}}^{L}$ denote the set of all sequences $x=(x_{mn})$ and $y=(y_{mn})$ such that *x is asymptotically double lacunary equivalent to y.*

Definition 2.10. The double sequence $\theta_{rs}=\{(m_r,n_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that $m_\circ=0, h_r=m_r-m_{r-1}\to\infty$ as $r\to\infty$ and $n_\circ=0, h_s=n_s-n_{s-1}\to\infty$ as $s\to\infty$.

Notations: $m_{r,s}=m_r m_s, h_{r,s}=h_r\overline{h}_s$ and θ_{rs} is determined by $I_{rs}=\{(m,n): m_{r-1}<m\le m_r \, and \, n_{s-1}<n\le n_s\}, q_r=\frac{m_r}{m_{r-1}}, \overline{q}_s=\frac{n_s}{n_{s-1}}$ and $q_{rs}=q_r\overline{q}_s$.

Definition 2.11. Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $P(\sigma)$ the sequence $\{P_{ab}(\sigma)\}$ such that $P_{ab}(\sigma)=1$ for $a,b\in\sigma$ and $P_{ab}(\sigma)=0$ otherwise. Further

$$P_{rs}=\{\sigma\in P:\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}P_{ab}(\sigma)\le r,s\}$$

that is P_{rs} is the set of those σ whose support has cardinality at most r,s and we get

$$\Phi=\left\{\phi=(\phi_{ab}):0<\phi_{11}\le\phi_{ab}\le\phi_{a+1,b+1}\ a,b\ \phi_{a+1,b+1}\le(a+1,b+1)\phi_{ab}\right\}.$$

We define

$$\tau_{rs}=\frac{1}{\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}.$$

Now we define the following definitions:

Definition 2.12. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two non-negative double sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be $\phi-$summable to 0 that is

$$\left[\chi_M^2,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]= lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}$$

$$\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]=0\right\}.$$

In this we write $\chi^2\to0$ and the set of all strongly $\phi-$summable sequences is denoted by $[\phi]$.

Definition 2.13. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative sequences $x=(x_{mn})$ and $y=(y_{mn})$ and let $E\subseteq\mathbb{N}\times\mathbb{N}$ is said to be the $\phi-$density of E.

$$\delta_\phi(E)=lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\left|\{m,n\in\sigma,\sigma\in P_{rs}:m,n\in E\}\right|.$$

It is clear that $\delta_\phi(E)\le\delta(E)$.

Definition 2.14. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be $\phi-$statistical convergent summable to $0\in\mathbb{R}$ if for each $\varepsilon>0$

$$\lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\left|\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\right\}\right|=0$$

In this we write $\chi^2\to0$ and it is denoted by St_ϕ.

Definition 2.15. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be Cesáro strong $M-$asymptotically double lacunary of multiple 0

$$\left[\chi_M^2,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]= \lim_{a,b\to\infty}\frac{1}{ab}\sum_{m=1}^{a}\sum_{n=1}^{b}\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]=0\right\}.$$

denoted by $(x_{mn})\cong(y_{mn})$ and simply Cesáro Orlicz asymptotically equivalent.

Definition 2.16. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be Cesáro strong $M-$asymptotically double lacunary $I-$ of multiple 0, provided that for every $\delta>0$

$$\left(a,b\in\mathbb{N}:\sum_{m=1}^{a}\sum_{n=1}^{b}\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\ge\delta\right\}\right)\in I.$$

Simply Cesáro asymptotically $I-$equivalent.

Definition 2.17. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two non-negative double sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be double lacunary ideal $\phi-$ of multiple 0, provided that

$$\lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}\left\{\left[M\left(((m+n)!\,|x_{mn},0|)^{1/m+n},\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right]=0\right\}.$$

Simply Cesáro asymptotically $\phi-$equivalent.

$$\left\{\lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}\left\{\left[M\left(((m+n)!\,|x_{mn},0|)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\delta\right\}\right\}\in I.$$

Simply Cesáro asymptotically $I-\phi-$equivalent.

Definition 2.18. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be asymptotically double lacunary $\phi-$ of multiple $0\in\mathbb{R}$, provided that for every $\varepsilon>0$

$$\lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\left|\left\{m,n\in\sigma,\sigma\in P_{rs}:\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|=0.$$

Simply asymptotically $\phi-$ equivalent.

Definition 2.19. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double non-negative sequences $x=(x_{mn})$ and $y=(y_{mn})$ are said to be asymptotically double lacunary $\phi-$ of multiple $0\in\mathbb{R}$, provided that for every $\varepsilon>0$ and for every $\delta>0$

$$\left\{\lim_{r,s\to\infty}\frac{1}{\phi_{rs}}\left|\left\{m,n\in\sigma,\sigma\in P_{rs}:\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$$

Simply asymptotically $I-\phi-$ equivalent.

3 Main Results

Theorem 3.1. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double sequences $x=(x_{mn})$ and $y=(y_{mn})$ then the two sequences are (a) $I-$equivalent $\Longrightarrow$ $I-$statistically equivalent
(b) $I-$statistically equivalent $\Longrightarrow$ $I-$equivalent, if M is finite.

Proof. Suppose that $I-$equivalent and let $\varepsilon>0$ be given we write

$$\frac{1}{ab}\sum_{m=1}^{a}\sum_{n=1}^{b}\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\right\}\geq$$
$$\frac{M(\varepsilon)}{ab}\left|\left\{m\leq a,n\leq b:\left[\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|.$$

Consequently for any $\gamma>0$, we have

$$\left\{a,b\in\mathbb{N}:\frac{1}{ab}\left|\left\{m\leq a,n\leq b:\left[\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|\geq\frac{\gamma}{M(\varepsilon)}\right\}$$
$$\subseteq\left\{a,b\in\mathbb{N}:\frac{1}{ab}\sum_{m=1}^{a}\sum_{n=1}^{b}\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\right\}\geq\gamma\right\}\in I.$$

Hence $I-$equivalent.

(b) Suppose that M is finite and $I-$statistically equivalent. Since M is finite then there exists a real number $N>0$ such that $sup_t M(t)\leq N$. Moreover for any $\varepsilon>0$ we can write

$$\frac{1}{ab}\sum_{m=1}^{a}\sum_{n=1}^{b}\left\{\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\right\}\leq$$
$$\frac{N}{ab}\left|\left\{m\leq a,n\leq b:\left[\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|+M(\varepsilon).$$

Now applying $\varepsilon\to 0$, then the result follows.

Theorem 3.2. Let M be an sequence of Orlicz functions and $\theta_{rs}=(m_r,n_s)$ be a double lacunary sequence; the two double sequences $x=(x_{mn})$ and $y=(y_{mn})$ then the two sequences and (ϕ_{rs}) be a non-decreasing sequence of positive real numbers such that $\phi_{rs}\to\infty$ as $r,s\to\infty$ and $\phi_{rs}\leq r,s$ for every $r,s\in\mathbb{N}$. Then statistically equivalent $\Longrightarrow$ $\phi-$statistically equivalent.

Proof. By definition of the sequences ϕ_{rs} it follows that $inf_{rs}\frac{rs}{rs-\phi_{rs}}\geq 1$. Then there exists a $t>0$ such that

$$\frac{rs}{\phi_{rs}}\leq\frac{1+t}{t}$$

suppose that two sequences are statistically equivalent then for every $\varepsilon>0$ and sufficiently large r,s we have

$$\frac{1}{\phi_{rs}}\left|\left\{m,n\in\sigma,\sigma\in P_{rs}:\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|=$$
$$\frac{1}{rs}\frac{rs}{\phi_{rs}}\left|\left\{m\leq a,n\leq b:\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n},\left\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right]\geq\varepsilon\right\}\right|-$$

$$\frac{1}{\phi_{rs}}|\{m\in\{1,2,\cdots,r\}-\sigma, n\in\{1,2,\cdots,s\}-\sigma, \sigma\in P_{rs}: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq \varepsilon\}| \leq$$

$$\frac{1+t}{t}\frac{1}{rs}\left|\left\{m\leq r, n\leq s: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq \varepsilon\right\}\right| -$$

$$\frac{1}{\phi_{rs}}|\{m_0\in\{1,2,\cdots,r\}-\sigma, n_0\in\{1,2,\cdots,s\}-\sigma, \sigma\in P_{rs}: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq \varepsilon\}|.$$

This completes the proof.

Theorem 3.3. Let M be an sequence of Orlicz functions and $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double sequences $x = (x_{mn})$ and $y = (y_{mn})$ and $m,n \in \mathbb{Z}$ such that $\phi_{rs} \leq [\phi_{rs}] + mn, sup_{rs}\frac{[\phi_{rs}]+mn}{\phi_{r-1s-1}} < \infty$ Then the two sequences are $\phi-$statistically equivalent $\Longrightarrow$ statistically equivalent.

Proof. If $sup_{rs}\frac{[\phi_{rs}]+mn}{\phi_{r-1s-1}} < \infty$, then there exists $N > 0$ such that $\frac{[\phi_{rs}]+mn}{\phi_{rs}-1} < N$ for all $r,s \geq 1$. Let a,b be an integers such that $\phi_{r-1,s-1} < a,b \leq \phi_{rs}$. Then for every $\varepsilon > 0$ we have

$$\frac{1}{ab}\left|\left\{m\leq a, n\leq b: \left[\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq \varepsilon\right\}\right| \leq$$

$$\frac{1}{ab}\left|\left\{m\leq a, n\leq b: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq M(\varepsilon)\right\}\right| \leq \frac{1}{[\phi_{rs}]+mn}\frac{[\phi_{rs}]+mn}{\phi_{r-1s-1}}$$

$$\left|\left\{m,n\leq \phi_{rs}: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq M(\varepsilon)\right\}\right| \leq \frac{1}{[\phi_{rs}]+mn}\frac{[\phi_{rs}]+mn}{\phi_{r-1s-1}}$$

$$\left|\left\{m,n\in\sigma, \sigma\in P_{|\phi_{rs}|+mn}: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq M(\varepsilon)\right\}\right| \leq \frac{N}{[\phi_{rs}]+mn}$$

$$\left|\left\{m,n\in\sigma, \sigma\in P_{|\phi_{rs}|+mn}: \left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \geq M(\varepsilon)\right\}\right|.$$

Theorem 3.4. Let M be an sequence of Orlicz functions and $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double sequences $x = (x_{mn})$ and $y = (y_{mn})$ and $m,n \in \mathbb{Z}$ then the two sequences are Cesáro equivalent $\Longrightarrow$ $\phi-$equivalent

Proof. From the definition of sequence (ϕ_{rs}) it follows that $inf_{rs}\frac{rs}{rs-\phi_{rs}} \geq 1$. Then there exists $t > 0$ such that

$$\frac{rs}{\phi_{rs}} \leq \frac{1+t}{t}.$$

Then the following relation

$$\frac{1}{\phi_{rs}}\sum_{m\in\sigma}\sum_{n\in\sigma,\sigma\in P_{rs}}\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] =$$

$$\frac{rs}{\phi_{rs}}\frac{1}{rs}\sum_{m=1}^{a}\sum_{n=1}^{b}\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] -$$

$$\frac{1}{\phi_{rs}}\sum_{m\in\{1,2,\cdots r\}\cdots\sigma}\sum_{n\in\{1,2,\cdots s\}\cdots\sigma,\sigma\in P_{rs}}\left[M\left(\left((m+n)!.\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] \leq$$

$$\frac{1+t}{t}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[M\left(\left((m+n)!\left|\frac{x_{mn}}{y_{mn}},0\right|\right)^{1/m+n}, \left\|(d(x_1,0), d(x_2,0),\cdots,d(x_{n-1},0)\|_p\right)\right)\right] -$$

$$\frac{1}{\phi_{rs}} \sum_{m_0 \in \{1,2,\cdots r\} \cdots \sigma} \sum_{n_0 \in \{1,2,\cdots s\} \cdots \sigma, \sigma \in P_{rs}} \left[M \left(\left((m+n)!. \left| \frac{x_{mn}}{y_{mn}}, 0 \right| \right)^{1/m+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right].$$

Since the two sequences are Cesáro equivalent and M is continuous letting $r,s \to \infty$ we get

$$\frac{1}{\phi_{rs}} \sum_{m \in \sigma} \sum_{n \in \sigma, \sigma \in P_{rs}} \left[M \left(\left((m+n)! \left| \frac{x_{mn}}{y_{mn}}, 0 \right| \right)^{1/m+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right] \to 0.$$

Hence two sequences are $\phi-$ equivalent.

Theorem 3.5. Let M be an sequence of Orlicz functions and $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double sequences $x = (x_{mn})$ and $y = (y_{mn})$ and $m,n \in \mathbb{Z}$ then the two sequences are (a) Cesáro equivalent $\Longrightarrow$ statistcally equivalent
(b) If M satisfies the Δ_2- condition and $(x_{mn}) \in \left[\Lambda_M^2, \| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)) \|_p \right]$ such that the two sequences are statistically equivalent $\Longrightarrow$ Cesáro equivalent

Proof. (a) Suppose that two sequences are Cesáro equivalent. Then for every $\varepsilon > 0$ we have

$$\frac{1}{ab} \left| \left\{ m \le a, n \le b : \left[\left(\left((m+n)! \left| \frac{x_{mn}}{y_{mn}}, 0 \right| \right)^{1/m+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right] \ge \varepsilon \right\} \right| \le$$
$$\frac{1}{ab} \left| \left\{ m \le a, n \le b : \left[M \left(\left((m+n)! \left| \frac{x_{mn}}{y_{mn}}, 0 \right| \right)^{1/m+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right] \ge M(\varepsilon) \right\} \right| \le$$
$$\frac{1}{ab} \sum_{m=1}^{a} \sum_{n=1}^{b} \left[M \left(\left((m+n)! \left| \frac{x_{mn}}{y_{mn}}, 0 \right| \right)^{1/m+n}, \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0) \|_p \right) \right) \right].$$

This completes the proof.

Proof. (b) Follows from the same technique of Theorem 3.1 and Theorem 3.4.

Theorem 3.6. Let M be an sequence of Orlicz functions and $\theta_{rs} = (m_r, n_s)$ be a double lacunary sequence; the two double sequences $x = (x_{mn})$ and $y = (y_{mn})$ and $m,n \in \mathbb{Z}$ then the two sequences are (a) $\phi-$ equivalent $\Longrightarrow$ $\phi-$statistcally equivalent
(b) If M satisfies the Δ_2- condition and $(x_{mn}) \in \left[\Lambda_M^2, \| (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)) \|_p \right]$ such that the two sequences are statistically equivalent $\Longrightarrow$ $\phi-$statistically equivalent.
(c) If M satisfies the Δ_2-condition, then

$$\phi\text{-equivalent} \bigcap \left[\Lambda_M^2, \| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0)) \|_p \right] =$$
$$\phi\text{-statistically equivalent} \bigcap \left[\Lambda_M^2, \| (d(x_1,0), d(x_2,0), ..., d(x_{n-1},0)) \|_p \right].$$

Proof. Follows from the same technique of Theorem 3.1 and Theorem 3.5.

4 Conclusion

we introduce the notion of Cesáro Orlicz asymptotically equivalent sequences of Musielak Orlicz function with χ^2 sequence spaces and establish their relationship with other classes.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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