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(p,q)-Baskakov-Kantorovich Operators

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Abstract: In the last decade the applications of quantum calculus in the field of approximation theory is an active area of research. The (p,q)-calculus is further extension of q-calculus, which provides a new direction for researchers. In the present article, we propose the (p,q)-variant of the Baskakov-Kantorovich operators, using (p,q)-integrals. We estimate moments and establish direct results, using linear approximating methods viz. Steklov mean and K-functionals in terms of modulus of continuity. Also, in a weighted space, we obtain a direct estimate.

Keywords: (p,q)-Baskakov operators, Kantorovich variant, direct estimates, Steklov mean, K-functional, modulus of continuity, weighted approximation

1 Introduction

The quantum calculus (q-calculus) in the field of approximation theory was discussed widely in the last two decades. Several generalizations to the q variants were recently presented in the books [5] and [11] related to convergence behaviours of different operators. Quantum calculus has many applications in special functions and many other areas (see [2], [6]). Also, Araci et al. [1] studied on the fermionic p-adic q-integral representation associated with weighted q-Bernstein and q-Genocchi polynomials. Further there is possibility of extension of the q-calculus to post-quantum calculus, namely the (p,q)-calculus. Actually such extension of quantum calculus can not be obtained directly by substitution of q by q/p in q-calculus. The q-calculus may be obtained by substituting $p=1$ in (p,q)-calculus. Sahai and Yadav [15] established some basic properties of (p,q)-calculus based on two parameters. Recently Mursaleen et al. [14] discussed some approximation properties of (p,q)-Bernstein-Stancu operators. Very recently the author [10] defined (p,q)-Szász-Mirakyan-Baskakov operators and established some approximation results. Some basic notations of (p,q)-calculus are mentioned below:

The (p,q)-numbers are defined as

$$\begin{aligned}[n]_{p,q} &:= p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+pq^{n-2}+q^{n-1}\\ &= \frac{p^n-q^n}{p-q}.\end{aligned}$$

Obviously, it may be seen that $[n]_{p,q}=p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ is the q-integer in quantum-calculus given by $[n]_{q/p}=\frac{1-(q/p)^n}{1-(q/p)}$. The (p,q)-factorial is defined by

$$[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q}, n\geq 1, [0]_{p,q}!=1.$$

The (p,q)-binomial coefficient is given by

$$\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, 0\leq k\leq n.$$

The (p,q)-power basis is defined by

$$(x\oplus a)^n_{p,q}=(x+a)(px+qa)(p^2x+q^2a)\cdots(p^{n-1}x+q^{n-1}a).$$

Definition 1.*The (p,q)-derivative of the function f is defined as*

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x}, x\neq 0$$

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As a special case when $p=1$, the (p,q)-derivative reduces to the $q-$derivative. The (p,q)-derivative fulfils the following product rules:

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x)$$
$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x).$$

Definition 2.*Let f be an arbitrary function and a be a real number. The (p,q)-integral of $f(x)$ on $[0,a]$ is defined as*

$$\int_0^a f(x)\,d_{p,q}x=(q-p)\,a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right)\quad \text{if}\quad \left|\frac{p}{q}\right|<1$$

and

$$\int_0^a f(x)\,d_{p,q}x=(p-q)\,a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}a\right)\quad \text{if}\quad \left|\frac{q}{p}\right|<1$$

In the year 2011, Aral and Gupta [3] proposed q-Baskakov operators, which was further extended to Durrmeyer variant in [12] by using q-integral. The (p,q)-analogue of Baskakov operators for $x\in[0,\infty)$ and $0<q<p\le 1$ may be defined as

$$B_{n,p,q}(f,x)=\sum_{k=0}^{\infty}b_{n,k}^{p,q}(x)\,f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \qquad (1)$$

where

$$b_{n,k}^{p,q}(x)=\begin{bmatrix} n+k-1\\ k\end{bmatrix}_{p,q} p^{k+n(n-1)/2}q^{k(k-1)/2}\frac{x^k}{(1\oplus x)_{p,q}^{n+k}}.$$

In case $p=1$, we get the q-Baskakov operators [3], [8]. If $p=q=1$, we get at once the well known Baskakov operators.

Definition 3.*For $x\in[0,\infty), 0<q<p\le 1$ the (p,q)-variant of Baskakov-Kantorovich operators are defined as*

$$K_n^{p,q}(f,x) = [n]_{p,q}\sum_{k=0}^{\infty}b_{n,k}^{p,q}(x)p^{-k}q^k \int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} f(t)d_{p,q}t \qquad (2)$$

where $b_{n,k}^{p,q}(x)$ is as defined in (1). For the special case of the operators (2), one may see [13].

In the present paper, we estimate the recurrence formula for moments of the (p,q)-Baskakov operators. For (p,q)-Baskakov-Kantorovich operators we estimate direct results using linear approximating methods viz. Steklov mean, K-functionals and also obtain approximation estimate in weighted space.

2 Moments

First we estimate the following Lorentz type lemma for (p,q)-Baskakov basis, which will be used in the sequel.

Lemma 1.*For $n,k\ge 0$, we have*

$$x(1+px)D_{p,q}b_{n,k}^{p,q}(x)=\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}-qx\right)\frac{[n]_{p,q}}{qp^{n-1}}b_{n,k}^{p,q}(qx).$$

*Proof.*By simple computation using the definition of (p,q)-derivative, we have

$$D_{p,q}\left(\frac{1}{(1\oplus x)_{p,q}^{n+k}}\right)=-\frac{p\,[n+k]_{p,q}}{(1\oplus px)_{p,q}^{n+k+1}},D_{p,q}x^k=[k]_{p,q}x^{k-1}.$$

Applying product rule

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x),$$

for (p,q)-derivative, we can write

$$\begin{aligned}
&D_{p,q}\left(\frac{x^k}{(1\oplus x)_{p,q}^{n+k}}\right)\\
&=[k]_{p,q}\frac{x^{k-1}}{(1\oplus qx)_{p,q}^{n+k}}-p^{k+1}[n+k]_{p,q}\frac{x^k}{(1\oplus px)_{p,q}^{n+k+1}}\\
&=[k]_{p,q}\frac{x^{k-1}}{(1\oplus qx)_{p,q}^{n+k}}-[n+k]_{p,q}\frac{x^k}{(1+px)p^{n-1}(1\oplus qx)_{p,q}^{n+k}}.
\end{aligned}$$

Thus using $[n+k]_{p,q}=p^n[k]_{p,q}+q^k[n]_{p,q}$, we get

$$\begin{aligned}
&x(1+px)D_{p,q}\left(\frac{x^k}{(1\oplus x)_{p,q}^{n+k}}\right)\\
&=\left[[k]_{p,q}(1+px)p^{n-1}-[n+k]_{p,q}x\right]\frac{x^k}{p^{n-1}(1\oplus qx)_{p,q}^{n+k}}\\
&=\left[[k]_{p,q}-\frac{q^k[n]_{p,q}x}{p^{n-1}}\right]\frac{x^k}{(1\oplus qx)_{p,q}^{n+k}}\\
&=\left[\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}-qx\right]\frac{[n]_{p,q}}{qp^{n-1}}\frac{(qx)^k}{(1\oplus qx)_{p,q}^{n+k}}.
\end{aligned}$$

Therefore, we have

$$x(1+px)D_{p,q}b_{n,k}^{p,q}(x)=\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}-qx\right)\frac{[n]_{p,q}}{qp^{n-1}}b_{n,k}^{p,q}(qx).$$

*Remark.*We may note done here that for the special case $p=q=1$ of the above lemma, we may capture at once the Lorentz type relation of the Baskakov operators, viz.

$$x(1+x)\frac{d}{dx}[b_{n,k}(x)]=(k-nx)b_{n,k}(x),$$

where the Baskakov basis is given by

$$b_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

The moments of (p,q)-Baskakov operators, satisfy the following:

Lemma 2. *If we define*

$$T^{p,q}_{n,m}(x) := B_{n,p,q}(e_m,x) = \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^m,$$

where $e_i = t^i, i = 0,1,2,\cdots$, then for $m \geq 1$, we have the following recurrence relation:

$$\begin{aligned}&[n]_{p,q}T^{p,q}_{n,m+1}(qx)\\ &= qp^{n-1}x(1+px)D_{p,q}[T^{p,q}_{n,m}(x)] + [n]_{p,q}qxT^{p,q}_{n,m}(qx).\end{aligned}$$

In particular, we have

$$B_{n,p,q}(e_0,x) = 1, B_{n,p,q}(e_1,x) = x$$

and

$$B_{n,p,q}(e_2,x) = x^2 + \frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right).$$

Proof. Using Lemma 1, we have

$$\begin{aligned}&qx(1+px)D_{p,q}[T^{p,q}_{n,m}(x)]\\ &= \sum_{k=0}^{\infty} qx(1+px)D_{p,q}b^{p,q}_{n,k}(x)\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^m\\ &= \sum_{k=0}^{\infty}\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}} - qx\right)\frac{[n]_{p,q}}{p^{n-1}}b^{p,q}_{n,k}(qx)\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^m\\ &= \frac{[n]_{p,q}}{p^{n-1}}T^{p,q}_{n,m+1}(qx) - \frac{[n]_{p,q}}{p^{n-1}}qxT^{p,q}_{n,m}(qx).\end{aligned}$$

This completes the proof of the recurrence relation. Obviously (p,q)-calculus may be related with the q-calculus and we may write

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} = p^{k(n-1)}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q/p}$$

and

$$(x\oplus a)^n_{p,q} = p^{n(n-1)/2}(x+a)^n_{q/p}.$$

Using the definition of q-Baskakov operators (see [3], [5]), we get $B_{n,p,q}(e_0,x) = 1$. The other consequences follow from recurrence relation.

Lemma 3. *For $x \in [0,\infty]$, $0 < q < p \leq 1$, we have*

1. $K^{p,q}_n(e_0,x) = 1$
2. $K^{p,q}_n(e_1,x) = \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}$
3. $K^{p,q}_n(e_2,x) = \frac{[n+1]_{p,q}x^2}{[n]_{p,q}q^3p^{2n-2}} + \frac{x}{p^{n-1}q[n]_{p,q}}\left[\frac{1}{q} + \frac{(2p+q)p}{[3]_{p,q}}\right] + \frac{1}{[3]_{p,q}[n]^2_{p,q}}.$

Proof. By (2), using $[k+1]_{p,q} = p^k + q[k]_{p,q}$ and Lemma 2, we have

$$\begin{aligned}K^{p,q}_n(e_0,x) &= [n]_{p,q}\sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)p^{-k}q^k\\ &\quad\int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} d_{p,q}t\\ &= [n]_{p,q}\sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)p^{-k}q^k\left[\frac{[k+1]_{p,q} - q[k]_{p,q}}{q^k[n]_{p,q}}\right]\\ &= B_{n,p,q}(e_0,x) = 1.\end{aligned}$$

By (2), using the identity

$$[k+1]_{p,q} = p^k + q[k]_{p,q} = q^k + p[k]_{p,q}$$

and applying Lemma 2, we have

$$\begin{aligned}&K^{p,q}_n(e_1,x)\\ &= [n]_{p,q}\sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)p^{-k}q^k\frac{1}{[2]_{p,q}}\left[\frac{[k+1]^2_{p,q} - q^2[k]^2_{p,q}}{q^{2k}[n]^2_{p,q}}\right]\\ &= \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)p^{-k}\frac{1}{[2]_{p,q}}\\ &\quad\left[\frac{([k+1]_{p,q} - q[k]_{p,q})([k+1]_{p,q} + q[k]_{p,q})}{q^k[n]_{p,q}}\right]\\ &= \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)p^{-k}\frac{1}{[2]_{p,q}}\left[\frac{p^k(q^k + p[k]_{p,q} + q[k]_{p,q})}{q^k[n]_{p,q}}\right]\\ &= \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)\frac{1}{[2]_{p,q}}\left[\frac{(q^k + [2]_{p,q}[k]_{p,q})}{q^k[n]_{p,q}}\right]\\ &= \frac{1}{[2]_{p,q}[n]_{p,q}}B_{n,p,q}(e_0,x) + \frac{1}{qp^{n-1}}B_{n,p,q}(e_1,x)\\ &= \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}.\end{aligned}$$

Again, using the identity

$$[k+1]_{p,q} = p^k + q[k]_{p,q} = q^k + p[k]_{p,q}$$

and by Lemma 2, we get

$$\begin{aligned}&K^{p,q}_n(e_2,x)\\ &= [n]_{p,q}\sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)p^{-k}q^k\frac{1}{[3]_{p,q}}\left[\frac{[k+1]^3_{p,q} - q^3[k]^3_{p,q}}{q^{3k}[n]^3_{p,q}}\right]\\ &= \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)\frac{1}{[3]_{p,q}}\left[\frac{([k+1]^2_{p,q} + q[k+1]_{p,q}[k]_{p,q} + q^2[k]^2_{p,q})}{q^{2k}[n]^2_{p,q}}\right]\\ &= \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)\frac{1}{[3]_{p,q}}\left[\frac{(p^2+q^2+pq)[k]^2_{p,q} + q^k(2p+q)p[k]_{p,q} + q^{2k}}{q^{2k}[n]^2_{p,q}}\right]\\ &= \sum_{k=0}^{\infty} b^{p,q}_{n,k}(x)\frac{1}{[3]_{p,q}}\left[\frac{[3]_{p,q}[k]^2_{p,q} + q^k(2p+q)p[k]_{p,q} + q^{2k}}{q^{2k}[n]^2_{p,q}}\right]\\ &= \frac{1}{q^2p^{2n-2}}B_{n,p,q}(e_2,x) + \frac{(2p+q)}{qp^{n-2}[3]_{p,q}[n]_{p,q}}B_{n,p,q}(e_1,x)\\ &\quad + \frac{1}{[3]_{p,q}[n]^2_{p,q}}B_{n,p,q}(e_0,x)\\ &= \frac{1}{q^2p^{2n-2}}\left[x^2 + \frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right] + \frac{(2p+q)x}{qp^{n-2}[3]_{p,q}[n]_{p,q}} + \frac{1}{[3]_{p,q}[n]^2_{p,q}}\\ &= \frac{[n+1]_{p,q}x^2}{[n]_{p,q}q^3p^{2n-2}} + \frac{x}{p^{n-1}q[n]_{p,q}}\left[\frac{1}{q} + \frac{(2p+q)p}{[3]_{p,q}}\right] + \frac{1}{[3]_{p,q}[n]^2_{p,q}}.\end{aligned}$$

3 Direct Estimates

By $C_B[0,\infty)$ we denote the class of all real valued continuous and bounded functions f on $[0,\infty)$. The norm $||.||_{C_B}$ is defined as

$$\|f\|_{C_B}=\sup_{x\in[0,\infty)}|f(x)|.$$

For $f\in C_B[0,\infty)$ the Steklov mean is defined as

$$f_h(t) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2f(t+u+v)-f(t+2(u+v))]\,dudv \quad (3)$$

By simple computation, it is observed that
(i) $\|f_h-f\|_{C_B}\leq\widetilde{\omega}_2(f,h)$.
(ii) If f is continuous and $f_h', f''\in C_B$ then

$$\|f_h'\|_{C_B}\leq\frac{5}{h}\widetilde{\omega}(f,h),\|f_h''\|_{C_B}\leq\frac{9}{h^2}\widetilde{\omega}_2(f,h),$$

where the first and second order modulus of continuity for $\delta\geq0$ are respectively defined as

$$\widetilde{\omega}(f,\delta)=\sup_{\substack{x,u,v\geq0\\|u-v|\leq\delta}}|f(x+u)-f(x+v)|$$

and

$$\widetilde{\omega}_2(f,\delta)=\sup_{\substack{x,u,v\geq0\\|u-v|\leq\delta}}|f(x+2u)-2f(x+u+v)+f(x+2v)|.$$

Theorem 1. *Let $q\in(0,1)$ and $p\in(q,1]$. The operator $K_n^{p,q}$ maps space C_B into C_B and*

$$\|K_n^{p,q}(f)\|_{C_B}\leq\|f\|_{C_B}.$$

Proof. Let $q\in(0,1)$ and $p\in(q,1]$. From Lemma 3, we have

$$\begin{aligned}&|K_n^{p,q}(f,x)|\\&\leq[n]_{p,q}\sum_{k=0}^{\infty}b_{n,k}^{p,q}(x)p^{-k}q^k\int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}}|f(t)|d_{p,q}t\\&\leq\sup_{x\in[0,\infty)}|f(x)|\sum_{k=0}^{\infty}b_{n,k}^{p,q}(x)p^{-k}q^k\int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}}d_{p,q}t\\&=\sup_{x\in[0,\infty)}|f(x)|K_n^{p,q}(1,x)=\|f\|_{C_B}.\end{aligned}$$

Theorem 2. *Let $q\in(0,1)$ and $p\in(q,1]$. If $f\in C_B$, then*

$$\begin{aligned}&|K_n^{p,q}(f,x)-f(x)|\\&\leq5\widetilde{\omega}\left(f,\frac{1}{\sqrt{[n]_{p,q}}}\right)\left(\frac{1}{[2]_{p,q}\sqrt{[n]_{p,q}}}+\left(\frac{1}{qp^{n-1}}-1\right)x\right)\\&+\frac{9}{2}\widetilde{\omega}_2\left(f,\frac{1}{\sqrt{[n]_{p,q}}}\right)\left[\left(\frac{[n+1]_{p,q}}{q^3p^{2n-2}}-\frac{2[n]_{p,q}}{qp^{n-1}}+[n]_{p,q}\right)x^2\right.\\&\left.+\left(\frac{1}{p^{n-1}q}\left[\frac{1}{q}+\frac{(2p+q)p}{[3]_{p,q}}\right]-\frac{2}{[2]_{p,q}}\right)x+\frac{1}{[3]_{p,q}[n]_{p,q}}+2\right].\end{aligned}$$

Proof. For $x\geq0$ and $n\in\mathbb{N}$ and using the Steklov function f_h defined by (3), we can write

$$\begin{aligned}&|K_n^{p,q}(f,x)-f(x)|\\&\leq K_n^{p,q}(|f-f_h|,x)+|K_n^{p,q}(f_h-f_h(x),x)|+|f_h(x)-f(x)|.\end{aligned}$$

First by Theorem 1 and property (i) of Steklov mean, we have

$$K_n^{p,q}(|f-f_h|,x)\leq\|K_n^{p,q}(f-f_h)\|_{C_B}\leq\|f-f_h\|_{C_B}\leq\widetilde{\omega}_2(f,h).$$

Also, by Taylor's expansion, we have

$$\begin{aligned}&|K_n^{p,q}(f_h-f_h(x),x)|\\&\leq|f_h'(x)|K_n^{p,q}(t-x,x)+\frac{1}{2}\|f''\|_{C_B}K_n^{p,q}\left((t-x)^2,x\right).\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}|K_n^{p,q}(f_h-f_h(x),x)|\leq&\frac{5}{h}\widetilde{\omega}(f,h)\left(\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{x}{qp^{n-1}}-x\right)\\&+\frac{9}{2h^2}\widetilde{\omega}_2(f,h)K_n^{p,q}\left((t-x)^2,x\right),\end{aligned}$$

where

$$\begin{aligned}&K_n^{p,q}\left((t-x)^2,x\right)\\&=\frac{[n+1]_{p,q}x^2}{[n]_{p,q}q^3p^{2n-2}}+\frac{x}{p^{n-1}q[n]_{p,q}}\left[\frac{1}{q}+\frac{(2p+q)p}{[3]_{p,q}}\right]\\&+\frac{1}{[3]_{p,q}[n]_{p,q}^2}-2x\left(\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{x}{qp^{n-1}}\right)+x^2\\&=\left(\frac{[n+1]_{p,q}}{[n]_{p,q}q^3p^{2n-2}}-\frac{2}{qp^{n-1}}+1\right)x^2\\&+\left(\frac{1}{p^{n-1}q[n]_{p,q}}\left[\frac{1}{q}+\frac{(2p+q)p}{[3]_{p,q}}\right]-\frac{2}{[2]_{p,q}[n]_{p,q}}\right)x\\&+\frac{1}{[3]_{p,q}[n]_{p,q}^2}\end{aligned}$$

for $x\geq0$, $h>0$. Setting $h=\sqrt{\frac{1}{[n]_{p,q}}}$, we get the desired result.

A different form to obtain the direct result is the applications of K-functional. The Peetre's $K-$functional is defined by

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\},$$

where $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [7, p. 177, Theorem 2.4], there exists a positive constant $C>0$ such that $K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}),\delta>0$, where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta},x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|$$

is the second order modulus of continuity of function $f\in C_B[0,\infty)$.
Also, for $f\in C_B[0,\infty)$ the first order modulus of continuity is given by

$$\omega(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta},x\in[0,\infty)}|f(x+h)-f(x)|.$$

Theorem 3. *Let $f \in C_B[0,\infty)$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$|K_n^{p,q}(f,x) - f(x)| \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

where

$$\delta_n(x) = \left\{K_n^{p,q}((t-x)^2,x) + (K_n^{p,q}((t-x),x))^2\right\}^{1/2}$$

and

$$\alpha_n(x) = \left|\frac{1}{[2]_{p,q}[n]_{p,q}} + \left(\frac{1}{qp^{n-1}} - 1\right)x\right|.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\overline{K}_n^{p,q}(f,x)$ defined by

$$\overline{K}_n^{p,q}(f,x) = K_n^{p,q}(f,x) + f(x) - f\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}\right).$$

It is observed that $\overline{K}_n^{p,q}(f,x)$ preserve linear functions. Let $x \in [0,\infty)$ and $g \in W^2$. Applying the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du,$$

we have

$$\begin{aligned}
&\overline{K}_n^{p,q}(g,x) - g(x)\\
&= \overline{K}_n^{p,q}\left(\int_x^t (t-u)g''(u)du, x\right)\\
&= K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du, x\right)\\
&\quad - \int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}}\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}} - u\right)g''(u)du\\
&= K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du, x\right)\\
&\quad - \int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}}\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}} - u\right)g''(u)du.
\end{aligned}$$

On the other hand,

$$\left|\int_x^t (t-u)g''(u)du\right| \leq \|g''\|\int_x^t |t-u|du \leq (t-x)^2\|g''\|,$$

and

$$\begin{aligned}
&\left|\int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}}\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}} - u\right)g''(u)du\right|\\
&\leq \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}} - x\right)^2\|g''\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&|\overline{K}_n^{p,q}(g,x) - g(x)|\\
&= \left|K_n^{p,q}\left(\int_x^t (t-u)g''(u)du, x\right)\right|\\
&+ \left|\int_x^{\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}}\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}} - u\right)g''(u)du\right|\\
&\leq \|g''\|K_n^{(p,q)}((t-x)^2,x) + \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}} - x\right)^2\|g''\|\\
&= \delta_n^2(x)\|g''\|.
\end{aligned}$$

Also, we have

$$|\overline{K}_n^{p,q}(f,x)| \leq |K_n^{p,q}(f,x)| + 2\|f\| \leq 3\|f\|.$$

Therefore,

$$\begin{aligned}
&|K_n^{p,q}(f,x) - f(x)|\\
&\leq |\overline{K}_n^{p,q}(f-g,x) - (f-g)(x)| + \left|f\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}\right) - f(x)\right|\\
&+ |\overline{K}_n^{p,q}(g,x) - g(x)|\\
&\leq |\overline{K}_n^{p,q}(f-g,x)| + |(f-g)(x)| + \left|f\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{x}{qp^{n-1}}\right) - f(x)\right|\\
&+ |\overline{K}_n^{p,q}(g,x) - g(x)|\\
&\leq 4\|f-g\| + \omega\left(f, \left|\frac{1}{[2]_{p,q}[n]_{p,q}} + \left(\frac{1}{qp^{n-1}} - 1\right)x\right|\right) + \delta_n^2(x)\|g''\|.
\end{aligned}$$

Finally taking the infimum on the right-hand side over all $g \in W^2$, we get

$$|K_n^{p,q}(f,x) - f(x)| \leq 4K_2(f,\delta_n^2(x)) + \omega(f,\alpha_n(x)).$$

By the property of $K-$functional, we have

$$|K_n^{p,q}(f,x) - f(x)| \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)).$$

This completes the proof of the theorem.

4 Weighted Approximation

We consider the following class of functions:

Let $H[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying $|f(x)| \leq M_f(1+x^2)$, where M_f is certain constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $H[0,\infty)$. Also, let $C_{x^2}^*[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)}|f(x)|(1+x^2)^{-1}$.

Finally, we discuss the weighted approximation theorem, where the approximation formula holds true on the interval $[0,\infty)$.

Theorem 4. *Let $p = p_n$ and $q = q_n$ satisfies $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \to 1$, $q_n \to 1$ and $q_n^n \to 1$ and $p_n^n \to 1$. For each $f \in C_{x^2}^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\|K_n^{p_n,q_n}(f)-f\|_{x^2}=0.$$

Proof. Using the methods of [9], in order to complete the proof of theorem, it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|K_n^{p_n,q_n}(e_\nu,x)-x^\nu\|_{x^2}=0,\quad \nu=0,1,2. \tag{4}$$

Since $K_n^{p_n,q_n}(e_0,x)=1$ the first condition of (4) is fulfilled for $\nu=0$. We can write

$$\|K_n^{p_n,q_n}(e_1,x)-x\|_{x^2} \leq \left(\frac{1}{[2]_{p_n,q_n}[n]_{p_n,q_n}}+\frac{(1-q_np_n^{n-1})x}{q_np_n^{n-1}}\right)\sup_{x\in[0,\,\infty)}\frac{1}{1+x^2}.$$

and

$$\left\|K_n^{p_n,q_n}(e_2,x)-x^2\right\|_{x^2} \leq \left(\frac{[n+1]_{p_n,q_n}x^2}{[n]_{p_n,q_n}q_n^3p_n^{2n-2}}+\frac{x}{p_n^{n-1}q_n[n]_{p_n,q_n}}\left[\frac{1}{q_n}+\frac{(2p_n+q_n)p_n}{[3]_{p_n,q_n}}\right]+\frac{1}{[3]_{p_n,q_n}[n]^2_{p_n,q_n}}-x^2\right)\sup_{x\in[0,\,\infty)}\frac{1}{1+x^2}$$

which implies that

$$\lim_{n\to\infty}\|K_n^{p_n,q_n}(e_\nu,x)-x^\nu\|_{x^2}=0,\nu=1,2.$$

Thus the proof is completed.

Remark. For $q\in(0,1)$ and $p\in(q,1]$ it is seen that $\lim_{n\to\infty}[n]_{p,q}=1/(p-q)$. In order to obtain convergence estimates of (p,q)-Baskakov-Kantorovich operators, we assume $p=(p_n)$, $q=(q_n)$ such that $0<q_n<p_n\leq 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $p_n^n\to 1$ and $q_n^n\to 1$ and $\lim_{n\to\infty}[n]_{p_n,q_n}=\infty$.

5 Conclusion

By considering the (p,q)-variant of the Baskakov-Kantorovich operators, we may have better results for suitable choices of p and q. Also, for special case $p=q=1$ of our operators, we capture the approximation properties of the usual Baskakov-Kantorovich operators. One may consider the other form of (p,q)-Baskakov-Kantorovich operators by extending the results of [4] to (p,q) setting, as the analysis is different, we may discuss it elsewhere.

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