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Application of Novel Schemes Based on Haar Wavelet Collocation Method for Burger and Boussinesq-Burger Equations

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Abstract: In this paper, efficient numerical schemes based on the Haar wavelet method are applied for finding numerical solutions of nonlinear Burger as well as Boussinesq-Burger equations. The numerical results are then compared with those of the exact solutions. The accuracy of the obtained solutions is quite high even if the number of collocation points is small.

Keywords: Haar wavelet collocation method, Boussinesq-Burger equation, Burger equation.

1 Introduction

Generalized Boussinesq-Burger equation [1] is a nonlinear partial differential equation of the form

$$u_t - \frac{1}{2}v_x + 2uu_x = 0, \quad (1)$$

$$v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0, \quad 0 \le x \le 1 \quad (2)$$

The Boussinesq-Burger equations arise in the study of fluid flow and describe the propagation of shallow water waves. Here x and t respectively represent the normalized space and time, $u(x,t)$ is the horizontal velocity field and $v(x,t)$ denotes the height of the water surface above a horizontal level at the bottom.
Consider the one-dimensional Burger equation [1]

$$u_t + uu_x - \nu u_{xx} = 0, \quad 0 \le x \le 1 \quad (3)$$

where ν is parameter.
The Burger's equation is a nonlinear homogeneous parabolic partial differential equation, which arises in many physical problems including one-dimensional turbulence, sound waves in viscous medium, shock waves in a viscous medium, waves in fluid filled viscous elastic tubes and magneto-hydrodynamic waves in a medium with finite electrical conductivity.

Various mathematical methods such as the Galerkin finite element method [2], spectral collocation method [3], quartic B-spline differential quadrature method [4], quartic B-splines collocation method [5], finite element method [6], fourth order finite difference method [7], explicit and exact explicit finite difference method [8] and least-squares quadratic B-splines finite element method [9] have been used in attempting to solve Burger's equations. Our aim in the present work is to implement the Haar wavelet method to stress its power in handling nonlinear equations, so that one can apply it to various types of nonlinearity.

Recently, there has been some attention devoted to search for better solution methods [20-22] for determining analytical approximate solutions of fluid flow problems. Moreover, the authors might likely to be interested to apply homotopy perturbation transform method [23, 24] for solving Burger's and Boussinesq-Burger equations in future.

This paper is organized as follows: in Section 1, introduction to Boussinesq-Burger and Burger equation is described. In Section 2, the mathematical preliminaries of Haar wavelet is presented. Sections 3 and 5 define the mathematical models of Burger and Boussinesq-Burger equations respectively. We applied the Haar wavelet method for solving Burger and Boussinesq-Burger

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equations in Sections 4 and 6 respectively. Convergence of Haar wavelet approximation is discussed in Section 7. The numerical results and discussions are discussed in Section 8 and Section 9 concludes the paper.

2 Haar wavelets and the operational matrices

The Haar wavelet family for $x \in [0,1)$ is defined as follows [10, 11]

$$h_i(x) = \begin{cases} 1, & x \in [\xi_1, \xi_2) \\ -1, & x \in [\xi_2, \xi_3) \\ 0, & \text{elsewhere} \end{cases} \quad (4)$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k+0.5}{m}, \quad \xi_3 = \frac{k+1}{m}$$

In these formulae integer $m = 2^J$, $j = 0,1,2,\ldots,J$ indicates the level of the wavelet; $k = 0,1,2,\ldots,m-1$ is the translation parameter. Maximum level of resolution is J. The index i is calculated from the formula $i = m+k+1$; in the case of minimal values $m = 1$, $k = 0$ and we have $i = 2$. The maximal value of $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which

$$h_i(x) = \begin{cases} 1, \text{ for } x \in [0,1) \\ 0, \quad \text{elsewhere} \end{cases} \quad (5)$$

In the following analysis, integrals of the wavelets are defined as

$$p_i(x) = \int_0^x h_i(x)\,dx$$

$$q_i(x) = \int_0^x p_i(x)\,dx$$

$$r_i(x) = \int_0^x q_i(x)\,dx$$

This can be done with the aid of (4)

$$p_i(x) = \begin{cases} x-\xi_1, \text{ for } x \in [\xi_1, \xi_2) \\ \xi_3 - x, \text{ for } x \in [\xi_2, \xi_3) \\ 0, \quad \text{elsewhere} \end{cases} \quad (6)$$

$$q_i(x) = \begin{cases} 0, & \text{for } x \in [0, \xi_1) \\ \frac{1}{2}(x-\xi_1)^2, & \text{for } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - x)^2, & \text{for } x \in [\xi_2, \xi_3) \\ \frac{1}{4m^2}, & \text{for } x \in [\xi_3, 1) \end{cases} \quad (7)$$

$$r_i(x) = \begin{cases} \frac{1}{6}(x-\xi_1)^3, & \text{for } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2}(x-\xi_2) + \frac{1}{6}(\xi_3 - x)^3, & \text{for } x \in [\xi_2, \xi_3) \\ \frac{1}{4m^2}(x-\xi_2), & \text{for } x \in [\xi_3, 1) \\ 0, & \text{elsewhere} \end{cases} \quad (8)$$

The collocation points are defined as

$$x_l = \frac{l-0.5}{2M}, \quad l = 1,2,\ldots,2M$$

It is expedient to introduce the $2M \times 2M$ matrices H, P, Q and R with the elements $H(i,l) = h_i(x_l)$, $P(i,l) = p_i(x_l)$, $Q(i,l) = q_i(x_l)$ and $R(i,l) = r_i(x_l)$.

3 Burger's Equation

Consider the generalized Burger's equation

$$u_t + uu_x - \nu u_{xx} = 0, \quad 0 \le x \le 1 \quad (9)$$

where $\nu(>0)$ can be interpreted as viscosity.
To show the effectiveness and accuracy of proposed scheme, we consider two test examples. The numerical solutions thus obtained are compared with those of analytical solutions as well as available numerical results. The initial condition associated with eq. (9) will be

$$u(x,t_0) = f(x), \quad 0 \le x \le 1 \quad (10)$$

with boundary conditions $u(0,t) = u(1,t) = 0$, $t > t_0$

4 Haar wavelet based scheme for Burger's equation

It is assumed that $\dot{u}''(x,t)$ can be expanded in terms of Haar wavelets as

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_s(i) h_i(x) \text{ for } t \in [t_s, t_{s+1}] \quad (11)$$

where "$\cdot$" and "$\prime$" stands for differentiation with respect to t and x respectively. Now, integrating eq. (11) with respect to t from t_s to t and twice with respect to x from 0 to x the following equations are obtained

$$u''(x,t) = (t-t_s) \sum_{i=1}^{2M} a_s(i) h_i(x) + u''(x,t_s) \quad (12)$$

$$u'(x,t) = (t-t_s) \sum_{i=1}^{2M} a_s(i) p_i(x) + u'(x,t_s) - u'(0,t_s) + u'(0,t) \quad (13)$$

$$u(x,t) = (t-t_s) \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x,t_s) - u(0,t_s) + x\left[u'(0,t) - u'(0,t_s)\right] + u(0,t) \quad (14)$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_s(i) q_i(x) + x\dot{u}'(0,t) + \dot{u}(0,t) \quad (15)$$

By using the boundary conditions at $x = 1$, and from eq. (15) and (14) respectively, we have

$$\dot{u}'(0,t) = -\sum_{i=1}^{2M} a_s(i) q_i(1) \quad (16)$$

and

$$u'(0,t) - u'(0,t_s) = -(t-t_s)\sum_{i=1}^{2M} a_s(i) q_i(1) \quad (17)$$

From eq. (7), it is obtained that

$$q_i(1) = \begin{cases} 0.5 & \text{if } i = 1 \\ \frac{1}{4m^2} & \text{if } i > 1 \end{cases} \quad (18)$$

Substituting eqs. (16), (17) and (18) in eqs. (13), (14) and (15) and discretising the results by assuming $x \to x_l$, $t \to t_{s+1}$, the following equations are obtained

$$u''(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i) h_i(x_l) + u''(x_l,t_s) \quad (19)$$

$$u'(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left[p_i(x_l) - q_i(1)\right] + u'(x_l,t_s) \quad (20)$$

$$u(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left[q_i(x_l) - x_l q_i(1)\right] + u(x_l,t_s) \quad (21)$$

$$\dot{u}(x_l,t_{s+1}) = \sum_{i=1}^{2M} a_s(i)\left[q_i(x_l) - x_l q_i(1)\right] \quad (22)$$

Substituting eqs. (19), (20), (21) and (22) in eq. (9), we have

$$\sum_{i=1}^{2M} a_s(i)\left[q_i(x_l) - x_l q_i(1)\right] =$$
$$\nu\left[(t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i) h_i(x_l) + u''(x_l,t_s)\right] -$$
$$\left[(t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left[q_i(x_l) - x_l q_i(1)\right] + u(x_l,t_s)\right] \times$$
$$\left[(t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left[p_i(x_l) - q_i(1)\right] + u'(x_l,t_s)\right] \quad (23)$$

From eq. (23), the wavelet coefficients $a_s(i)$ can be successively calculated using mathematical software. This process starts with
$u(x_l,t_0) = f(x_l)$
$u'(x_l,t_0) = f'(x_l)$
$u''(x_l,t_0) = f''(x_l)$

Example 1. Consider Burger's equation with the following initial and boundary conditions [8]

$$u(x,0) = \sin(\pi x), \ \ 0 \le x \le 1 \quad (24)$$
$$u(0,t) = u(1,t) = 0, \ \ t > 0.$$

The exact solution of eq. (9) is given by [8]

$$u(x,t) = \frac{2\pi\nu \sum_{n=1}^{\infty} A_n n \sin(n\pi x)\exp(-n^2\pi^2\nu t)}{A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\exp(-n^2\pi^2\nu t)}, \quad (25)$$

where

$$A_0 = \int_0^1 \exp\left(\frac{-1}{2\pi\nu}(1-\cos(\pi x))\right)dx,$$
$$A_n = 2\int_0^1 \exp\left(\frac{-1}{2\pi\nu}(1-\cos(\pi x))\right)dx,$$

The numerical solutions of the example 1 are presented for $\nu = 0.01$ with $\triangle t = 0.001$ taking $M = 64$ in Table 1 and Figs. 1 and 2. The results are compared with Refs. [8, 9, 12] and consequently it is found that the present method is much better than the results presented in [8, 9, 12]. The Figs. 1 and 2 are in good agreement with the results obtained by learned researcher Jiwari in [13].

Example 2. In this example, we consider Burger's equation with initial condition in the following form

$$u(x,0) = \frac{2\pi\nu \sin(\pi x)}{a + \cos(\pi x)}, \quad a > 1 \quad (26)$$

The exact solution of eq. (9) is given by [14]

$$u(x,t) = \frac{2\pi\nu \exp(-\pi^2\nu t)\sin(\pi x)}{a + \exp(-\pi^2\nu t)\cos(\pi x)}, \quad a > 1 \quad (27)$$

In case of example 2, Tables 2 and 3 show the L_2 and L_∞ errors at different values of a, ν and M. Moreover, the results are compared with Refs. [15, 16] and it has been observed that the present method is more accurate and efficient than the other numerical solutions. The physical behaviour of solutions at different time stages are shown in Fig. 3 and Fig. 4.

5 Boussinesq-Burger's Equation

Consider the general Boussinesq-Burger equation [17] of the form

$$u_t - \frac{1}{2}v_x + 2uu_x = 0, \quad (28)$$

$$v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0, \quad 0 \le x \le 1 \quad (29)$$

with initial conditions

$$u(x,0) = \frac{-1}{4} - \frac{1}{4}\tanh\left(\frac{x - \log 2}{2}\right) \quad (30)$$

$$v(x,0)=\frac{-1}{8}\operatorname{sec}h^2\left(\frac{-x+\log 2}{2}\right) \tag{31}$$

The exact solutions of eq. (28) and (29) is given by [18]

$$u(x,t)=\frac{-1}{4}-\frac{1}{4}\tanh\left(\frac{x+\frac{t}{2}-\log 2}{2}\right) \tag{32}$$

$$v(x,t)=\frac{-1}{8}\operatorname{sec}h^2\left(\frac{-x-\frac{t}{2}+\log 2}{2}\right) \tag{33}$$

These exact solutions satisfies the following boundary conditions

$$u(0,t)=\frac{-1}{4}-\frac{1}{4}\tanh\left(\frac{\frac{t}{2}-\log 2}{2}\right),$$

$$u(1,t)=\frac{-1}{4}-\frac{1}{4}\tanh\left(\frac{1+\frac{t}{2}-\log 2}{2}\right) \tag{34}$$

$$v(0,t)=\frac{-1}{8}\operatorname{sec}h^2\left(\frac{-\frac{t}{2}+\log 2}{2}\right),$$

$$v(1,t)=\frac{-1}{8}\operatorname{sec}h^2\left(\frac{-1-\frac{t}{2}+\log 2}{2}\right) \tag{35}$$

6 Application of Haar wavelet to Boussinesq-Burgers equation

The Haar wavelet solutions of $u(x,t)$ and $v(x,t)$ is sought by assuming that $\dot{u}'''(x,t)$ and $\dot{v}'(x,t)$ can be expanded in terms of Haar wavelets as

$$\dot{u}'''(x,t)=\sum_{i=1}^{2M}a_s(i)h_i(x) \tag{36}$$

$$\dot{v}'(x,t)=\sum_{i=1}^{2M}b_s(i)h_i(x) \text{ for } t\in[t_s,t_{s+1}] \tag{37}$$

where "$\cdot$" and "$\prime$" stands for differentiation with respect to t and x respectively.
Now, integrating eq. (36) with respect to t from t_s to t and thrice with respect to x from 0 to x the following equations are obtained

$$u'''(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)h_i(x)+u'''(x,t_s) \tag{38}$$

$$u''(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)p_i(x)+u''(x,t_s)-u''(0,t_s)+u''(0,t) \tag{39}$$

$$u'(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)q_i(x)+u'(x,t_s)-u'(0,t_s)+x\left[u''(0,t)-u''(0,t_s)\right]+u'(0,t) \tag{40}$$

$$u(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)r_i(x)+u(x,t_s)-u(0,t_s)+\frac{x^2}{2}\left[u''(0,t)-u''(0,t_s)\right]+x\left[u'(0,t)-u'(0,t_s)\right]+u(0,t) \tag{41}$$

$$\dot{u}(x,t)=\sum_{i=1}^{2M}a_s(i)r_i(x)+\frac{x^2}{2}\dot{u}''(0,t)+x\dot{u}'(0,t)+\dot{u}(0,t) \tag{42}$$

Integrating eq. (37) with respect to t from t_s to t and once with respect to x from 0 to x, the following equations are obtained

$$v'(x,t)=(t-t_s)\sum_{i=1}^{2M}b_s(i)h_i(x)+v'(x,t_s) \tag{43}$$

$$v(x,t)=(t-t_s)\sum_{i=1}^{2M}b_s(i)p_i(x)+v(x,t_s)-v(0,t_s)+v(0,t) \tag{44}$$

$$\dot{v}(x,t)=\sum_{i=1}^{2M}b_s(i)p_i(x)+\dot{v}(0,t) \tag{45}$$

Discretising the above results by assuming $x\to x_l, t\to t_{s+1}$, from eqs. (43), (44) and (45), we obtain

$$v'(x_l,t_{s+1})=(t_{s+1}-t_s)\sum_{i=1}^{2M}b_s(i)h_i(x_l)+v'(x_l,t_s) \tag{46}$$

$$v(x_l,t_{s+1})=(t_{s+1}-t_s)\sum_{i=1}^{2M}b_s(i)p_i(x_l)+v(x_l,t_s)-v(0,t_s)+v(0,t_{s+1}) \tag{47}$$

$$\dot{v}(x_l,t_{s+1})=\sum_{i=1}^{2M}b_s(i)p_i(x_l)+\dot{v}(0,t_{s+1}) \tag{48}$$

Using finite difference method

$$\dot{u}(0,t)=\frac{u(0,t)-u(0,t_s)}{t-t_s}$$

Equation (42) becomes

$$\dot{u}(x,t)=\sum_{i=1}^{2M}a_s(i)r_i(x)+\frac{x^2}{2}\left[\frac{u''(0,t)-u''(0,t_s)}{t-t_s}\right]+x\left[\frac{u'(0,t)-u'(0,t_s)}{t-t_s}\right]+\left[\frac{u(0,t)-u(0,t_s)}{t-t_s}\right] \tag{49}$$

By using the boundary condition at $x=1$, eq. (40) becomes

$$u'(1,t) = (t-t_s)\sum_{i=1}^{2M} a_s(i)q_i(1) + u'(1,t_s) - u'(0,t_s) + \left[u''(0,t) - u''(0,t_s)\right] + u'(0,t)$$

This implies

$$u''(0,t) - u''(0,t_s) = -(t-t_s)\sum_{i=1}^{2M} a_s(i)q_i(1) + u'(1,t) - u'(1,t_s) - u'(0,t) + u'(0,t_s) \quad (50)$$

Substituting eq. (50) in eqs. (39), (40), (41) and (49) and discretising the resultant results by assuming $x \to x_l, t \to t_{s+1}$, we obtain

$$u'''(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)h_i(x_l) + u'''(x_l,t_s) \quad (51)$$

$$u''(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)p_i(x_l) + u''(x_l,t_s) - (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)q_i(1) + \left[u'(1,t_{s+1}) - u'(1,t_s)\right] - \left[u'(0,t_{s+1}) - u'(0,t_s)\right] \quad (52)$$

$$u'(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)q_i(x_l) + u'(x_l,t_s) - u'(0,t_s) + u'(0,t_{s+1}) - x_l(t_{s+1}-t_s)\times \sum_{i=1}^{2M} a_s(i)q_i(1) + x_l\left[u'(1,t_{s+1}) - u'(1,t_s)\right] - x_l\left[u'(0,t_{s+1}) - u'(0,t_s)\right] \quad (53)$$

$$u(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)r_i(x_l) + u(x_l,t_s) - u(0,t_s) + u(0,t_{s+1}) + x_l(u'(0,t_{s+1}) - u'(0,t_s)) - \frac{x_l^2}{2}(t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)q_i(1) + \frac{x_l^2}{2}\left[u'(1,t_{s+1}) - u'(1,t_s)\right] - \frac{x_l^2}{2}\left[u'(0,t_{s+1}) - u'(0,t_s)\right] \quad (54)$$

$$\dot{u}(x_l,t_{s+1}) = \sum_{i=1}^{2M} a_s(i)r_i(x_l) + \frac{x_l}{t_{s+1}-t_s}(u'(0,t_{s+1}) - u'(0,t_s)) + \frac{1}{t_{s+1}-t_s}\left[u(0,t_{s+1}) - u(0,t_s)\right] - \frac{x_l^2}{2}\sum_{i=1}^{2M} a_s(i)q_i(1) + \frac{x_l^2}{2(t_{s+1}-t_s)}\times \left[u'(1,t_{s+1}) - u'(1,t_s)\right] - \frac{x_l^2}{2(t_{s+1}-t_s)}\left[u'(0,t_{s+1}) - u'(0,t_s)\right] \quad (55)$$

Substituting the above equations in eq. (28) and eq. (29), we have

$$\sum_{i=1}^{2M} a_s(i)\left[r_i(x_l) - \frac{x_l^2}{2}q_i(1)\right] + \frac{x_l^2}{2(t_{s+1}-t_s)}\times \left(\left(u'(1,t_{s+1}) - u'(1,t_s)\right) - \left(u'(0,t_{s+1}) - u'(0,t_s)\right)\right) + \frac{x_l}{t_{s+1}-t_s}\left[u'(0,t_{s+1}) - u'(0,t_s)\right] + \frac{1}{t_{s+1}-t_s}\left[u(0,t_{s+1}) - u(0,t_s)\right] = \frac{1}{2}\left[(t_{s+1}-t_s)\sum_{i=1}^{2M} b_s(i)h_i(x_l) + v'(x_l,t_s)\right] - 2((t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left(r_i(x_l) - \frac{x_l^2}{2}q_i(1)\right) + u(x_l,t_s) - u(0,t_s) + u(0,t_{s+1}) + x_l(u'(0,t_{s+1}) - u'(0,t_s)) + \frac{x_l^2}{2}((u'(1,t_{s+1}) - u'(1,t_s)) - (u'(0,t_{s+1}) - u'(0,t_s)))) ((t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)[q_i(x_l) - x_l q_i(1)] + u'(x_l,t_s) - u'(0,t_s) + u'(0,t_{s+1}) + x_l((u'(1,t_{s+1}) - u'(1,t_s)) - (u'(0,t_{s+1}) - u'(0,t_s)))) \quad (56)$$

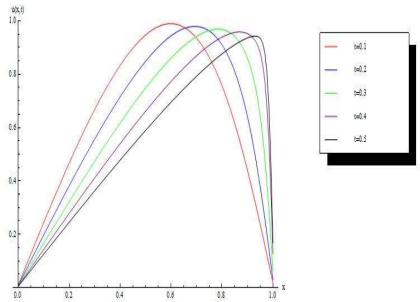

Fig. 1: Behaviour of numerical solutions for Burger's equation (example 1) when $\nu = 0.01$ and $\triangle t = 0.001$ at times $t = 0.1, 0.2, 0.3, 0.4$ and 0.5.

$$\sum_{i=1}^{2M} b_s(i)\left[p_i(x_l) + \dot{v}(x_l, t_{s+1})\right] =$$
$$\frac{1}{2}\left[(t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)h_i(x_l) + u'''(x_l,t_s)\right]$$
$$-2((t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left(r_i(x_l) - \frac{x_l^2}{2}q_i(1)\right)$$
$$+u(x_l,t_s) - u(0,t_s) + u(0,t_{s+1}) +$$
$$x_l(u'(0,t_{s+1}) - u'(0,t_s)) + \frac{x_l^2}{2}((u'(1,t_{s+1})$$
$$-u'(1,t_s)) - (u'(0,t_{s+1}) - u'(0,t_s))))\times$$
$$((t_{s+1}-t_s)\sum_{i=1}^{2M} b_s(i)h_i(x_l) + v'(x_l,t_s))$$
$$-2((t_{s+1}-t_s)\sum_{i=1}^{2M} b_s(i)p_i(x_l) + v(x_l,t_s)$$
$$-v(0,t_s) + v(0,t_{s+1}))\times$$
$$((t_{s+1}-t_s)\sum_{i=1}^{2M} a_s(i)\left(q_i(x_l) - x_l q_i(1)\right) +$$
$$u'(x_l,t_s) - u'(0,t_s) + u'(0,t_{s+1}) - u'(0,t_{s+1})$$
$$+u'(0,t_s) + x_l((u'(1,t_{s+1}) - u'(1,t_s))))) \qquad (57)$$

From the above two eqs. (56) and (57), the wavelet coefficients $a_s(i)$ and $b_s(i)$ can be successively calculated using mathematical software. This process starts with

$u(x_l,t_0) = \frac{-1}{4} - \frac{1}{4}\tanh\left(\frac{x_l - \log 2}{2}\right)$

$u'(x_l,t_0) = \frac{-1}{8}\sec h^2\left(\frac{x_l - \log 2}{2}\right)$

$u''(x_l,t_0) = \frac{1}{8}\sec h^2\left(\frac{x_l - \log 2}{2}\right)\tanh\left(\frac{x_l - \log 2}{2}\right)$

$v(x_l,t_0) = \frac{-1}{8}\sec h^2\left(\frac{-x_l + \log 2}{2}\right)$

$v'(x_l,t_0) = \frac{-1}{8}\sec h^2\left(\frac{-x_l + \log 2}{2}\right)\tanh\left(\frac{-x_l + \log 2}{2}\right)$

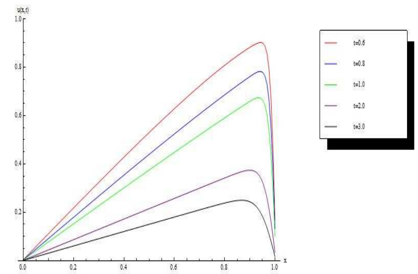

Fig. 2: Behaviour of numerical solutions for Burger's equation (example 1) when $\nu = 0.01$ and $\triangle t = 0.001$ at times $t = 0.6, 0.8, 1.0, 2.0$ and 3.0.

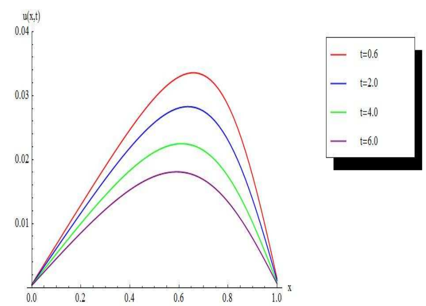

Fig. 3: Behaviour of numerical solutions for Burger's equation (example 2) when $\nu = 0.01$ and $\triangle t = 0.001$ at times $t = 0.6, 2.0, 4.0$ and 6.0.

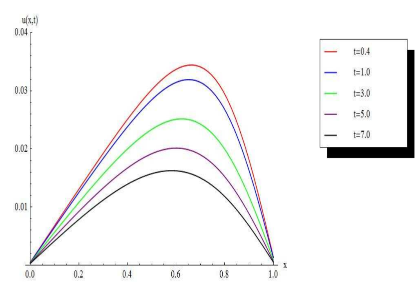

Fig. 4: Behaviour of numerical solutions for Burger's equation (example 2) when $\nu = 0.01$ and $\triangle t = 0.001$ at times $t = 0.4, 1.0, 3.0, 5.0$ and 7.0.

7 Convergence of Haar wavelet approximation

The convergence of the method may be discussed on the same lines as given by learned researcher Saha Ray [19].

Theorem 7.1 Let $f(x) \in L^2(R)$ be a continuous function defined on $[0,1)$. Then the error at Jth level may be defined as

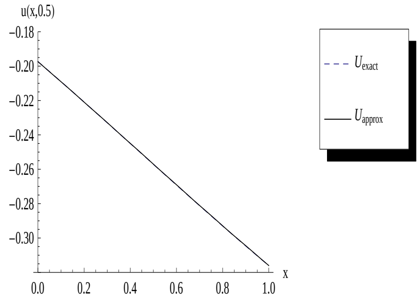

Fig. 5: Comparison of Numerical solution and exact solution of Boussinesq-Burger equation when $t = 0.5$.

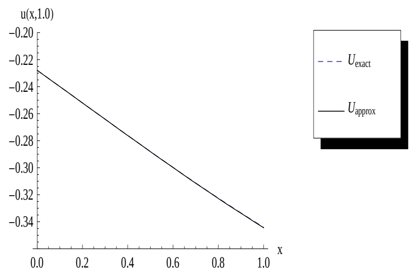

Fig. 6: Comparison of Numerical solution and exact solution of Boussinesq-Burger equation when $t = 1.0$.

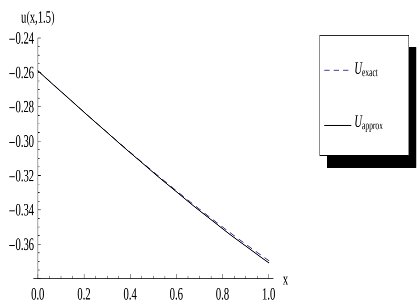

Fig. 7: Comparison of Numerical solution and exact solution of Boussinesq-Burger equation when $t = 1.5$.

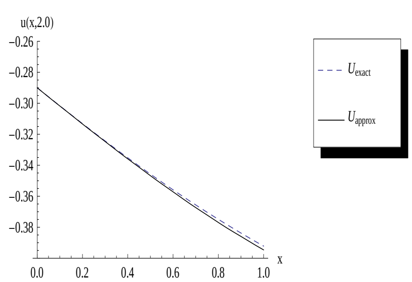

Fig. 8: Comparison of Numerical solution and exact solution of Boussinesq-Burger equation when $t = 2.0$.

$$
\begin{aligned}
E_J(x) &= |f(x) - f_J(x)| \\
&= \left| f(x) - \sum_{i=1}^{2M} a_i h_i(x) \right| \\
&= \left| \sum_{i=2M}^{\infty} a_i h_i(x) \right|
\end{aligned}
$$

The error norm for $E_J(x)$ is obtained as

$$\|E_J(x)\|_2 \leqslant \frac{K^2}{12} 2^{-2J} \tag{58}$$

where $|f'(x)| \leqslant K, \forall x \in [0,1)$ and $K > 0$ and M is a positive number related to the Jth level of resolution of the wavelet given by $M = 2^J$.

Proof:

The proof of the theorem 7.1 can be found in Ref. [19]. From eq. (58), it can be observed that the error bound is inversely proportional to the level of resolution J. So, more accurate result can be obtained by increasing the level of resolution.

8 Numerical Results and Discussions

The error function is given by

$$
\begin{aligned}
Error\,function &= \left\| u_{approx}(x_l,t) - u_{exact}(x_l,t) \right\| \\
&= \sqrt{\sum_{l=1}^{2M} \left(\left| u_{approx}(x_l,t) - u_{exact}(x_l,t) \right| \right)^2}
\end{aligned}
$$

$$
\begin{aligned}
Global\ error\ estimate &= R.M.S.\ error \\
&= \frac{1}{\sqrt{2M}} \left\| u_{approx}(x_l,t) - u_{exact}(x_l,t) \right\| \\
&= \frac{1}{\sqrt{2M}} \sqrt{\sum_{l=1}^{2M} \left(\left| u_{approx}(x_l,t) - u_{exact}(x_l,t) \right| \right)^2}
\end{aligned} \tag{59}
$$

In order to measure the accuracy of the numerical scheme error norm L_2 and L_∞ are calculated using the following formula

$$
\begin{aligned}
L_2 &= R.M.S.\ error \\
&= \frac{1}{\sqrt{2M}} \sqrt{\sum_{l=1}^{2M} \left(\left| u_{approx}(x_l,t) - u_{exact}(x_l,t) \right| \right)^2}
\end{aligned} \tag{60}
$$

Table 1: Comparison with present method solution and other numerical methods for Burger's equation (example 1) at different values of t with $a = 2$, $\nu = 0.01$ and $\triangle t = 0.001$.

x	t	$EFDM$ [8] $\triangle t = 0.001$	$EEFDM$ [8] $\triangle t = 0.001$	$Least-square$ $quadratic$ $B-spline\ FEM$ [9] $\triangle t = 0.0001$	$Crank-Nicolson$ $method$ $\triangle t = 0.01$	$Present$ $Method$ $\triangle t = 0.001$	$Exact$
0.25	0.4	0.34244	0.34164	0.34244	0.34229	0.34224	0.34191
	0.6	0.26905	0.26890	0.27536	0.26902	0.26924	0.26896
	0.8	0.22145	0.22150	0.22752	—	0.22170	0.22148
	1.0	0.18813	0.18825	0.19375	0.18817	0.18837	0.18819
	3.0	0.07509	0.07515	0.07754	0.07511	0.07516	0.07511
0.5	0.4	0.67152	0.65606	0.66543	0.66797	0.65106	0.66071
	0.6	0.53406	0.52658	0.53525	0.53211	0.52984	0.52942
	0.8	0.44143	0.43743	0.44526	—	0.43953	0.43914
	1.0	0.37568	0.37336	0.38047	0.37500	0.37476	0.37442
	3.0	0.15020	0.15015	0.15362	0.15018	0.15027	0.15018
0.75	0.4	0.94675	0.90111	0.91201	0.93680	0.90980	0.91026
	0.6	0.78474	0.75862	0.77132	0.77724	0.76745	0.76724
	0.8	0.65659	0.64129	0.65254	—	0.64778	0.64740
	1.0	0.56135	0.55187	0.56157	0.55833	0.55647	0.55605
	3.0	0.22502	0.22454	0.22874	0.22485	0.22497	0.22481

Table 2: Comparison of L_2 and L_∞ errors with other numerical methods for Burger's equation (example 2) taking $a = 100$, $\nu = 0.01$ and at $t = 1$.

N	Rahman [15]		Mittal and Jain [16]		M	Present Method $\triangle t = 0.01$		Present Method $\triangle t = 0.001$	
	L_2	L_∞	L_2	L_∞		L_2	L_∞	L_2	L_∞
10	3.455E-7	4.881E-7	3.284E-7	4.624E-7	4	3.267E-8	4.634E-8	1.498E-8	2.157E-8
20	1.013E-7	1.431E-7	8.192E-8	1.164E-7	8	2.288E-8	3.239E-8	5.235E-9	7.452E-9
40	4.003E-8	5.668E-8	2.047E-8	2.907E-8	16	2.042E-8	2.889E-8	2.779E-9	3.939E-9
80	4.003E-8	3.499E-8	5.119E-9	7.271E-9	32	1.981E-8	2.802E-8	2.165E-9	3.064E-9

Table 3: Comparison of L_2 and L_∞ errors with other numerical methods for Burger's equation (example 2) taking $a = 100$, $\nu = 0.005$ and at $t = 1$.

N	Rahman [15]		Mittal and Jain [16]		M	Present Method $\triangle t = 0.01$		Present Method $\triangle t = 0.001$	
	L_2	L_∞	L_2	L_∞		L_2	L_∞	L_2	L_∞
10	8.819E-8	1.246E-7	8.631E-8	1.215E-7	4	4.266E-9	6.056E-9	1.942E-9	2.799E-9
20	2.403E-8	3.394E-8	2.153E-8	3.062E-8	8	2.999E-9	4.246E-9	6.809E-10	9.701E-10
40	7.942E-9	1.125E-8	5.378E-9	7.644E-9	16	2.681E-9	3.793E-9	3.632E-10	5.152E-10
80	3.918E-9	5.549E-9	1.345E-9	7.644E-9	32	2.601E-9	3.679E-9	2.839E-10	4.018E-10

$$L_\infty = max\left|u_{approx}(x_l,t) - u_{exact}(x_l,t)\right| \tag{61}$$

The following Table 1 shows the comparison of exact solutions with the approximate solutions of different numerical methods for Burger's equation. Agreement between present numerical results and exact solutions appears very satisfactory through illustration in Table 1. In the following Table 1, J has been taken as 6 i.e. $M = 64$ with $\nu = 0.01$ and different values of t. Similarly Tables 2 and 3 show the comparison of L_2 and L_∞ errors with other numerical methods for $\nu = 0.01$ and 0.005 with $\alpha = 100$ and $t = 1$. From Tables 2 and 3, it has been observed that the present method is more accurate and efficient than the other numerical methods presented in Refs. [15, 16].

The following Tables show the comparisons of the exact solutions with the approximate solutions of Boussinesq-Burger equation at different collocation points. In the following Tables 4-6, J has been taken as 4 i.e. $M = 16$ and $\triangle t$ is taken as 0.0001.

The R.M.S. error between the numerical solutions and the exact solutions of $u(x,t)$ for Boussinesq-Burger equations for $t = 0.5, 1.0$ and 1.5 are 0.000142255, 0.000216937 and 0.000935793 respectively and for $v(x,t)$ the R.M.S. error is found to be 0.0118472, 0.0236667 and 0.0346156 respectively.

Figures 1-4 cite the behaviour of numerical solutions obtained for Burger's equation at $\nu = 0.01$ and different values of t. Similarly in case of Boussinesq-Burger's

Table 4: The absolute errors in the solution of Boussinesq-Burger equation at various collocation points of x with $t = 0.5$.

x	u_{approx}	v_{approx}	u_{exact}	v_{exact}	$\|u_{exact} - u_{approx}\|$	$\|v_{exact} - v_{approx}\|$
0.015625	-0.197367	-0.118653	-0.197359	-0.119458	8.3598E-6	8.05119E-4
0.046875	-0.201157	-0.119271	-0.201104	-0.120218	5.29589E-5	9.47342E-4
0.078125	-0.204944	-0.122 098	-0.204872	-0. 120927	7.19897E-5	1.17066E-3
0.109375	-0.20869	-0.12456	-0.208662	-0.121582	2.82735E-5	2.97813E-3
0.140625	-0.212446	-0.124475	-0.212471	-0.122183	2.42459E-5	2.29235E-3
0.171875	-0.216279	-0.123392	-0.216297	-0.122728	1.87152E-5	6.63754E-4
0.203125	-0.220164	-0.124375	-0.22014	-0.123217	2.37371E-5	1.15814E-3
0.234375	-0.224021	-0.127389	-0.223998	-0.123648	2.32677E-5	3.74124E-3
0.265625	-0.227829	-0.129227	-0.227868	-0.12402	3.87521E-5	5.2064E-3
0.296875	-0.23166	-0.128271	-0.231749	-0.124334	8.83945E-5	3.93697E-3
0.328125	-0.23557	-0.127024	-0.235638	-0.124587	6.77895E-5	2.4362E-3
0.359375	-0.239511	-0.128457	-0.239535	-0.124781	2.37165E-5	3.67564E-3
0.390625	-0.243396	-0.131492	-0.243436	-0.124914	4.01933E-5	6.57804E-3
0.421875	-0.24723	-0.132477	-0.247341	-0.124986	1.11453E-4	7.49073E-3
0.453125	-0.251101	-0.130649	-0.251247	-0.124997	1.46677E-4	5.65231E-3
0.484375	-0.255048	-0.129425	-0.255153	-0.124947	1.04846E-4	4.47776E-3
0.515625	-0.258996	-0.131357	-0.259056	-0.124836	5.93702E-5	6.52131E-3
0.546875	-0.262864	-0.134199	-0.262954	-0.124664	9.07736E-5	9.53507E-3
0.578125	-0.266683	-0.134123	-0.266847	-0.124432	1.63839E-4	9.691E-3
0.609375	-0.270555	-0.1315	-0.270731	-0.12414	1.76238E-4	7.35999E-3
0.640625	-0.274493	-0.130521	-0.274605	-0.123789	1.11967E-4	6.73224E-3
0.671875	-0.278397	-0.132961	-0.278467	-0.123379	6.97724E-5	9.58204E-3
0.703125	-0.282198	-0.135353	-0.282315	-0.122911	1.17448E-4	1.2442E-2
0.734375	-0.285962	-0.13405	-0.286148	-0.122387	1.86803E-4	1.16634E-2
0.765625	-0.289792	-0.130795	-0.289964	-0.121806	1.71833E-4	8.98884E-3
0.796875	-0.293671	-0.130309	-0.293761	-0.12117	8.93953E-5	9.13864E-3
0.828125	-0.297477	-0.133207	-0.297537	-0.120481	5.94947E-5	1.2726E-2
0.859375	-0.301163	-0.13486	-0.30129	-0.119739	1.2673E-4	1.51218E-2
0.890625	-0.304831	-0.132242	-0.30502	-0.118946	1.88492E-4	1.32964E-2
0.921875	-0.308579	-0.128677	-0.308724	-0.118103	1.44573E-4	1.05737E-2
0.953125	-0.312353	-0.129219	-0.312401	-0.117212	4.80031E-5	1.20066E-2
0.984375	-0.316031	-0.13366	-0.316049	-0.116275	1.80034E-5	1.73849E-2

equation, the Figures 5-8 cite the comparison graphically between the numerical and exact solutions for different values of t.

9 Conclusion

In this paper, the Boussinesq-Burger and Burger's equations have been solved by Haar wavelet method. The obtained results are then compared with the exact solutions as well as solutions available in open literature. These have been cited in Tables and also graphically. These results demonstrated in Tables justify the accuracy and efficiency of the proposed schemes based on Haar wavelet. The numerical schemes are reliable and convenient for solving Boussinesq-Burger and Burger's equations. The main advantages of these schemes are its simplicity, applicability and less computational errors. Moreover, the errors may be reduced significantly if we increase level of resolution which prompts more number of collocation points.

Acknowledgement

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Table 5: The absolute errors in the solution of Boussinesq-Burger equation at various collocation points of x with $t = 1.0$.

x	u_{approx}	v_{approx}	u_{exact}	v_{exact}	$\lvert u_{exact} - u_{approx}\rvert$	$\lvert v_{exact} - v_{approx}\rvert$
0.015625	-0.227876	-0.123436	-0.227868	-0.12402	8.38233E-6	5.84734E-4
0.046875	-0.231802	-0.124054	-0.231749	-0.124334	5.3262E-5	2.79865E-4
0.078125	-0.235711	-0.12688	-0.235638	-0. 124587	7.30976E-5	2.293E-3
0.109375	-0.239566	-0.129343	-0.239535	-0.124781	3.09406E-5	4.56229E-3
0.140625	-0.243417	-0.129258	-0.243436	-0.124914	1.9061E-5	4.34445E-3
0.171875	-0.247331	-0.128175	-0.247341	-0.124986	9.87667E-6	3.18901E-3
0.203125	-0.251285	-0.129158	-0.251247	-0.124997	3.75151E-5	4.16092E-3
0.234375	-0.255196	-0.132172	-0.255153	-0.124947	4.33927E-5	7.22497E-3
0.265625	-0.259045	-0.13401	-0.259056	-0.124836	1.07795E-5	9.1736E-3
0.296875	-0.262903	-0.133054	-0.262954	-0.124664	5.10099E-5	8.38921E-3
0.328125	-0.266827	-0.131807	-0.266847	-0.124432	1.93935E-5	7.37414E-3
0.359375	-0.270768	-0.133239	-0.270731	-0.12414	3.72955E-5	9.09897E-3
0.390625	-0.27464	-0.136275	-0.274605	-0.123789	3.50153E-5	1.24855E-2
0.421875	-0.278446	-0.137259	-0.278467	-0.123379	2.05208E-5	1.38802E-2
0.453125	-0.282277	-0.135432	-0.282315	-0.122911	3.85771E-5	1.25206E-2
0.484375	-0.28617	-0.134208	-0.286148	-0.122387	2.17555E-5	1.18209E-2
0.515625	-0.290051	-0.13614	-0.289964	-0.121806	8.69255E-5	1.43344E-2
0.546875	-0.293837	-0.138982	-0.293761	-0.12117	7.62424E-5	1.78123E-2
0.578125	-0.297561	-0.138906	-0.297537	-0.120481	2.47286E-5	1.84257E-2
0.609375	-0.301325	-0.136283	-0.30129	-0.119739	3.44909E-5	1.65447E-2
0.640625	-0.305141	-0.135304	-0.30502	-0.118946	1.21288E-4	1.63586E-2
0.671875	-0.30891	-0.137744	-0.308724	-0.118103	1.86102E-4	1.96411E-2
0.703125	-0.312561	-0.140136	-0.312401	-0.117212	1.60845E-4	2.2924E-2
0.734375	-0.316162	-0.138833	-0.316049	-0.116275	1.1339E-4	2.25577E-2
0.765625	-0.319817	-0.135577	-0.319667	-0.115293	1.49407E-4	2.02845E-2
0.796875	-0.323506	-0.135092	-0.323254	-0.114268	2.51678E-4	2.08239E-2
0.828125	-0.327108	-0.137989	-0.326809	-0.113201	2.99826E-4	2.47885E-2
0.859375	-0.330578	-0.139643	-0.330329	-0.112095	2.48857E-4	2.75487E-2
0.890625	-0.334015	-0.137025	-0.333814	-0.11095	2.00973E-4	2.60745E-2
0.921875	-0.337519	-0.13346	-0.337263	-0.10977	2.55958E-4	2.36893E-2
0.953125	-0.341035	-0.134002	-0.340674	-0.108556	3.6035E-4	2.54455E-2
0.984375	-0.344442	-0.138443	-0.344047	-0.10731	3.94479E-4	3.11327E-2

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Table 6: The absolute errors in the solution of Boussinesq-Burger equation at various collocation points of x with $t = 1.5$.

x	u_{approx}	v_{approx}	u_{exact}	v_{exact}	$\lvert u_{exact} - u_{approx}\rvert$	$\lvert v_{exact} - v_{approx}\rvert$
0.015625	-0.259065	-0.124493	-0.259056	-0.124836	8.80296E-6	3.42744E-4
0.046875	-0.263011	-0.125112	-0.262954	-0.124664	5.70662E-5	4.47176E-4
0.078125	-0.26693	-0.127938	-0.266847	-0. 124432	8.3701E-5	3.50573E-3
0.109375	-0.270783	-0.130401	-0.270731	-0.12414	5017642E-5	6.26042E-3
0.140625	-0.27462	-0.130316	-0.274605	-0.123789	1.53796E-5	6.52672E-3
0.171875	-0.278509	-0.129233	-0.278467	-0.123379	4015244E-5	5.85325E-3
0.203125	-0.282425	-0.130215	-0.282315	-0.122911	1.09137E-4	7.30402E-3
0.234375	-0.286287	-0.13323	-0.286148	-0.122387	1.38385E-4	1.08429E-2
0.265625	-0.290075	-0.135067	-0.289964	-0.121806	1.10593E-4	1.32615E-2
0.296875	-0.29386	-0.134111	-0.293761	-0.12117	9.95855E-5	1.29412E-2
0.328125	-0.2977	-0.132864	-0.297537	-0.120481	1.63072E-4	1.23836E-2
0.359375	-0.301544	-0.134297	-0.30129	-0.119739	2.54057E-4	1.45584E-2
0.390625	-0.305308	-0.137332	-0.30502	-0.118946	2.88254E-4	1.83867E-2
0.421875	-0.308995	-0.138317	-0.308724	-0.118103	2.71105E-4	2.0214E-2
0.453125	-0.312694	-0.13649	-0.312401	-0.117212	2.9305E-4	1.92774E-2
0.484375	-0.316444	-0.135265	-0.316049	-0.116275	3.94684E-4	1.89901E-2
0.515625	-0.320169	-0.137198	-0.319667	-0.115293	5.02116E-4	2.19049E-2
0.546875	-0.323789	-0.14004	-0.323254	-0.114268	5.343E-4	2.57723E-2
0.578125	-0.327335	-0.139964	-0.326809	-0.113201	5.25882E-4	2.6763E-2
0.609375	-0.330908	-0.137341	-0.330329	-0.112095	5.78576E-4	2.52464E-2
0.640625	-0.334522	-0.136362	-0.333814	-0.11095	7.07734E-4	2.54116E-2
0.671875	-0.338077	-0.138802	-0.337263	-0.10977	8.13912E-4	2.90315E-2
0.703125	-0.341503	-0.141194	-0.340674	-0.108556	8.28591E-4	3.26377E-2
0.734375	-0.344867	-0.13989	-0.344047	-0.10731	8.19196E-4	3.25803E-2
0.765625	-0.348272	-0.136635	-0.347381	-0.106034	8.9094E-4	3.06012E-2
0.796875	-0.3517	-0.136149	-0.350674	-0.104729	1.02614E-3	3.14198E-2
0.828125	-0.35503	-0.139047	-0.353926	-0.103399	1.10396E-3	3.56484E-2
0.859375	-0.358215	-0.140701	-0.357136	-0.102044	1.07891E-3	3.86573E-2
0.890625	-0.361357	-0.138083	-0.360304	-0.100666	1.05272E-3	3.74164E-2
0.921875	-0.364553	-0.134517	-0.363428	-0.0992682	1.1247E-3	3.5249E-2
0.953125	-0.367749	-0.135059	-0.366508	-0.0978518	1.2409E-3	3.72076E-2
0.984375	-0.370825	-0.1395	-0.369543	-0.0964187	1.28116E-3	4.30817E-2

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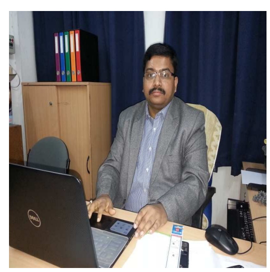

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