Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100415

Some Asymptotic Approximation Properties of One Boundary-Value Problem with Singular Interior Point

K. Aydemir[1,*] *and O. Sh. Mukhtarov*[1,2]

[1] Faculty of Science, Giresun University, 28100 Giresun, Turkey
[2] Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan

Received: 28 Mar. 2016, Revised: 14 May 2016, Accepted: 15 May 2016
Published online: 1 Jul. 2016

Abstract: In this study by applying an own technique we investigate some asymptotic approximation properties of new type discontinuous boundary-value problems, which consists of a Sturm-Liouville equation together with eigenparameter-dependent boundary and transmission conditions.

Keywords: Boundary-value problems, eigenvalue, eigenfunction, asymptotic formulas, singular point.

1 Introduction

Sturm-Liouville problems which contained spectral parameter in boundary conditions form an important part of the spectral theory of boundary value problems. This type problems has a lot of applications in mechanics and physics (see [7,9,15] and references cited therein). Variety theoretic question of such type problems was intensively studied for quite a long time. In the recent years, there has been increasing interest of this kind problems which also may have discontinuities in the solution or its derivative at interior points (see [1,2,3,4,5, 6,8,10,11,12,16,17]). Such problems are connected with discontinuous material properties, such as heat and mass transfer, vibrating string problems when the string loaded additionally with points masses, diffraction problems [9, 15] and varied assortment of physical transfer problems. In this paper we shall investigated some asymptotic approximation properties of one discontinuous Sturm-Liouville problem for which the eigenvalue parameter takes part in both differential equation and boundary conditions and two supplementary transmission conditions at one interior point are added to boundary conditions. In particular, we find asymptotic approximation formulas for eigenvalues and corresponding eigenfunctions. The problems with transmission conditions arise in mechanics, such as thermal conduction problems for a thin laminated plate, which studied in [13]. This class of problems essentially differs from the classical case, and its investigation requires a specific approach based on the method of separation of variables. Note that, eigenfunctions of our problem are discontinuous at the one inner point of the considered interval, in general.

2 Statement of the problem and construction of the fundamental solutions

Let us consider the boundary value problem, consisting of the differential equation

$$Ty := -y''(x) + q(x)y(x) = \lambda y(x) \tag{1}$$

on $[a,c)\cup(c,b]$, with eigenparameter- dependent boundary conditions

$$\tau_1(y) := \alpha_{10}y(a) + \alpha_{11}y'(a) = 0, \tag{2}$$

$$\tau_2(y) := \alpha_{20}y(b) - \alpha_{21}y'(b) + \lambda(\alpha'_{20}y(b) - \alpha'_{21}y'(b)) = 0 \tag{3}$$

and the transmission conditions

$$\tau_3(y) := \beta^-_{11}y'(c-) + \beta^-_{10}y(c-) + \beta^+_{11}y'(c+) + \beta^+_{10}y(c+) = 0, \tag{4}$$

$$\tau_4(y) := \beta^-_{21}y'(c-) + \beta^-_{20}y(c-) + \beta^+_{21}y'(c+) + \beta^+_{20}y(c+) = 0, \tag{5}$$

* Corresponding author e-mail: kadriyeaydemr@gmail.com

where the potential $q(x)$ is real-valued function, which continuous in each of the intervals $[a,c)$ and$(c,b]$ and has a finite limits $q(c\mp 0)$, λ is a complex spectral parameter, α_{ij}, $\beta_{ij}^{\pm}$, $(i=1,2$ and $j=0,1)$, α_{ij}' $(i=2$ and $j=0,1)$ are real numbers. This problem differs from the usual regular Sturm-Liouville problem in the sense that the eigenvalue parameter λ are contained in both differential equation and boundary conditions and two supplementary transmission conditions at one interior point are added to boundary conditions. Let

$$A_0 = \begin{bmatrix} \alpha_{21} & \alpha_{20} \\ \alpha_{21}' & \alpha_{20}' \end{bmatrix} \text{ and } A = \begin{bmatrix} \beta_{10}^- & \beta_{11}^- & \beta_{10}^+ & \beta_{11}^+ \\ \beta_{20}^- & \beta_{21}^- & \beta_{20}^+ & \beta_{21}^+ \end{bmatrix}.$$

Denote the determinant of the matrix A_0 by Δ_0 and the determinant of the k-th and j-th columns of the matrix A by Δ_{kj}. Note that throughout this study we shall assume that $\Delta_0 > 0$, $\Delta_{12} > 0$ and $\Delta_{34} > 0$. With a view to constructing the characteristic function we shall define two basic solution $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ by the following procedure.

At first, let us consider the initial-value problem on the left part $[a,c)$ of the considered interval $[a,c)\cup(c,b]$

$$-y'' + q(x)y = \lambda y, \quad x \in [a,c) \tag{6}$$

$$y(a) = \alpha_{11}, \ y'(a) = -\alpha_{10} \tag{7}$$

By virtue of well-known existence and uniqueness theorem of ordinary differential equation theory this initial-value problem for each λ has a unique solution $\varphi_1(x,\lambda)$. Moreover [[14], theorem 7] this solution is an entire function of λ for each fixed $x\in[a,c)$. By using we shall investigate the differential equation (6) on $(c,b]$ together with special type initial conditions

$$y(c) = \frac{1}{\Delta_{12}}(\Delta_{23}\varphi_1(c,\lambda) + \Delta_{24}\varphi_1'(c,\lambda)) \tag{8}$$

$$y'(c) = \frac{-1}{\Delta_{12}}(\Delta_{13}\varphi_1(c,\lambda) + \Delta_{14}\varphi_1'(c,\lambda)). \tag{9}$$

Define a sequence of functions $y_n(x,\lambda), n=0,1,2,\ldots$ on interval $(c,b]$ by the following equations:

$$\begin{aligned} y_0(x,\lambda) &= \frac{1}{\Delta_{12}}[(\Delta_{23}+c\Delta_{13})\varphi_1(c,\lambda) + (\Delta_{24}+c\Delta_{14})\varphi_1'(c,\lambda) \\ &\quad + (-\Delta_{13}\varphi_1(c,\lambda) + \Delta_{14}\varphi_1'(c,\lambda))x] \\ y_n(x,\lambda) &= y_0(x,\lambda) + \int_c^x (x-z)(q(z)-\lambda)y_{n-1}(z,\lambda)dz, \end{aligned}$$

for n=1,2,.... It is easy to see that each of $y_n(x,\lambda)$ is an entire function of λ for each $(c,b]$ Consider the series

$$y_0(x,\lambda) + \sum_{n=1}^{\infty}(y_n(x,\lambda) - y_{n-1}(x,\lambda)) \tag{10}$$

Denoting $q_1 = \max_{x\in(c,b]}|q(x)|$ and $Y(\lambda) = \max_{x\in(c,b]}|y_0(x,\lambda)|$, we get $|y_n(x,\lambda)-y_{n-1}(x,\lambda)| \le \frac{1}{(2n)!}Y(\lambda)(q_1+|\lambda|^n)(x-c)^{2n}$ for each $n=1,2,\ldots$.

Because of this inequality the series (10) is uniformly convergent with respect to the variable x on $(c,b]$, and with respect to the variable λ on every closed bar $|\lambda|\le R$. Let $\varphi_2(x,\lambda)$ be the sum of the series (10). Consequently $\varphi_2(x,\lambda)$ is an entire function of λ for each fixed $x\in(c,b]$. Since for $n\ge 2$

$$y_n'(x,\lambda) - y_{n-1}'(x,\lambda) = \int_0^x (q(z)-\lambda)(y_{n-1}(z,\lambda) - y_{n-2}(z,\lambda))dz$$

and

$$y_n''(x,\lambda) - y_{n-1}''(x,\lambda) = (q(x)-\lambda)(y_{n-1}(x,\lambda) - y_{n-2}(x,\lambda))$$

the first and second differentiated series also converge uniformly with respect to x. Taking into account the last equality we have

$$\begin{aligned} \varphi_2''(x,\lambda) &= y_1''(x,\lambda) + \sum_{n=2}^{\infty}(y_n''(x,\lambda) - y_{n-1}''(x,\lambda)) \\ &= (q(x)-\lambda)y_1(x,\lambda) \\ &\quad + \sum_{n=2}^{\infty}(q(x)-\lambda)(y_n(x,\lambda) - y_{n-1}(x,\lambda)) \\ &= (q(x)-\lambda)\varphi_2(x,\lambda), \end{aligned}$$

so $\varphi_2(x,\lambda)$ satisfies the equation (6). Moreover, since each $y_n(x,\lambda)$ satisfies the initial conditions (8) and (9), then the function $\varphi_2(x,\lambda)$ satisfies the initial conditions (8) and (9). Consequently, the function $\varphi(x,\lambda)$ defined by

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda) & \text{for } x\in[a,c) \\ \varphi_2(x,\lambda) & \text{for } x\in(c,b]. \end{cases} \tag{11}$$

satisfies equation (1), the first boundary condition (2) and the both transmission conditions (4) and (5). By applying the same technique we can prove that for any $\lambda\in C$ the differential equation (1) has such solution

$$\psi(x,\lambda) = \begin{cases} \psi_1(x,\lambda) & \text{for } x\in[a,c) \\ \psi_2(x,\lambda) & \text{for } x\in(c,b]. \end{cases} \tag{12}$$

which satisfies the initial condition (3), the both transmission conditions (4) − (5) for each $x\in[a,c)\cup(c,b]$ and is an entire function of λ for each fixed $x\in[a,c)\cup(c,b]$. Below, for shorting we shall use also notations; $\varphi_i(x,\lambda) := \varphi_{i\lambda}$, $\psi_i(x,\lambda) := \psi_{i\lambda}$.

3 Some asymptotic approximation formulas for fundamental solutions

Let $\lambda = s^2$. By applying the method of variation of parameters we can prove that the next integral and integro-differential equations are hold for $k=0$ and

$k=1.$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = \alpha_{11}\frac{d^k}{dx^k}\cos[s(x-a)] - \frac{a_{10}}{s}\frac{d^k}{dx^k}\sin[s(x-a)] + \frac{1}{s}\int_a^x \frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\varphi_1(z,\lambda)dz \quad (13)$$

$$\frac{d^k}{dx^k}\psi_{1\lambda}(x) = -\frac{1}{\Delta_{34}}(\Delta_{14}\psi_2(c,\lambda)+\Delta_{24}\psi_2'(c,\lambda)) \times\frac{d^k}{dx^k}\cos[s(x-c)] + \frac{1}{s\Delta_{34}}(\Delta_{13}\psi_2(c,\lambda) + \Delta_{23}\psi_2'(c,\lambda))\frac{d^k}{dx^k}\sin[s(x-c)] + \frac{1}{s}\int_x^c \times\frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\psi_1(z,\lambda)dz \quad (14)$$

for $x\in[a,c)$ and

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = \frac{1}{\Delta_{12}}(\Delta_{23}\varphi_1(c,\lambda)+\Delta_{24}\varphi_1'(c,\lambda)) \times\frac{d^k}{dx^k}\cos[s(x-c)] - \frac{1}{s\Delta_{12}}(\Delta_{13}\varphi_1(c,\lambda) + \Delta_{14}\varphi_1'(c,\lambda))\frac{d^k}{dx^k}\sin[s(x-c)] + \frac{1}{s}\int_c^x \times\frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\varphi_2(z,\lambda)dz \quad (15)$$

$$\frac{d^k}{dx^k}\psi_{2\lambda}(x) = (\alpha_{21}+\lambda\alpha_{21}')\frac{d^k}{dx^k}\cos[s(x-b)] + \frac{1}{s}(\alpha_{20}+\lambda\alpha_{20}')\frac{d^k}{dx^k}\sin[s(x-b)] + \frac{1}{s}\int_x^b \frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\psi_1(z,\lambda)dz \quad (16)$$

for $x\in(c,b]$. Now we are ready to prove the following theorems.

Theorem 1.*Let $\lambda=s^2$, $Ims=t$. Then if $\alpha_{11}\neq 0$*

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = \alpha_{11}\frac{d^k}{dx^k}\cos[s(x-a)] + O(|s|^{k-1}e^{|t|(x-a)}) \quad (17)$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = -\frac{\Delta_{24}}{\Delta_{12}}\alpha_{11}s\sin[s(c-a)]\frac{d^k}{dx^k}\cos[s(x-c)] + O(|s|^k e^{|t|(x-a)}) \quad (18)$$

as $|\lambda|\to\infty$, while if $\alpha_{11}=0$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -\frac{\alpha_{10}}{s}\frac{d^k}{dx^k}\sin[s(x-a)] + O(|s|^{k-2}e^{|t|(x-a)}) \quad (19)$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = -\frac{\Delta_{24}}{\Delta_{12}}\alpha_{10}\cos[s(c-a)]\frac{d^k}{dx^k}\cos[s(x-c)] + O(|s|^{k-1}e^{|t|(x-a)}) \quad (20)$$

as $|\lambda|\to\infty$ ($k=0,1$). Each of this asymptotic equalities hold uniformly for x.

*Proof.*The asymptotic formulas for (17) in (18) follows immediately from the Titchmarsh's Lemma on the asymptotic behavior of $\varphi_\lambda(x)$ ([14], Lemma 1.7). But the corresponding formulas for $\varphi_{2\lambda}(x)$ need individual consideration. Let $\alpha_{11}\neq 0$. Substituting (17) in (15) we have the next "asymptotic integral equation"

$$\varphi_{2\lambda}(x) = \frac{1}{\Delta_{12}}\alpha_{11}\Big[\Delta_{23}\cos s(c-a)\cos s(x-c) - \Delta_{24}s\sin s(c-a)\cos s(x-c) - \frac{\Delta_{13}}{s}\cos s(c-a)\sin s(x-c) + \frac{\Delta_{14}}{s}\sin s(c-a)\sin s(x-c)\Big] + \frac{1}{s}\int_c^x \sin[s(x-z)]q(z)\phi_2(z,\lambda)dz + O(e^{|t|(x-a)}) \quad (21)$$

Multiplying by $e^{-|t|(x-a)}$ and denoting $Y(x,\lambda)=e^{-|t|(x-a)}\varphi_{2\lambda}(x)$ we get

$$Y(x,\lambda) = \frac{1}{\Delta_{12}}\alpha_{11}e^{-|t|(x-a)}\Big[\Delta_{23}\cos s(c-a)\cos s(x-c) - \Delta_{24}s\sin s(c-a)\cos s(x-c) - \frac{\Delta_{13}}{s}\cos s(c-a)\sin s(x-c) + \frac{\Delta_{14}}{s}\sin s(c-a)\sin s(x-c)\Big] + \frac{1}{s}\int_c^x \sin[s(x-z)]q(z)e^{-|t|(x-z)}Y(z,\lambda)dz + O(1) \quad (22)$$

Denoting $Y(\lambda)=\max_{x\in[c,b]}|Y(x,\lambda)|$ and $\widetilde{q}=\int_c^b|q(z)|dz$ from the last equation we have $Y(\lambda)=O(1)$ as$|\lambda|\to\infty$, so $\varphi_{2\lambda}(x)=O(e^{|t|(x-a)})$. Substituting in (21) we obtain (18) for the case $k=0$. The case $k=1$ of the (18) follows at once on differentiating (15) and making the same procedure as in the case $k=0$. The proof of (19) in (20) is similar.

Similarly we can easily obtain the following Theorem for $\psi_i(x,\lambda)(i=1,2)$.

Theorem 2.*Let $\lambda=s^2$, $Ims=t$. Then if $\alpha_{21}'\neq 0$*

$$\frac{d^k}{dx^k}\psi_{2\lambda}(x) = \alpha_{21}'s^2\frac{d^k}{dx^k}\cos[s(b-x)] + O(|s|^{k+1}e^{|t|(b-x)}) \quad (23)$$

$$\frac{d^k}{dx^k}\psi_{1\lambda}(x) = -\frac{\Delta_{24}}{\Delta_{34}}\alpha_{21}'s^3\sin[s(b-c)]\frac{d^k}{dx^k}\cos[s(x-c)] + O(|s|^{k+2}e^{|t|(b-x)}) \quad (24)$$

as $|\lambda|\to\infty$, while if $\alpha_{21}'=0$

$$\frac{d^k}{dx^k}\psi_{2\lambda}(x) = -a_{20}'s\frac{d^k}{dx^k}\sin[s(b-x)] + O(|s|^k e^{|t|(b-x)}) \quad (25)$$

$$\frac{d^k}{dx^k}\psi_{1\lambda}(x) = -\frac{\Delta_{24}}{\Delta_{34}}\alpha_{20}'s^2\cos[s(b-c)]\frac{d^k}{dx^k}\cos[s(x-c)] + O(|s|^{k+1}e^{|t|(b-x)}) \quad (26)$$

as $|\lambda|\to\infty$ ($k=0,1$). Each of this asymptotic equalities hold uniformly for x.

4 Asymptotic behaviour of eigenvalues and corresponding eigenfunctions

It is well-known from ordinary differential equation theory that the Wronskians $W[\varphi_{1\lambda},\psi_{1\lambda}]_x$ and $W[\varphi_{2\lambda},\psi_{2\lambda}]_x$ are independent of variable x. Denoting $w_i(\lambda)=W[\varphi_{i\lambda},\psi_{i\lambda}]_x$ we have

$$\begin{aligned} w_2(\lambda) &= \varphi_2(c,\lambda)\psi_2'(c,\lambda)-\varphi_2'(,\lambda)\psi_2(c,\lambda) \\ &= \frac{\Delta_{34}}{\Delta_{12}}(\varphi_1(c,\lambda)\psi_1'(c,\lambda)-\varphi_1'(c,\lambda)\psi_1(c,\lambda)) \\ &= \frac{\Delta_{34}}{\Delta_{12}}w_1(\lambda) \end{aligned}$$

Denote $\omega(\lambda):=\Delta_{34}\omega_1(\lambda)=\Delta_{12}\ \omega_2(\lambda)$.

By the same technique as in [11] we can prove the following theorem

Theorem 3. *The eigenvalues of the problem (1)-(5) are consist of the zeros of the function $w(\lambda)$.*

Now by modifying the standard method we prove that all eigenvalues of the problem $(1)-(5)$ are real.

Theorem 4. *All eigenvalues of the problem $(1)-(5)$ are real.*

Proof. Let λ_0 be eigenvalue and y_0 be eigenfunction corresponding to this eigenvalue. Denoting $R_b(y):=\alpha_{20}y(b)-\alpha_{21}y'(b)$, $R_b'(y):=\alpha_{20}'y(b)-\alpha_{21}'y'(b)$ by two partial integration we have

$$\begin{aligned} &\Delta_{12}\int_a^c(\lambda_0y_0)(x)\overline{y_0(x)}dx+\Delta_{34}\int_c^b(\lambda_0y_0)(x)\overline{y_0(x)}dx \\ &+\frac{\Delta_{34}}{\Delta_0}R_b(y_0)\overline{R_b'(y_0)}-\{\Delta_{12}\int_a^c y_0(x)\overline{(\lambda_0y_0)(x)}dx \\ &+\Delta_{34}\int_c^b y_0(x)\overline{(\lambda_0y_0)(x)}dx+\frac{\Delta_{34}}{\Delta_0}R_b'(y_0)\overline{R_b(y_0)}\} \\ =\ & W[y_0,\overline{y_0};c-]-\Delta_{12}\,W[y_0,\overline{y_0};a]+\Delta_{34}\,W[y_0,\overline{y_0};b] \\ &-\Delta_{34}\,W[y_0,\overline{y_0};c+]+\frac{\Delta_{34}}{\Delta_0}R_b(y_0)\overline{R_b'(y_0)} \\ &-\frac{\Delta_{34}}{\Delta_0}R_b'(y_0)\overline{R_b(y_0)} \end{aligned} \tag{27}$$

From the boundary conditions (2) it is follows obviously that

$$W(y_0,\overline{y_0};a)=0. \tag{28}$$

The direct calculation gives

$$R_b(y_0)\overline{R_b'(y_0)}-R_b'(y_0)\overline{R_b(y_0)}=-\Delta_0W(y_0,\overline{y_0};b) \tag{29}$$

and

$$W(y_0,\overline{y_0};c-)=\tfrac{\Delta_{34}}{\Delta_{12}}\,W(y_0,\overline{y_0};c+). \tag{30}$$

Substituting (28), (29) and (30) in (27) we obtain the equality

$$(\lambda_0-\overline{\lambda_0})[\Delta_{12}\int_a^c(y_0(x))^2dx+\Delta_{34}\int_c^b(y_0(x))^2dx]=0$$

Thus, we get $\lambda_0=\overline{\lambda_0}$ since $\Delta_{12}>0$ and $\Delta_{34}>0$. Consequently all eigenvalues of the problem $(1)-(5)$ are real.

Since the Wronskians of $\varphi_{2\lambda}(x)$ and $\psi_{2\lambda}(x)$ are independent of x, in particular, by putting $x=a$ we have

$$\begin{aligned} w(\lambda) &= \varphi_1(a,\lambda)\psi_1'(a,\lambda)-\varphi_1'(a,\lambda)\psi_1(a,\lambda) \\ &= \alpha_{11}\psi_1'(a,\lambda)+\alpha_{10}\psi_1(a,\lambda). \end{aligned} \tag{31}$$

Let $\lambda=s^2$, $Ims=t$. By substituting (23) and (26) in (31) we obtain easily the following asymptotic representations

(i) If $\alpha_{21}'\neq0$ and $\alpha_{11}\neq0$, then

$$\begin{aligned} w(\lambda) &= \Delta_{24}\alpha_{11}\alpha_{21}'s^4\sin[s(b-c)]\sin[s(a-c)] \\ &+O\left(|s|^3e^{|t|(b-a)}\right) \end{aligned} \tag{32}$$

(ii) If $\alpha_{21}'\neq0$ and $\alpha_{11}=0$, then

$$\begin{aligned} w(\lambda) &= -\Delta_{24}\alpha_{10}\alpha_{21}'s^3\sin[s(b-c)]\cos[s(a-c)] \\ &+O\left(|s|^2e^{|t|(b-a)}\right) \end{aligned} \tag{33}$$

(iii) If $\alpha_{21}'=0$ and $\alpha_{11}\neq0$, then

$$\begin{aligned} w(\lambda) &= \Delta_{24}\alpha_{11}\alpha_{20}'s^3\cos[s(b-c)]\sin[s(a-c)] \\ &+O\left(|s|^2e^{|t|(b-a)}\right) \end{aligned} \tag{34}$$

(iv) If $\alpha_{21}'=0$ and $\alpha_{11}=0$, then

$$\begin{aligned} w(\lambda) &= -\Delta_{24}\alpha_{10}\alpha_{20}'s^2\cos[s(b-c)]\cos[s(a-c)] \\ &+O\left(|s|e^{|t|(b-a)}\right) \end{aligned} \tag{35}$$

Now we are ready to derived the needed asymptotic formulas for eigenvalues and eigenfunctions.

Theorem 5. *The boundary-value-transmission problem (1)-(5) has an precisely numerable many real eigenvalues, whose behavior may be expressed by two sequence $\{\lambda_{n,1}\}$ and $\{\lambda_{n,2}\}$ with following asymptotic as $n\to\infty$* ***(i)*** *If $\alpha_{21}'\neq0$ and $\alpha_{11}\neq0$, then*

$$s_{n,1}=(n-2)\frac{\pi}{b-c}+O\left(\frac{1}{n}\right),\ s_{n,2}=\frac{n\pi}{a-c}+O\left(\frac{1}{n}\right), \tag{36}$$

(ii) *If $\alpha_{21}'\neq0$ and $\alpha_{11}=0$, then*

$$s_{n,1}=(n+\frac{1}{2})\frac{\pi}{b-c}+O\left(\frac{1}{n}\right),\ s_{n,2}=\frac{\pi}{a-c}(n-1)+O\left(\frac{1}{n}\right), \tag{37}$$

(iii) *If $\alpha_{21}'=0$ and $\alpha_{11}\neq0$, then*

$$s_{n,1}=(n-1)\frac{\pi}{b-c}+O\left(\frac{1}{n}\right),\ s_{n,2}=\frac{\pi}{a-c}(n+\frac{1}{2})+O\left(\frac{1}{n}\right), \tag{38}$$

(iv) *If $\alpha_{21}'=0$ and $\alpha_{11}=0$, then*

$$s_{n,1}=(n-\frac{1}{2})\frac{\pi}{b-c}+O\left(\frac{1}{n}\right),\ s_{n,2}=\frac{\pi}{a-c}(n+\frac{1}{2})+O\left(\frac{1}{n}\right) \tag{39}$$

where $\lambda_{n,1}=s_{n,1}^2$, $\lambda_{n,2}=s_{n,2}^2$

*Proof.*Let $\alpha'_{21} \neq 0$ and $\alpha_{11} \neq 0$. Denote by $\widetilde{w_1}(s)$ and $\widetilde{w_2}(s)$ the leading term and O-term of the right of (32) by $\widetilde{w_1}(s)$ and $\widetilde{w_2}(s)$ respectively, i.e. we let $w(s^2) = \widetilde{w_1}(s) + \widetilde{w_2}(s)$. Let $\{\varepsilon_{n,1}\}$ and $\{\varepsilon_{n,2}\}$ be any two sequence, for which $0 < \varepsilon_{n,i} < \frac{1}{2}$ and let $\Gamma_{n,i}(i=1,2)$ are the bounds of the domains

$$\{s = \sigma + it \mid |t|, |\sigma| \leq \pi(n + \varepsilon_{n,i})\}\,(i=1,2).$$

We can choose the sequences $\{\varepsilon_{n,1}\}$ and $\{\varepsilon_{n,2}\}$ so that

$$\pi(n+\varepsilon_{n,1}) \neq \pi m \text{ and } \pi(n+\varepsilon_{n,2}) \neq \pi m$$

for every n and m. Taking in view this and the fact that $0 < \varepsilon_{n,i} < \frac{1}{2}$, it is easy to show that $|\widetilde{w_1}(s)| > |\widetilde{w_2}(s)|$on both $\Gamma_{n,1}$ and $\Gamma_{n,2}$ for sufficiently large n. By applying Rouche's theorem on a sufficiently large contours $\Gamma_{n,i}$ it follows that $w(\lambda)$ has the same number zeros inside the contours $\Gamma_{n,i}$ as the leading term $\widetilde{w_1}(s)$. Hence, if $\lambda_0 < \lambda_1 < \lambda_2 \ldots$ are the zeros of $w(\lambda)$ and $s_n = \lambda_n$, we have the needed asymptotic formulas (36). The proofs of the other formulas are similar.

Using this asymptotic expression of eigenvalues we can easily obtain the corresponding asymptotic expressions for eigenfunctions of the problem (1)-(5). Denote the corresponding eigenfunction of the problem by

$$\widetilde{\varphi}_{n,i} = \left\{ \begin{array}{ll} \varphi_{1\lambda_{n,i}}(x) & \text{for } x \in [a,c) \\ \varphi_{2\lambda_{n,i}}(x,) & \text{for } x \in (c,b]. \end{array} \right.$$

Recalling that $\varphi_{\lambda_{n,i}}(x)$ is an eigenfunction according to the eigenvalue λ_n, by putting (36) in the (17) for $k=0$ we get

$$\widetilde{\varphi}_{n,1}(x) = \begin{cases} \alpha_{11}\cos\left[(n-2)\pi\frac{(x-a)}{(b-c)}\right] + O\left(\frac{1}{n}\right) \quad \text{for } x \in [a,c) \\ -\alpha_{11}\Delta_{24}\frac{\pi(n-2)}{\Delta_{12}(b-c)}\sin\left[(n-1)\pi\frac{(c-a)}{(b-c)}\right] \\ \times\cos\left[(n-2)\pi\frac{(x-c)}{(b-c)}\right] + O(1) \;\; \text{for } x \in (c,b] \end{cases}$$

if $\alpha'_{21} \neq 0$ and $\alpha_{11} \neq 0$. Similarly, by putting (36) in the (18) for $k=0$ yields

$$\widetilde{\varphi}_{n,2}(x) = \begin{cases} \alpha_{11}\cos\left[n\pi\frac{(x-a)}{(a-c)}\right] + O\left(\frac{1}{n}\right) \quad \text{for } x \in [a,c) \\ -\alpha_{11}\Delta_{24}\frac{n\pi}{\Delta_{12}(a-c)}\sin[n\pi]\cos\left[n\pi\frac{(x-c)}{(a-c)}\right] \\ +O(1) \;\; \text{for } x \in (c,b] \end{cases}$$

Similar expressions are as follows:
If $\alpha'_{21} \neq 0$ and $\alpha_{11} = 0$, then

$$\widetilde{\varphi}_{n,1}(x) = \begin{cases} -\alpha_{10}\frac{(b-c)}{\pi(n+\frac{1}{2})}\sin\left[\pi(n+\frac{1}{2})\frac{(x-a)}{(b-c)}\right] + O\left(\frac{1}{n^2}\right) \\ \text{for } x \in [a,c) \\ -\alpha_{10}\frac{\Delta_{24}}{\Delta_{12}}\cos\left[\pi(n+\frac{1}{2})\frac{(c-a)}{(b-c)}\right]\cos\left[\pi(n+\frac{1}{2})\frac{(x-c)}{(b-c)}\right] \\ +O\left(\frac{1}{n}\right) \quad \text{for } x \in (c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x) = \begin{cases} -\alpha_{10}\frac{(a-c)}{\pi(n-1)}\sin\left[(n-1)\pi\frac{(x-a)}{(a-c)}\right] + O\left(\frac{1}{n^2}\right) \\ \text{for } x \in [a,c) \\ -\alpha_{10}\frac{\Delta_{24}}{\Delta_{12}}\cos[\pi(n-1)]\cos\left[\pi(n-1)\frac{(x-c)}{(a-c)}\right] \\ +O\left(\frac{1}{n}\right) \quad \text{for } x \in (c,b] \end{cases}$$

If $\alpha'_{21} = 0$ and $\alpha_{11} \neq 0$, then

$$\widetilde{\varphi}_{n,1}(x) = \begin{cases} \alpha_{11}\cos\left[\pi(n-1)\frac{(x-a)}{(b-c)}\right] + O\left(\frac{1}{n}\right) \;\; \text{for } x \in [a,c) \\ -\alpha_{11}\Delta_{24}\frac{\pi(n-1)}{\Delta_{12}(b-c)}\sin\left[(n-1)\pi\frac{(c-a)}{(b-c)}\right] \\ \times\cos\left[(n-1)\pi\frac{(x-c)}{(b-c)}\right] + O(1) \;\; \text{for } x \in (c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x) = \begin{cases} \alpha_{11}\cos\left[\pi(n+\frac{1}{2})\frac{(x-a)}{(a-c)}\right] + O\left(\frac{1}{n}\right) \\ \quad \text{for } x \in [a,c) \\ -\alpha_{11}\pi\frac{\Delta_{24}(n+\frac{1}{2})}{\Delta_{12}(a-c)}\sin\left[\pi(n+\frac{1}{2})\right]\cos\left[(n+\frac{1}{2})\pi\frac{(x-c)}{(a-c)}\right] \\ +O(1) \;\; \text{for } x \in (c,b] \end{cases}$$

If $\alpha'_{21} = 0$ and $\alpha_{11} = 0$, then

$$\widetilde{\varphi}_{n,1}(x) = \begin{cases} -\alpha_{10}\frac{(b-c)}{(n-\frac{1}{2})\pi}\sin\left[(n-\frac{1}{2})\pi\frac{(x-a)}{(b-c)}\right] + O\left(\frac{1}{n^2}\right) \\ \text{for } x \in [a,c) \\ -\alpha_{10}\frac{\Delta_{24}}{\Delta_{12}}\cos\left[(n-\frac{1}{2})\pi\frac{(c-a)}{(b-c)}\right]\cos\left[(n-\frac{1}{2})\pi\frac{(x-c)}{(b-c)}\right] \\ +O\left(\frac{1}{n}\right) \quad \text{for } x \in (c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x) = \begin{cases} -\alpha_{10}\frac{(a-c)}{(n+\frac{1}{2})\pi}\sin\left[(n+\frac{1}{2})\pi\frac{(x-a)}{(a-c)}\right] + O\left(\frac{1}{n^2}\right), \\ \text{for } x \in [a,c) \\ -\alpha_{10}\frac{\Delta_{24}}{\Delta_{12}}\cos\left[(n+\frac{1}{2})\pi\right]\cos\left[(n+\frac{1}{2})\pi\frac{(x-c)}{(a-c)}\right] \\ +O\left(\frac{1}{n}\right) \;\; \text{for } x \in (c,b] \end{cases}$$

All this asymptotic approximations are hold uniformly for x.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] N. Altınışık, M. Kadakal and O. Sh. Mukhtarov, Eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problems with eigenparameter dependent boundary conditions, Acta Mathematica Hungarica, **102**, 159-175(2004).

[2] J. Ao, J. Sun and M. Zhang, Matrix representations of Sturm-Liouville problems with transmission conditions, Comput. Math. Appl., **63**, 1335-1348(2012).

[3] K. Aydemir and O. Sh. Mukhtarov, Green's Function Method for Self-Adjoint Realization of Boundary-Value Problems with Interior Singularities, Abstract and Applied Analysis, **2013**, Article ID 503267, 7 pages, 2013. doi:10.1155/2013/503267.

[4] E. Bairamov and E. Ugurlu, On the characteristic values of the real component of a dissipative boundary value transmission problem, Appl. Math. and Comp. **218**, 9657-9663(2012).

[5] A. Boumenir, Sampling the miss-distance and transmission function, J. Math. Anal. Appl., **310**, 197-208(2005).

[6] B. Chanane, *Sturm-Liouville problems with impulse effects*, Appl. Math. Comput., **190/1**, 610-626(2007).

[7] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edin. **77A**, 293-308(1977).

[8] M. Kandemir, O. Sh. Mukhtarov and Y.Y. Yakubov, Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter, Mediterr, J. Math. **6**, 317-338(2009).

[9] A. V. Likov and Y. A. Mikhailov, *The heory of Heat and Mass Transfer*, Qosenergaizdat(In Russian,1963).

[10] O. Sh. Mukhtarov and K. Aydemir, New type Sturm-Liouville problems in associated Hilbert spaces , Journal of Function Spaces and Applications, vol. 2014, Article ID 606815.

[11] O. Sh. Mukhtarov and M. Kadakal, Some spectral properties of one Sturm-Liouville type problem with discontinuous weight, Sib. Math. J., 46, 681-694(2005).

[12] O. Sh. Mukhtarov and S. Yakubov, *Problems for ordinary differential equations with transmission conditions*, Appl. Anal., Vol. 81, 1033-1064(2002).

[13] A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics, Oxford and New York, Pergamon, 1963.

[14] E. C. Titchmarsh, *Eigenfunctions Expansion Associated with Second Order Differential Equations I*, second edn. Oxford Univ. Press, London, 1962.

[15] I. Titeux, Ya. Yakubov, *Completeness of root functions for thermal condition in a strip with piecewise continuous coefficients*, Math. Models Methods Appl. Sci. **7**, 1035-1050(1997).

[16] E. Uğurlu and E. Bairamov, Dissipative operators with impulsive conditions, Math Chem, **51** 1670-1680(2013).

[17] A. Wang, Jiong Sun, Xiaoling Hao and Siqin Yao, Asymptotic behavior of a differential operator with discontinuities at two points, Math. Methods Appl. Sci. **34**, 373-383(2011).

Kadriye Aydemir received the PhD degree in Mathematics for Applied Mathematics at University of Gaziosmanpaşa. Her research interests are boundary value problems in the areas of applied mathematics. She is referee several international journals in the frame of Pure and Applied Mathematics. She has published research articles in reputed International Journals of Mathematical and Physical Sciences.

Oktay Sh. Mukhtarov is Professor in Department of Mathematics at Gaziosmanpaşa University. He received the PhD degree in ”Operator Theory” at University of Mathematics and Mechanical(Azerbaijan). He is referee and editor of several international journals in the frame of Pure and Applied mathematics. His main research interests are: Boundary value problem, operator theory and functional analysis.