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Solution of Higher-Order, Multipoint, Nonlinear Boundary Value Problems with High-Order Robin-Type Boundary Conditions by the Adomian Decomposition Method

Randolph Rach[1], *Jun-Sheng Duan*[2] *and Abdul-Majid Wazwaz*[3,*]

[1] Emeritus, 316 South Maple Street, Hartford, MI 49057-1225, U.S.A.
[2] School of Sciences, Shanghai Institute of Technology, Shanghai 201418, P.R. China
[3] Department of Mathematics, Saint Xavier University, Chicago, IL 60655, U.S.A.

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Abstract: In this paper, we propose a new modified recursion scheme for the approximate solution of higher-order, multipoint, nonlinear boundary value problems with higher-order Robin-type boundary conditions by the Adomian decomposition method. Our new approach utilizes all of the boundary conditions to derive an equivalent nonlinear Fredholm-Volterra integral equation before establishing the new modified recursion scheme for the solution. We solve several complex numerical examples obtaining a rapidly convergent sequence of analytic functions as the solution. In all cases investigated, we achieved an approximately exponential rate of convergence, thus confirming that only a few terms of the Adomian decomposition series can provide an accurate engineering model for parametric simulations.

Keywords: Adomian decomposition method, Adomian polynomials, boundary value problem, Robin boundary condition, nonlinear differential equation

1 Introduction

Boundary value problems (BVPs) for nonlinear ordinary differential equations are extensively applied in science and engineering. For example, the temperature distribution of convective straight fins with temperature-dependent thermal conductivity is modeled by a second-order nonlinear BVP [1]. The diffusion of oxygen in a spherical cell with Michaelis–Menten kinetics is modeled by a second-order nonlinear BVP [2]. The magneto hydrodynamics Jeffery–Hamel flow problem leads to a third-order nonlinear BVP [3]. The Euler-Bernoulli beam is described by a fourth-order nonlinear BVP, e.g. the beam-type nanoscale electromechanical system actuator [4]. In all of these applications, Robin and Robin-type boundary conditions can often be involved [2, 5–9].

The Adomian decomposition method (ADM) is a well-known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc. [10–20]. The ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering.

In the ADM, the solution $u(x)$ is represented by the Adomian decomposition series and the nonlinearity $Nu(x)$ is represented by the series of the Adomian polynomials that are tailored to the particular nonlinear function as

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \text{ and } Nu(x) = \sum_{n=0}^{\infty} A_n(x), \tag{1}$$

respectively, where the Adomian polynomials are dependent upon the solution components from $u_0(x)$ through $u_n(x)$, inclusively, i.e. $A_n(x) = A_n(u_0(x), \ldots, u_n(x))$. Adomian and Rach [21] published the definitional formula for the Adomian polynomials in 1983 for the simple nonlinearity $Nu(x) = F(x, u(x))$ and the differential multivariable nonlinearity

* Corresponding author e-mail: wazwaz@sxu.edu

$Nu = F(x,u(x),u'(x),u^{(p-1)}(x))$, or one-variable nonlinearity and differential multivariable nonlinearity, respectively, where F is assumed to be analytic, as

$$A_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}F(x,\sum_{n=0}^{\infty}\lambda^n u_n(x))\bigg|_{\lambda=0}, \quad (2)$$

$$A_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}F(x,\sum_{n=0}^{\infty}\lambda^n u_n(x),\sum_{n=0}^{\infty}\lambda^n u_n'(x),\ldots,\sum_{n=0}^{\infty}\lambda^n u_n^{(p-1)}(x))\bigg|_{\lambda=0}, \quad (3)$$

where λ is a grouping parameter of convenience, and similarly for more complex nonlinearities. We observe that only the dependent variables, such as the solution $u(x)$ and its derivatives, are parameterized while the independent variables such as x are not parameterized.

The Adomian polynomials are in the form

$$A_0 = F(x,u_0), \text{ and } A_n = \sum_{k=1}^{n} C_n^k F^{(k)}(x,u_0), \quad n \geq 1, \quad (4)$$

etc. Several convenient algorithms to readily generate the Adomian polynomials have been developed by Adomian and Rach [21, 22], Rach [23, 24], Wazwaz [16, 25], Abdelwahid [26], and several others [27–30]. Recently, Duan [31–34] has developed several new algorithms and subroutines for fast generation of the one-variable and the multi-variable Adomian polynomials.

For fast computer generation, we especially recommend Duan's new Corollary 3 algorithm [33] to generate the coefficients C_n^k and hence the Adomian polynomials quickly and to high orders as

$$\begin{aligned} &C_n^1 = u_n, \text{ for } n \geq 1, \text{ and} \\ &C_n^k = \tfrac{1}{n}\textstyle\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1}, \text{ for } 2 \leq k \leq n. \end{aligned} \quad (5)$$

It does not involve the differentiation operator, but only requires the analytic operations of addition and multiplication, which is eminently convenient for symbolic implementation by MATHEMATICA, MAPLE or MATLAB as well as for debugging. Furthermore, it has been timed to be one of the fastest subroutines on record using a commercially available laptop computer [33].

We note that the solution components $u_n(x)$ may be determined by one of several advantageous recursion schemes, which differ from one another by the choice of the initial solution component $u_0(x)$, beginning with the classic Adomian recursion scheme [35–38].

In next section, we first consider the second-order and third-order BVPs with Robin and Robin-type boundary conditions, respectively, as

$$u^{(2)}(x) + f\left(x,u^{(0)}(x),u^{(1)}(x)\right) = 0,$$

$$\alpha_{1,1}u^{(1)}(\xi_1) + \alpha_{1,2}u^{(0)}(\xi_1) = \beta_1,$$
$$\alpha_{2,1}u^{(1)}(\xi_2) + \alpha_{2,2}u^{(0)}(\xi_2) = \beta_2,$$

where $\xi_1 < \xi_2$, and

$$u^{(3)}(x) + f\left(x,u^{(0)}(x),u^{(1)}(x),u^{(2)}(x)\right) = 0,$$

$$\alpha_{1,1}u^{(2)}(\xi_1) + \alpha_{1,2}u^{(1)}(\xi_1) + \alpha_{1,3}u^{(0)}(\xi_1) = \beta_1,$$
$$\alpha_{2,1}u^{(2)}(\xi_2) + \alpha_{2,2}u^{(1)}(\xi_2) + \alpha_{2,3}u^{(0)}(\xi_2) = \beta_2,$$
$$\alpha_{3,1}u^{(2)}(\xi_3) + \alpha_{3,2}u^{(1)}(\xi_3) + \alpha_{3,3}u^{(0)}(\xi_3) = \beta_3,$$

where $\xi_1 \leq \xi_2 \leq \xi_3$ and $\xi_1 < \xi_3$. We emphasize that our formulation permits the number of distinct boundary points to be less than or equal to the order of the differential equation.

Then we consider the general case:

$$u^{(p)}(x) + f\left(x,u^{(0)}(x),\ldots,u^{(p-1)}(x)\right) = 0,$$

$$\sum_{i=1}^{p}\alpha_{j,i}u^{(p-i)}(\xi_j) = \beta_j, \; j = 1,2,\ldots,p, \; p \geq 2,$$

where $\xi_1 \leq \cdots \leq \xi_j \leq \cdots \leq \xi_p$ and $\xi_1 < \xi_p$.

2 Derivation of the equivalent integral equations and decomposition of the solutions

Case 1: the second-order differential equation

We begin by considering the second-order nonlinear differential equation

$$u^{(2)}(x) + f\left(x,u^{(0)}(x),u^{(1)}(x)\right) = 0, \quad (6)$$

subject to the classic Robin boundary conditions

$$\alpha_{1,1}u^{(1)}(\xi_1) + \alpha_{1,2}u^{(0)}(\xi_1) = \beta_1, \quad (7)$$
$$\alpha_{2,1}u^{(1)}(\xi_2) + \alpha_{2,2}u^{(0)}(\xi_2) = \beta_2, \quad (8)$$

where we have assumed that $\xi_1 < \xi_2$, i.e. there are two distinct boundary points. We assume that Eqs. (7) and (8) are two linearly independent equations in four unknowns. Also the nonlinear function f is assumed to be analytic in all of its arguments.

We rewrite Eq. (6) in Adomian's operator-theoretic form as

$$L^2u(x) + Nu(x) = 0, \quad (9)$$

where

$$L^2 = \frac{d^2}{dx^2}, \; Nu(x) = f\left(x,u^{(0)}(x),u^{(1)}(x)\right).$$

Applying the inverse operator

$$L_1^{-2}v(x) = \int_{\xi_1}^{x}\int_{\xi_1}^{x}v(x)\,dxdx$$

to both sides of Eq. (9) yields

$$L_1^{-2}L^2u(x) = -L_1^{-2}N(x),$$

where

$$L_1^{-2}L^2u(x) = u(x) - \Phi(x),$$

$$\Phi(x) = u^{(0)}(\xi_1) + u^{(1)}(\xi_1)\Delta_1, \quad \Delta_1 = x - \xi_1.$$

Thus

$$u(x) = \Phi(x) - L_1^{-2}Nu(x), \tag{10}$$

or, upon substitution, we obtain

$$u(x) = u^{(0)}(\xi_1) + u^{(1)}(\xi_1)\Delta_1 - L_1^{-2}Nu(x), \tag{11}$$

which is the equivalent nonlinear Volterra integral equation for the solution $u(x)$ with two undetermined constants of integration, $u^{(0)}(\xi_1)$ and $u^{(1)}(\xi_1)$. Calculating the first-order derivative yields

$$u^{(1)}(x) = u^{(1)}(\xi_1) - L_1^{-1}Nu(x), \text{ where } L_1^{-1}(\cdot) = \int_{\xi_1}^{x}(\cdot)\,dx.$$

Next, we evaluate the formulas for the solution and its first-order derivative at $x = \xi_2$ as

$$u(\xi_2) = u^{(0)}(\xi_1) + u^{(1)}(\xi_1)\Delta_{2,1} - L_{1,2}^{-2}Nu(x), \tag{12}$$

$$u^{(1)}(\xi_2) = u^{(1)}(\xi_1) - L_{1,2}^{-1}Nu(x), \tag{13}$$

where

$$\Delta_{2,1} = \xi_2 - \xi_1, \; L_{1,2}^{-2}(\cdot) = \int_{\xi_1}^{\xi_2}\int_{\xi_1}^{x}(\cdot)\,dxdx,$$

$$L_{1,2}^{-1}(\cdot) = \int_{\xi_1}^{\xi_2}(\cdot)\,dx.$$

Substituting Eqs. (12) and (13) into the remaining boundary condition (8) yields

$$(\alpha_{2,1} + \alpha_{2,2}\Delta_{2,1})u^{(1)}(\xi_1) + \alpha_{2,2}u^{(0)}(\xi_1)$$
$$= \beta_2 + \alpha_{2,1}L_{1,2}^{-1}Nu(x) + \alpha_{2,2}L_{1,2}^{-2}Nu(x).$$

By denoting

$$\begin{aligned}\tilde{\alpha}_{2,1} &= \alpha_{2,1} + \alpha_{2,2}\Delta_{2,1},\\ \tilde{\beta}_2 &= \beta_2 + \alpha_{2,1}L_{1,2}^{-1}Nu(x) + \alpha_{2,2}L_{1,2}^{-2}Nu(x),\end{aligned} \tag{14}$$

we obtain two linearly independent equations solely in terms of the two unknowns $u^{(0)}(\xi_1)$ and $u^{(1)}(\xi_1)$ in a similar pattern as the boundary conditions (7) and (8) as

$$\alpha_{1,1}u^{(1)}(\xi_1) + \alpha_{1,2}u^{(0)}(\xi_1) = \beta_1, \tag{15}$$

$$\tilde{\alpha}_{2,1}u^{(1)}(\xi_1) + \alpha_{2,2}u^{(0)}(\xi_1) = \tilde{\beta}_2. \tag{16}$$

Or equivalently

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \tilde{\alpha}_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} u^{(1)}(\xi_1) \\ u^{(0)}(\xi_1) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \tilde{\beta}_2 \end{pmatrix}.$$

Introducing the partitioned matrix system coefficient α and the partitioned vector input β, we have

$$\alpha\boldsymbol{u} = \beta,$$

where

$$\alpha = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \tilde{\alpha}_{2,1} & \alpha_{2,2} \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} u^{(1)}(\xi_1) \\ u^{(0)}(\xi_1) \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \tilde{\beta}_2 \end{pmatrix}.$$

From our assumption that the boundary conditions (7) and (8) are linearly independent, it follows that $\det(\alpha) \neq 0$, thus $\boldsymbol{u} = \alpha^{-1}\beta$, where

$$\begin{aligned}\alpha^{-1} &= \frac{\text{adj}(\alpha)}{\det(\alpha)} = \frac{1}{\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\tilde{\alpha}_{2,1}} \begin{pmatrix} \alpha_{2,2} & -\alpha_{1,2} \\ -\tilde{\alpha}_{2,1} & \alpha_{1,1} \end{pmatrix}\\ &= \begin{pmatrix} \bar{\alpha}_{1,1} & \bar{\alpha}_{1,2} \\ \bar{\alpha}_{2,1} & \bar{\alpha}_{2,2} \end{pmatrix},\end{aligned}$$

and where the inverse matrix elements $\bar{\alpha}_{i,j}$ are computed by an appropriate algorithm, such as by the Laplace expansion, Gauss-Jordan elimination, Gauss-Seidel iteration, and LU or Cholesky decomposition where applicable, or by a native routine implemented within an available computer algebra system such as MATHEMATICA, MAPLE or MATLAB.

For our 2×2 matrix, we readily obtain

$$\begin{pmatrix} u^{(1)}(\xi_1) \\ u^{(0)}(\xi_1) \end{pmatrix} = \begin{pmatrix} \bar{\alpha}_{1,1} & \bar{\alpha}_{1,2} \\ \bar{\alpha}_{2,1} & \bar{\alpha}_{2,2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \tilde{\beta}_2 \end{pmatrix},$$

i.e.

$$\begin{aligned}u^{(1)}(\xi_1) &= \bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\tilde{\beta}_2,\\ u^{(0)}(\xi_1) &= \bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\tilde{\beta}_2,\end{aligned} \tag{17}$$

which therefore determines the values of the constants of integration $u^{(1)}(\xi_1)$ and $u^{(0)}(\xi_1)$ by formula.

Substituting Eq. (17) into Eq. (11), we have

$$\begin{aligned}u(x) = &\left\{\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\tilde{\beta}_2\right\} + \left\{\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\tilde{\beta}_2\right\}\Delta_1\\ &-L_1^{-2}Nu(x).\end{aligned}$$

Inserting $\tilde{\beta}_2$ in Eq. (14) leads to

$$\begin{aligned}u(x) &= \bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2 + (\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\Delta_1\\ &+\bar{\alpha}_{2,2}\alpha_{2,1}L_{1,2}^{-1}Nu(x) + \bar{\alpha}_{2,2}\alpha_{2,2}L_{1,2}^{-2}Nu(x) - L_1^{-2}Nu(x)\\ &+\left(\bar{\alpha}_{1,2}\alpha_{2,1}L_{1,2}^{-1}Nu(x) + \bar{\alpha}_{1,2}\alpha_{2,2}L_{1,2}^{-2}Nu(x)\right)\Delta_1.\end{aligned}$$

Upon appropriate algebraic manipulations, we deduce that

$$\begin{aligned}u(x) &= (\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2) + (\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\Delta_1\\ &+(\bar{\alpha}_{2,2} + \bar{\alpha}_{1,2}\Delta_1)\left(\alpha_{2,1}L_{1,2}^{-1}Nu(x) + \alpha_{2,2}L_{1,2}^{-2}Nu(x)\right)\\ &-L_1^{-2}Nu(x),\end{aligned} \tag{18}$$

which is the equivalent nonlinear Fredholm-Volterra integral equation for the solution without any undetermined constants of integration.

Decomposing the solution and nonlinearity as

$$u(x) = \sum_{n=0}^{\infty}u_n(x), \quad Nu(x) = \sum_{n=0}^{\infty}A_n(x), \tag{19}$$

where $A_n(x) = A_n(u_0(x),\ldots,u_n(x))$ are the Adomian polynomials, and substituting the decomposition series (19) into Eq. (18) yields

$$\sum_{n=0}^{\infty} u_n(x) = (\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2) + (\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\Delta_1$$
$$-L_1^{-2}\sum_{n=0}^{\infty} A_n(x) + (\bar{\alpha}_{2,2} + \bar{\alpha}_{1,2}\Delta_1)\left(\alpha_{2,1}L_{1,2}^{-1}\sum_{n=0}^{\infty} A_n(x)\right.$$
$$\left.+\alpha_{2,2}L_{1,2}^{-2}\sum_{n=0}^{\infty} A_n(x)\right).$$

For simple nonlinearities such as $Nu = u^2, u^3, uu'$, e.g., positive integer powers, polynomials and product nonlinearities, etc., we can use the classic Adomian recursion scheme as

$$\begin{aligned}
&u_0(x) = (\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2) + (\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\Delta_1,\\
&u_{n+1}(x) = -L_1^{-2}A_n(x)\\
&+(\bar{\alpha}_{2,2} + \bar{\alpha}_{1,2}\Delta_1)\left(\alpha_{2,1}L_{1,2}^{-1}A_n(x) + \alpha_{2,2}L_{1,2}^{-2}A_n(x)\right),
\end{aligned} \tag{20}$$

$n \geq 0$, to calculate the solution components. We note that each such approximate solution

$$\phi_n(x) = \sum_{m=0}^{n-1} u_m(x) \tag{21}$$

satisfies all of the boundary conditions, i.e. when using Adomian's choice for the initial solution component $u_0(x)$.

For more complicated nonlinearities such as $Nu = e^u, \sin(u), \sqrt{1+u^2}$, e.g., exponential, sinusoidal, radical, negative-power, and even decimal-power nonlinearities, etc., we instead use one of the parameterized recursion schemes, e.g.,

$$\begin{aligned}
&u_0(x) = c,\\
&u_1(x) = -c(1-q) + (\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2)\\
&+(\bar{\alpha}_{2,2} + \bar{\alpha}_{1,2}\Delta_1)\left(\alpha_{2,1}L_{1,2}^{-1}A_0(x) + \alpha_{2,2}L_{1,2}^{-2}A_0(x)\right)\\
&+(\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\Delta_1 - L_1^{-2}A_0(x),\\
&u_{n+1}(x) = -c(1-q)q^n - L_1^{-2}A_n(x)\\
&+(\bar{\alpha}_{2,2} + \bar{\alpha}_{1,2}\Delta_1)\left(\alpha_{2,1}L_{1,2}^{-1}A_n(x) + \alpha_{2,2}L_{1,2}^{-2}A_n(x)\right),
\end{aligned} \tag{22}$$

where $n \geq 1$, c and q are two predetermined constants and $0 \leq q < 1$. For example, by the mean value theorem of integral calculus we can take c as the average value of the original solution component u_0 in the classic Adomian recursion scheme over the domain, i.e.

$$\begin{aligned}
c &= \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} (\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2 + (\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\Delta_1)dx\\
&= \bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2 + (\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2)\frac{\xi_2 - \xi_1}{2}.
\end{aligned} \tag{23}$$

For a problem without an a priori exact closed-form analytic solution, the error analysis can be considered by calculating the sequence of error remainder functions

$$ER_n(x) = \phi_n^{(2)}(x) + f\left(x, \phi_n^{(0)}(x), \phi_n^{(1)}(x)\right),$$

and the maximal error remainder parameters

$$MER_n = \max_{\xi_1 \leq x \leq \xi_2} |ER_n(x)|,$$

for $n \geq 0$. We remark that the logarithmic plot of the MER_n versus the index n provides a reliable measure of the rate of convergence, e.g., a nearly linear relation with a negative slope demonstrates an approximate exponential rate of convergence.

Case 2: the third-order differential equation

Consider the third-order nonlinear differential equation

$$u^{(3)}(x) + f\left(x, u^{(0)}(x), u^{(1)}(x), u^{(2)}(x)\right) = 0, \tag{24}$$

subject to the set of three Robin-type boundary conditions

$$\alpha_{1,1}u^{(2)}(\xi_1) + \alpha_{1,2}u^{(1)}(\xi_1) + \alpha_{1,3}u^{(0)}(\xi_1) = \beta_1, \tag{25}$$
$$\alpha_{2,1}u^{(2)}(\xi_2) + \alpha_{2,2}u^{(1)}(\xi_2) + \alpha_{2,3}u^{(0)}(\xi_2) = \beta_2, \tag{26}$$
$$\alpha_{3,1}u^{(2)}(\xi_3) + \alpha_{3,2}u^{(1)}(\xi_3) + \alpha_{3,3}u^{(0)}(\xi_3) = \beta_3, \tag{27}$$

where $\xi_1 \leq \xi_2 \leq \xi_3$ and $\xi_1 < \xi_3$. We assume that Eqs. (25)–(27) are three linearly independent equations in nine unknowns.

Using Adomian's operator-theoretic form, we have

$$L^3 u(x) = -Nu(x), \tag{28}$$

where

$$L^3 = \frac{d^3}{dx^3}, \quad Nu(x) = f\left(x, u^{(0)}(x), u^{(1)}(x), u^{(2)}(x)\right).$$

Applying the inverse operator

$$L_1^{-3}v(x) = \int_{\xi_1}^{x}\int_{\xi_1}^{x}\int_{\xi_1}^{x} v(x)\,dxdxdx$$

to both sides of Eq. (28) yields

$$L_1^{-3}L^3 u(x) = -L_1^{-3}Nu(x),$$

where

$$L_1^{-3}L^3 u(x) = u(x) - \Phi(x),$$

and

$$\Phi(x) = u^{(0)}(\xi_1) + u^{(1)}(\xi_1)\Delta_1 + u^{(2)}(\xi_1)\frac{\Delta_1^2}{2!}.$$

Thus we have

$$u(x) = \Phi(x) - L_1^{-3}Nu(x), \tag{29}$$

or, upon substitution, we obtain

$$u(x) = u^{(0)}(\xi_1) + u^{(1)}(\xi_1)\Delta_1 + u^{(2)}(\xi_1)\frac{\Delta_1^2}{2!} - L_1^{-3}Nu(x), \tag{30}$$

which is the equivalent nonlinear Volterra integral equation for the solution $u(x)$ with three undetermined constants of integration, $u^{(0)}(\xi_1)$, $u^{(1)}(\xi_1)$ and $u^{(2)}(\xi_1)$.

From Eq. (30), we calculate the first- and second-order derivatives as

$$u^{(1)}(x) = u^{(1)}(\xi_1) + u^{(2)}(\xi_1)\Delta_1 - L_1^{-2}Nu(x), \tag{31}$$

$$u^{(2)}(x) = u^{(2)}(\xi_1) - L_1^{-1}Nu(x), \tag{32}$$

where

$$L_1^{-2}v(x) = \int_{\xi_1}^{x}\int_{\xi_1}^{x} v(x)\,dxdx,\ L_1^{-1}v(x) = \int_{\xi_1}^{x} v(x)\,dx.$$

Next, we evaluate the formulas for the solution and its first- and second-order derivatives at $x = \xi_j$, for $j = 2, 3$, as

$$u(\xi_j) = u^{(0)}(\xi_1) + u^{(1)}(\xi_1)\Delta_{j,1} + u^{(2)}(\xi_1)\frac{\Delta_{j,1}^2}{2!} - L_{1,j}^{-3}Nu(x), \tag{33}$$

$$u^{(1)}(\xi_j) = u^{(1)}(\xi_1) + u^{(2)}(\xi_1)\Delta_{j,1} - L_{1,j}^{-2}Nu(x), \tag{34}$$

$$u^{(2)}(\xi_j) = u^{(2)}(\xi_1) - L_{1,j}^{-1}Nu(x), \tag{35}$$

where $\Delta_{j,1} = \xi_j - \xi_1$ and

$$L_{1,j}^{-3}(\cdot) = \int_{\xi_1}^{\xi_j}\int_{\xi_1}^{x}\int_{\xi_1}^{x}(\cdot)\,dxdxdx, \tag{36}$$

$$L_{1,j}^{-2}(\cdot) = \int_{\xi_1}^{\xi_j}\int_{\xi_1}^{x}(\cdot)dxdx, \tag{37}$$

$$L_{1,j}^{-1}(\cdot) = \int_{\xi_1}^{\xi_j}(\cdot)\,dx. \tag{38}$$

Substituting Eqs. (33)–(35) into the boundary conditions (26) and (27), for $j = 2, 3$, yields

$$\begin{aligned}&u^{(2)}(\xi_1)\left(\alpha_{j,1} + \alpha_{j,2}\Delta_{j,1} + \alpha_{j,3}\tfrac{\Delta_{j,1}^2}{2!}\right)\\ &+u^{(1)}(\xi_1)\left(\alpha_{j,2} + \alpha_{j,3}\Delta_{j,1}\right) + u^{(0)}(\xi_1)\,\alpha_{j,3} =\\ &\beta_j + \alpha_{j,1}L_{1,j}^{-1}Nu(x) + \alpha_{j,2}L_{1,j}^{-2}Nu(x) + \alpha_{j,3}L_{1,j}^{-3}Nu(x).\end{aligned}$$

By denoting

$$\begin{aligned}&\tilde{\alpha}_{j,1} = \alpha_{j,1} + \alpha_{j,2}\Delta_{j,1} + \alpha_{j,3}\tfrac{\Delta_{j,1}^2}{2!},\ \tilde{\alpha}_{j,2} = \alpha_{j,2} + \alpha_{j,3}\Delta_{j,1},\\ &\tilde{\beta}_j = \beta_j + \alpha_{j,1}L_{1,j}^{-1}Nu(x) + \alpha_{j,2}L_{1,j}^{-2}Nu(x) + \alpha_{j,3}L_{1,j}^{-3}Nu(x),\end{aligned} \tag{39}$$

we obtain three linearly independent equations solely in terms of the three unknowns $u^{(0)}(\xi_1)$, $u^{(1)}(\xi_1)$ and $u^{(2)}(\xi_1)$ in a similar pattern as the boundary conditions (25), (26) and (27) as

$$\alpha_{1,1}u^{(2)}(\xi_1) + \alpha_{1,2}u^{(1)}(\xi_1) + \alpha_{1,3}u^{(0)}(\xi_1) = \beta_1, \tag{40}$$

$$\tilde{\alpha}_{2,1}u^{(2)}(\xi_1) + \tilde{\alpha}_{2,2}u^{(1)}(\xi_1) + \alpha_{2,3}u^{(0)}(\xi_1) = \tilde{\beta}_2, \tag{41}$$

$$\tilde{\alpha}_{3,1}u^{(2)}(\xi_1) + \tilde{\alpha}_{3,2}u^{(1)}(\xi_1) + \alpha_{3,3}u^{(0)}(\xi_1) = \tilde{\beta}_3. \tag{42}$$

The matrix form is

$$\left(\begin{array}{cc:c}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \hdashline \tilde{\alpha}_{21} & \tilde{\alpha}_{22} & \alpha_{23}\\ \tilde{\alpha}_{31} & \tilde{\alpha}_{32} & \alpha_{33}\end{array}\right)\begin{pmatrix}u^{(2)}(\xi_1)\\ u^{(1)}(\xi_1)\\ u^{(0)}(\xi_1)\end{pmatrix} = \left(\begin{array}{c}\beta_1\\ \hdashline \tilde{\beta}_2\\ \tilde{\beta}_3\end{array}\right),$$

Introducing the partitioned matrix system coefficient α and the partitioned vector input β, we have

$$\alpha\boldsymbol{u} = \beta,$$

where

$$\alpha = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \tilde{\alpha}_{21} & \tilde{\alpha}_{22} & \alpha_{23}\\ \tilde{\alpha}_{31} & \tilde{\alpha}_{32} & \alpha_{33}\end{pmatrix},\quad \boldsymbol{u} = \begin{pmatrix}u^{(2)}(\xi_1)\\ u^{(1)}(\xi_1)\\ u^{(0)}(\xi_1)\end{pmatrix},\quad \beta = \begin{pmatrix}\beta_1\\ \tilde{\beta}_2\\ \tilde{\beta}_3\end{pmatrix}.$$

If $\det(\alpha) \neq 0$, then $\boldsymbol{u} = \alpha^{-1}\beta$, where we define

$$\alpha^{-1} = \frac{\mathrm{adj}(\alpha)}{\det(\alpha)} = \begin{pmatrix}\bar{\alpha}_{1,1} & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,3}\\ \bar{\alpha}_{2,1} & \bar{\alpha}_{2,2} & \bar{\alpha}_{2,3}\\ \bar{\alpha}_{3,1} & \bar{\alpha}_{3,2} & \bar{\alpha}_{3,3}\end{pmatrix},$$

where the inverse matrix elements $\bar{\alpha}_{i,j}$ are rather computed by an appropriate algorithm.

For our 3×3 matrix, we readily obtain

$$\begin{pmatrix}u^{(2)}(\xi_1)\\ u^{(1)}(\xi_1)\\ u^{(0)}(\xi_1)\end{pmatrix} = \begin{pmatrix}\bar{\alpha}_{1,1} & \bar{\alpha}_{1,2} & \bar{\alpha}_{1,3}\\ \bar{\alpha}_{2,1} & \bar{\alpha}_{2,2} & \bar{\alpha}_{2,3}\\ \bar{\alpha}_{3,1} & \bar{\alpha}_{3,2} & \bar{\alpha}_{3,3}\end{pmatrix}\begin{pmatrix}\beta_1\\ \tilde{\beta}_2\\ \tilde{\beta}_3\end{pmatrix},$$

or

$$u^{(2)}(\xi_1) = \bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\tilde{\beta}_2 + \bar{\alpha}_{1,3}\tilde{\beta}_3, \tag{43}$$

$$u^{(1)}(\xi_1) = \bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\tilde{\beta}_2 + \bar{\alpha}_{2,3}\tilde{\beta}_3, \tag{44}$$

$$u^{(0)}(\xi_1) = \bar{\alpha}_{3,1}\beta_1 + \bar{\alpha}_{3,2}\tilde{\beta}_2 + \bar{\alpha}_{3,3}\tilde{\beta}_3, \tag{45}$$

which therefore determines the values of the constants of integration $u^{(2)}(\xi_1)$, $u^{(1)}(\xi_1)$ and $u^{(0)}(\xi_1)$ by formula.

Substituting Eqs. (43)–(45) into Eq. (30), we have

$$\begin{aligned}u(x) = {}&\bar{\alpha}_{3,1}\beta_1 + \bar{\alpha}_{3,2}\tilde{\beta}_2 + \bar{\alpha}_{3,3}\tilde{\beta}_3\\ &+(\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\tilde{\beta}_2 + \bar{\alpha}_{2,3}\tilde{\beta}_3)\Delta_1\\ &+(\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\tilde{\beta}_2 + \bar{\alpha}_{1,3}\tilde{\beta}_3)\tfrac{\Delta_1^2}{2!} - L_1^{-3}Nu(x),\end{aligned} \tag{46}$$

where $\tilde{\beta}_2$ and $\tilde{\beta}_3$ are given in Eq. (39). Upon appropriate algebraic manipulations, we deduce that

$$\begin{aligned}&u(x) = (\bar{\alpha}_{3,1}\beta_1 + \bar{\alpha}_{3,2}\beta_2 + \bar{\alpha}_{3,3}\beta_3)\\ &+(\bar{\alpha}_{2,1}\beta_1 + \bar{\alpha}_{2,2}\beta_2 + \bar{\alpha}_{2,3}\beta_3)\Delta_1\\ &+(\bar{\alpha}_{1,1}\beta_1 + \bar{\alpha}_{1,2}\beta_2 + \bar{\alpha}_{1,3}\beta_3)\tfrac{\Delta_1^2}{2} - L_1^{-3}Nu(x)\\ &+\left(\bar{\alpha}_{3,2} + \bar{\alpha}_{2,2}\Delta_1 + \bar{\alpha}_{1,2}\tfrac{\Delta_1^2}{2}\right)\times\\ &\left(\alpha_{2,1}L_{1,2}^{-1}Nu(x) + \alpha_{2,2}L_{1,2}^{-2}Nu(x) + \alpha_{2,3}L_{1,2}^{-3}Nu(x)\right)\\ &+\left(\bar{\alpha}_{3,3} + \bar{\alpha}_{2,3}\Delta_1 + \bar{\alpha}_{1,3}\tfrac{\Delta_1^2}{2}\right)\times\\ &\left(\alpha_{3,1}L_{1,3}^{-1}Nu(x) + \alpha_{3,2}L_{1,3}^{-2}Nu(x) + \alpha_{3,3}L_{1,3}^{-3}Nu(x)\right),\end{aligned} \tag{47}$$

which is the equivalent nonlinear Fredholm-Volterra integral equation for the solution $u(x)$ without any undetermined constants of integration.

Next the decomposition series of the solution and the nonlinearity and the recursion scheme are similarly obtained as in the last case. Here we omit the details to avoid self-evident repetitions.

Case 3: the general pth-order differential equation for $p \geq 2$

Consider the pth-order nonlinear differential equation

$$u^{(p)}(x) + f\left(x, u(x), \ldots, u^{(p-1)}(x)\right) = 0, \tag{48}$$

subject to the set of p Robin-type boundary conditions

$$\sum_{i=1}^{p} \alpha_{j,i} u^{(p-i)}(\xi_j) = \beta_j, \; j = 1, 2, \ldots, p, \tag{49}$$

where $\xi_1 \leq \cdots \leq \xi_j \leq \cdots \leq \xi_p$ and $\xi_1 < \xi_p$. Eq. (49) denotes p linearly independent equations in p^2 unknowns.

Using Adomian's operator-theoretic notation, we write Eq. (48) as

$$L^p u(x) = -Nu(x), \tag{50}$$

where $L^p = \frac{d^p}{dx^p}$, $Nu(x) = f\left(x, u(x), \ldots, u^{(p-1)}(x)\right)$. Applying the p-fold integral operator

$$L_1^{-p} v(x) = \underbrace{\int_{\xi_1}^{x} \cdots \int_{\xi_1}^{x}}_{p-fold} v(x) \underbrace{dx \cdots dx}_{p-fold}$$

to both sides of Eq. (50), we have

$$L_1^{-p} L^p u(x) = -L_1^{-p} Nu(x),$$

where

$$L_1^{-p} L^p u(x) = u(x) - \Phi(x), \tag{51}$$

$$\Phi(x) = \sum_{\nu=0}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_1^{\nu}}{\nu!}. \tag{52}$$

So we obtain

$$u(x) = \sum_{\nu=0}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_1^{\nu}}{\nu!} - L_1^{-p} Nu(x), \tag{53}$$

which is the equivalent nonlinear Volterra integral equation for the solution $u(x)$ with p undetermined constants of integration, $u^{(0)}(\xi_1), \ldots, u^{(p-1)}(\xi_1)$.

Calculating the kth-order derivatives of $u(x)$, for $0 \leq k \leq p-1$, yields

$$\begin{aligned} u^{(k)}(x) &= \frac{d^k}{dx^k} \sum_{\nu=0}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_1^{\nu}}{\nu!} - \frac{d^k}{dx^k} L_1^{-p} Nu(x) \\ &= \sum_{\nu=k}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_1^{\nu-k}}{(\nu-k)!} - L_1^{-(p-k)} Nu(x), \end{aligned} \tag{54}$$

where

$$L_1^{-(p-k)} v(x) = \underbrace{\int_{\xi_1}^{x} \cdots \int_{\xi_1}^{x}}_{(p-k)-fold} v(x) \underbrace{dx \cdots dx}_{(p-k)-fold}.$$

Next we calculate the derivatives $u^{(k)}(\xi_j)$, for $k = 0, 1, \ldots, p-1$ and $j = 2, 3, \ldots, p$, as

$$u^{(k)}(\xi_j) = \sum_{\nu=k}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_{j,1}^{\nu-k}}{(\nu-k)!} - L_{1,j}^{-(p-k)} Nu(x),$$

where the operator $L_{1,j}^{-(p-k)}$ is defined as

$$L_{1,j}^{-(p-k)} v(x) = \int_{\xi_1}^{\xi_j} \underbrace{\int_{\xi_1}^{x} \cdots \int_{\xi_1}^{x}}_{(p-k-1)-fold} v(x) \underbrace{dx \cdots dx}_{(p-k-1)-fold} dx.$$

For the substitution $k := (p-i)$, then we have $\nu - k := \nu + i - p$ and $p - k := i$, and

$$u^{(p-i)}(\xi_j) = \sum_{\nu=p-i}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_{j,1}^{\nu+i-p}}{(\nu+i-p)!} - L_{1,j}^{-i} Nu(x),$$
$$i = 1, \ldots, p, \; j = 2, \ldots, p. \tag{55}$$

Substituting the last equation into the boundary conditions (49) at ξ_j, for $j = 2, \ldots, p$, we have

$$\sum_{i=1}^{p} \alpha_{j,i} \left[\sum_{\nu=p-i}^{p-1} u^{(\nu)}(\xi_1) \frac{\Delta_{j,1}^{\nu+i-p}}{(\nu+i-p)!} - L_{1,j}^{-i} Nu(x) \right] = \beta_j,$$
$$j = 2, \ldots, p. \tag{56}$$

Using the following summation formula

$$\sum_{i=1}^{p} \sum_{\nu=p-i}^{p-1} C_{i,\nu} = \sum_{i=1}^{p} \sum_{l=1}^{i} C_{i,p-l} = \sum_{l=1}^{p} \sum_{i=l}^{p} C_{i,p-l}$$

$$= C_{p,0} + \sum_{l=1}^{p-1} \sum_{\nu=l}^{p} C_{\nu,p-l} = C_{p,0} + \sum_{i=1}^{p-1} \sum_{\nu=i}^{p} C_{\nu,p-i},$$

we obtain p linearly independent equations solely in terms of the p unknowns $u^{(0)}(\xi_1), u^{(1)}(\xi_1), \ldots, u^{(p-1)}(\xi_1)$ as

$$\text{for } j = 1, \; \sum_{i=1}^{p} \alpha_{1,i} u^{(p-i)}(\xi_1) = \beta_1, \tag{57}$$

for $j = 2, \ldots, p$,

$$\left\{ \sum_{i=1}^{p-1} \tilde{\alpha}_{j,i} u^{(p-i)}(\xi_1) \right\} + \alpha_{j,p} u^{(0)}(\xi_1) = \tilde{\beta}_j, \tag{58}$$

where

$$\tilde{\beta}_j = \beta_j + \sum_{\nu=1}^{p} \alpha_{j,\nu} L_{1,j}^{-\nu} Nu(x), \tilde{\alpha}_{j,i} = \sum_{\nu=i}^{p} \alpha_{j,\nu} \frac{\Delta_{j,1}^{\nu-i}}{(\nu-i)!}. \tag{59}$$

Or equivalently, we have the matrix form

$$\alpha \boldsymbol{u} = \beta, \tag{60}$$

where

$$\alpha=\left(\begin{array}{ccccc|c}\alpha_{1,1}&\alpha_{1,2}&\alpha_{1,3}&\cdots&\alpha_{1,p-1}&\alpha_{1,p}\\ \hline \tilde{\alpha}_{2,1}&\tilde{\alpha}_{2,2}&\tilde{\alpha}_{2,3}&\cdots&\tilde{\alpha}_{2,p-1}&\alpha_{2,p}\\ \tilde{\alpha}_{3,1}&\tilde{\alpha}_{3,2}&\tilde{\alpha}_{3,3}&\cdots&\tilde{\alpha}_{3,p-1}&\alpha_{3,p}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ \tilde{\alpha}_{p,1}&\tilde{\alpha}_{p,2}&\tilde{\alpha}_{p,3}&\cdots&\tilde{\alpha}_{p,p-1}&\alpha_{p,p}\end{array}\right)$$

$$\boldsymbol{u}=\begin{pmatrix}u^{(p-1)}(\xi_1)\\ u^{(p-2)}(\xi_1)\\ u^{(p-3)}(\xi_1)\\ \vdots\\ u^{(0)}(\xi_1)\end{pmatrix},\quad \beta=\begin{pmatrix}\beta_1\\ \hline \tilde{\beta}_2\\ \tilde{\beta}_3\\ \vdots\\ \tilde{\beta}_p\end{pmatrix}.$$

If $\det(\alpha)\neq 0$, then $\boldsymbol{u}=\alpha^{-1}\beta$, where the inverse matrix $\alpha^{-1}=(\bar{\alpha}_{i,j})$ is computed by an appropriate algorithm or native routine implemented within an available computer algebra system such as MATHEMATICA, MAPLE or MATLAB.

We calculate for $v=0,1,\ldots,p-1$,

$$u^{(v)}(\xi_1)=\bar{\alpha}_{p-v,\,1}\beta_1+\sum_{j=2}^{p}\bar{\alpha}_{p-v,\,j}\tilde{\beta}_j, \tag{61}$$

which therefore determines the values of the constants of integration $u^{(p-1)}(\xi_1)$, $u^{(p-2)}(\xi_1)$, $\ldots$, $u^{(0)}(\xi_1)$ by formula. Substituting these values into Eq. (53), we have

$$u(x)=\sum_{v=0}^{p-1}\left\{\bar{\alpha}_{p-v,\,1}\beta_1+\sum_{j=2}^{p}\bar{\alpha}_{p-v,\,j}\tilde{\beta}_j\right\}\frac{\Delta_1^v}{v!}-L_1^{-p}Nu(x),$$

Inserting the $\tilde{\beta}_j$, for $j=2,\ldots,p$, from Eq. (59) into the last previous equation leads to

$$u(x)=\sum_{v=0}^{p-1}\left\{\bar{\alpha}_{p-v,1}\beta_1+\sum_{j=2}^{p}\bar{\alpha}_{p-v,j}\times\right.$$
$$\left.\left[\beta_j+\sum_{k=1}^{p}\alpha_{j,\,k}L_{1,j}^{-k}Nu(x)\right]\right\}\frac{\Delta_1^v}{v!}-L_1^{-p}Nu(x).$$

Upon appropriate algebraic manipulations, we deduce that

$$u(x)=\sum_{j=1}^{p}\left(\beta_j\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)-L_1^{-p}Nu(x)$$
$$+\sum_{j=2}^{p}\left(\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)\left(\sum_{k=1}^{p}\alpha_{j,k}L_{1,j}^{-k}Nu(x)\right), \tag{62}$$

which is the equivalent nonlinear Fredholm-Volterra integral equation for the solution without any undetermined constants of integration.

Next we decompose the solution and the nonlinearity as

$$u(x)=\sum_{n=0}^{\infty}u_n(x),\quad Nu(x)=\sum_{n=0}^{\infty}A_n(x),$$
$$A_n(x)=A_n(u_0(x),\ldots,u_n(x)), \tag{63}$$

and substitute them into Eq. (62) to obtain

$$\sum_{n=0}^{\infty}u_n(x)=\sum_{j=1}^{p}\left(\beta_j\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)-L_1^{-p}\sum_{n=0}^{\infty}A_n(x)$$
$$+\sum_{j=2}^{p}\left(\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)\left(\sum_{k=1}^{p}\alpha_{j,k}L_{1,j}^{-k}\sum_{n=0}^{\infty}A_n(x)\right).$$

For simple nonlinearities such as the quadratic nonlinearity, product nonlinearity, etc., we can use the classic Adomian recursion scheme as

$$u_0(x)=\sum_{j=1}^{p}\left\{\beta_j\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right\},$$
$$u_{n+1}(x)=\sum_{j=2}^{p}\left\{\left(\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)\left(\sum_{k=1}^{p}\alpha_{j,k}L_{1,j}^{-k}A_n(x)\right)\right\}$$
$$-L_1^{-p}A_n(x),\ n\geq 0, \tag{64}$$

to calculate the solution components. We note that each approximate solution

$$\phi_n(x)=\sum_{m=0}^{n-1}u_m(x),\ n=1,2,\ldots, \tag{65}$$

satisfies all of the boundary conditions.

For complicated nonlinearities such as $Nu=e^u,\sin(u),\sqrt{1+u^2}$, etc., we instead use one of the parameterized recursion schemes, e.g.,

$$u_0(x)=c,$$
$$u_1(x)=-c(1-q)+\sum_{j=1}^{p}\left(\beta_j\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)-L_1^{-p}A_0(x)$$
$$+\sum_{j=2}^{p}\left(\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)\left(\sum_{k=1}^{p}\alpha_{j,k}L_{1,j}^{-k}A_0(x)\right),$$
$$u_{n+2}(x)=-c(1-q)q^{n+1}-L_1^{-p}A_{n+1}(x)$$
$$+\sum_{j=2}^{p}\left(\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right)\left(\sum_{k=1}^{p}\alpha_{j,k}L_{1,j}^{-k}A_{n+1}(x)\right),\ n\geq 0, \tag{66}$$

where c and q are two predetermined constants and $0\leq q<1$. For example, we can take c as the average value of the initial solution component u_0 in the classic Adomian recursion scheme over the domain, i.e.

$$c=\frac{1}{\xi_p-\xi_1}\int_{\xi_1}^{\xi_p}\sum_{j=1}^{p}\left\{\beta_j\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{\Delta_1^v}{v!}\right\}dx$$
$$=\sum_{j=1}^{p}\left\{\beta_j\sum_{v=0}^{p-1}\bar{\alpha}_{p-v,j}\frac{(\xi_p-\xi_1)^v}{(v+1)!}\right\}, \tag{67}$$

For the error analysis for a problem without an a priori exact closed-form analytic solution, we can consider the sequence of error remainder functions

$$ER_n(x)=\phi_n^{(p)}(x)+f\left(x,\phi_n^{(0)}(x),\ldots,\phi_n^{(p-1)}(x)\right), \tag{68}$$

and the sequence of maximal error remainder parameters

$$MER_n=\max_{\xi_1\leq x\leq\xi_p}|ER_n(x)|. \tag{69}$$

3 Numeric examples

Example 1. Consider the third-order linear differential equation

$$u'''(x)+u''(x)-u'(x)-u(x)=0, \tag{70}$$

subject to the set of three Robin-type boundary conditions

$$u''(0)-u'(0)-u(0)=1,$$
$$u''(0)-u(0)=-1,$$
$$u'(1)+u(1)=10.$$

The exact solution is

$$u^*(x)=\left(\frac{x}{2}-\frac{1-20e-10e^2}{4e^2}\right)e^{-x}-\frac{1-20e}{4e^2}e^x. \tag{71}$$

The equivalent Fredholm-Volterra integral equation is

$$u(x) = \tfrac{31}{5}-2x+\tfrac{13x^2}{5}+\left(\tfrac{2}{5}+\tfrac{x^2}{5}\right)(L_{1,3}^{-2}Nu+L_{1,3}^{-3}Nu) -L_1^{-3}Nu,$$

where $Nu=u''(x)-u'(x)-u(x)$ and

$$L_{1,3}^{-3}(\cdot)=\int_0^1\int_0^x\int_0^x(\cdot)\,dxdxdx,\; L_{1,3}^{-2}(\cdot)=\int_0^1\int_0^x(\cdot)\,dxdx.$$

The recursion scheme is

$$u_0(x)=\tfrac{31}{5}-2x+\tfrac{13x^2}{5},$$
$$u_{n+1}(x)=\left(\tfrac{2}{5}+\tfrac{x^2}{5}\right)(L_{1,3}^{-2}A_n+L_{1,3}^{-3}A_n)-L_1^{-3}A_n,$$

where $n\geq 0$ and

$$A_n=u_n''(x)-u_n'(x)-u_n(x),\; n\geq 0.$$

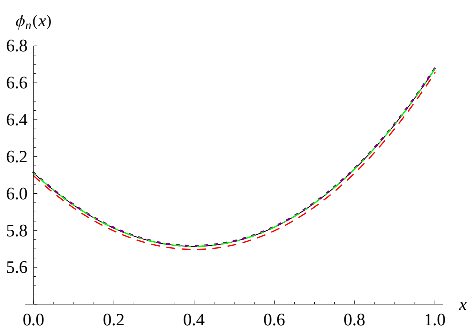

Fig. 1: Curves of the exact solution $u^*(x)$ (solid line), and the approximate solutions $\phi_2(x)$ (dash line), $\phi_3(x)$ (dot line) and $\phi_4(x)$ (dot-dash line).

In Fig. 1, we plot the curves of the exact solution $u^*(x)$ and the approximate solutions $\phi_2(x)$, $\phi_3(x)$ and $\phi_4(x)$, where $\phi_3(x)$, $\phi_4(x)$ and the exact solution overlap.

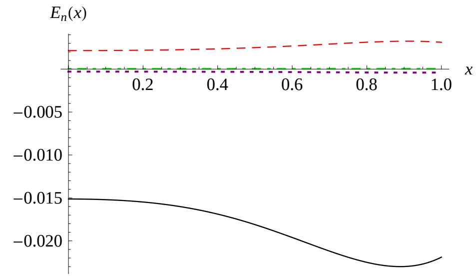

Fig. 2: Curves of the error functions $E_2(x)$ (solid line), $E_3(x)$ (dash line), $E_4(x)$ (dot line) and $E_5(x)$ (dot-dash line).

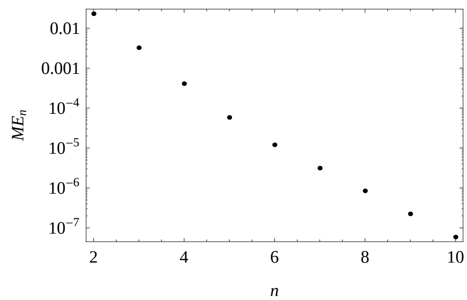

Fig. 3: The logarithmic plots of the maximal error parameters ME_n, $n=2$ through 10.

We consider the error function

$$E_n(x)=\phi_n(x)-u^*(x)$$

and the maximal error parameters

$$ME_n=\max_{0\leq x\leq 1}|E_n(x)|.$$

In Fig. 2, we plot the curves of the error functions $E_n(x)$ for $n=2,3,4$ and 5. In Fig. 3, we display the logarithmic plots of the maximal error parameters ME_n, $n=2$ through 10, where the points are distributed almost on a straight line thus indicating an approximate exponential rate of convergence.

Example 2. Consider the third-order nonlinear differential equation

$$u'''(x)-2-u''(x)+x\sinh(u(x))=0, \tag{72}$$

subject to the set of three Robin-type boundary conditions

$$u''(0)-u'(0)+u(0)=-2,$$
$$u''(0.5)+2u'(0.5)+2u(0.5)=2,$$
$$u''(1)-3u'(1)+u(1)=-1.$$

For this BVP, we calculate that

$$\alpha=\begin{pmatrix}1 & -1 & 1\\ 9/4 & 3 & 2\\ -3/2 & -2 & 1\end{pmatrix},\; \alpha^{-1}=\frac{1}{49}\begin{pmatrix}28 & -4 & -20\\ -21 & 10 & 1\\ 0 & 14 & 21\end{pmatrix},$$

and the equivalent nonlinear Fredholm-Volterra integral equation as

$$\begin{aligned} u(x) &= \tfrac{1}{7} + \tfrac{61}{49}x - \tfrac{22}{49}x^2 - L_1^{-3}Nu \\ &+ \tfrac{1}{49}(14+10x-2x^2)\left(L_{1,2}^{-1}Nu + 2L_{1,2}^{-2}Nu + 2L_{1,2}^{-3}Nu\right) \\ &+ \tfrac{1}{49}(21+x-10x^2)\left(L_{1,3}^{-1}Nu - 3L_{1,3}^{-2}Nu + L_{1,3}^{-3}Nu\right), \end{aligned}$$

where $Nu = -2 - u''(x) + x\sinh(u(x))$. We decompose the solution and the nonlinearity as

$$u(x) = \sum_{n=0}^{\infty} u_n(x),\ Nu(x) = \sum_{n=0}^{\infty} B_n(x),$$

where

$$B_0 = -2 - u_0''(x) + xA_0,\ B_n = -u_n''(x) + xA_n,$$

and the A_n are the Adomian polynomials for the analytic function $\sinh(u(x))$, i.e.

$$\begin{aligned} A_0 &= \sinh(u_0), \\ A_1 &= u_1\cosh(u_0), \\ A_2 &= \frac{1}{2}u_1^2\sinh(u_0) + u_2\cosh(u_0), \\ A_3 &= u_2u_1\sinh(u_0) + \frac{1}{6}u_1^3\cosh(u_0) + u_3\cosh(u_0), \\ &\ldots, \end{aligned}$$

and design the parameterized recursion scheme accordingly as

$$\begin{aligned} u_0(x) &= c, \\ u_1(x) &= -c(1-q) + \tfrac{1}{7} + \tfrac{61}{49}x - \tfrac{22}{49}x^2 - L_1^{-3}B_0 \\ &+ \tfrac{1}{49}(14+10x-2x^2)\left(L_{1,2}^{-1}B_0 + 2L_{1,2}^{-2}B_0 + 2L_{1,2}^{-3}B_0\right) \\ &+ \tfrac{1}{49}(21+x-10x^2)\left(L_{1,3}^{-1}B_0 - 3L_{1,3}^{-2}B_0 + L_{1,3}^{-3}B_0\right), \\ u_{n+1}(x) &= -c(1-q)q^n - L_1^{-3}B_n \\ &+ \tfrac{1}{49}(14+10x-2x^2)\left(L_{1,2}^{-1}B_n + 2L_{1,2}^{-2}B_n + 2L_{1,2}^{-3}B_n\right) \\ &+ \tfrac{1}{49}(21+x-10x^2)\left(L_{1,3}^{-1}B_n - 3L_{1,3}^{-2}B_n + L_{1,3}^{-3}B_n\right),\ n \geq 1, \end{aligned}$$

where we take $c = \int_0^1 \left(\frac{1}{7} + \frac{61}{49}x - \frac{22}{49}x^2\right)dx = \frac{181}{294}$.

Table 1: The maximal error remainder parameters MER_n in Example 2.

n	2	3	4	5	6
MER_n	1.06115	0.135321	0.073565	0.0383467	0.0174748
n	7	8	9	10	
MER_n	0.00825477	0.00397323	0.00191811	0.000930779	

We take $q = 0.1$ to compute the solution approximants. The solution approximants $\phi_n(x)$, $n = 2,3,4,5$, are plotted in Fig. 4, where the last three curves overlap. In Fig. 5, we plot the error remainder functions $ER_n(x)$ for $n = 2,3,4,5$. The maximal error remainder parameters MER_n, $n = 2$ through 10, are listed

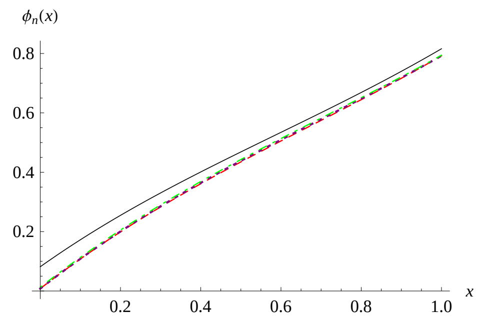

Fig. 4: Curves of $\phi_2(x)$ (solid line), $\phi_3(x)$ (dash line), $\phi_4(x)$ (dot line) and $\phi_5(x)$ (dot-dash line).

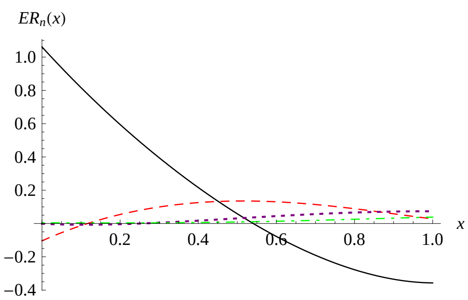

Fig. 5: Curves of $ER_2(x)$ (solid line), $ER_3(x)$ (dash line), $ER_4(x)$ (dot line) and $ER_5(x)$ (dot-dash line).

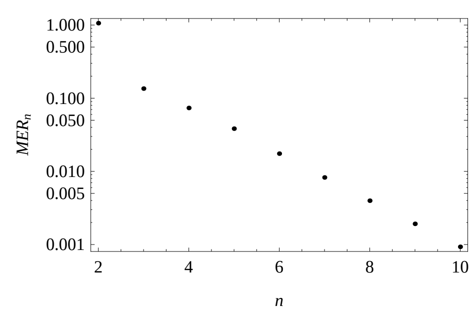

Fig. 6: Logarithmic plots of MER_n for $n = 2$ through 10.

in Table 1. The logarithmic plots of these values are displayed in Fig. 6, where the last 7 points are distributed almost on a straight line thus indicating an approximate exponential rate of convergence.

Example 3. The squeezing flow and heat transfer between two parallel disks with velocity slip and temperature jump [39] lead to the following nonlinear BVP in dimensionless form

$$f^{(4)}(\eta) - S\eta f'''(\eta) - (3S+M^2)f''(\eta) + 2Sf(\eta)f'''(\eta) = 0, \tag{73}$$

$$\begin{aligned} &f(0) = 0,\ -\beta f''(0) + f'(0) = 0, \\ &f(1) = 1/2,\ \beta f''(1) + f'(1) = 0. \end{aligned}$$

where η is a similarity variable, $f(\eta)$ characterizes the axial velocity, S is the squeeze number, M is the Hartman number, and β is the dimensionless slip parameter.

For this fourth-order nonlinear BVP, $0=\xi_1=\xi_2<\xi_3=\xi_4=1$, $\alpha_{1,4}=\alpha_{2,3}=\alpha_{3,4}=\alpha_{4,3}=1$, $\alpha_{2,2}=-\beta$, $\alpha_{4,2}=\beta$, $\alpha_{i,j}=0$ for other i,j, $\beta_1=\beta_2=\beta_4=0, \beta_3=1/2$, and

$$Nf(\eta)=-S\eta f'''(\eta)-(3S+M^2)f''(\eta)+2Sf(\eta)f'''(\eta).$$

We calculate that

$$\alpha=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & -\beta & 1 & 0\\ \frac{1}{6} & \frac{1}{2} & 1 & 1\\ \beta+\frac{1}{2} & \beta+1 & 1 & 0 \end{pmatrix},$$
$$\alpha^{-1}=\frac{1}{D}\begin{pmatrix} 1+2\beta & \frac{1}{2}+\beta & -1-2\beta & \frac{1}{2}+\beta\\ -\frac{1}{2}-\beta & -\frac{1}{3}-\beta & \frac{1}{2}+\beta & -\frac{1}{6}\\ -\frac{\beta}{2}-\beta^2 & \frac{1}{12}+\frac{\beta}{3} & \frac{\beta}{2}+\beta^2 & -\frac{\beta}{6}\\ D & 0 & 0 & 0 \end{pmatrix},$$

where

$$D=\frac{1}{12}+\frac{2\beta}{3}+\beta^2.$$

The equivalent nonlinear Fredholm-Volterra integral equation is

$$\begin{aligned} f(\eta)=&\tfrac{1}{2D}\left(\tfrac{\beta+2\beta^2}{2}\eta+\tfrac{1+2\beta}{4}\eta^2-\tfrac{1+2\beta}{6}\eta^3\right)-L_1^{-4}Nf(\eta)\\ &+\tfrac{1}{D}\left(\tfrac{\beta+2\beta^2}{2}\eta+\tfrac{1+2\beta}{4}\eta^2-\tfrac{1+2\beta}{6}\eta^3\right)L_{1,3}^{-4}Nf(\eta)\\ &+\tfrac{1}{D}\left(\tfrac{1+2\beta}{12}\eta^3-\tfrac{\beta}{6}\eta-\tfrac{\eta^2}{12}\right)\left(\beta L_{1,4}^{-2}Nf(\eta)+L_{1,4}^{-3}Nf(\eta)\right). \end{aligned}$$

We decompose the solution and the nonlinearity as

$$f(\eta)=\sum_{n=0}^{\infty}f_n(\eta),\ Nf(\eta)=\sum_{n=0}^{\infty}A_n(\eta),$$

and design the recursion scheme

$$\begin{aligned} &f_0(\eta)=\tfrac{1}{2D}\left(\tfrac{\beta+2\beta^2}{2}\eta+\tfrac{1+2\beta}{4}\eta^2-\tfrac{1+2\beta}{6}\eta^3\right),\\ &f_{n+1}(\eta)=\tfrac{1}{D}\left(\tfrac{\beta+2\beta^2}{2}\eta+\tfrac{1+2\beta}{4}\eta^2-\tfrac{1+2\beta}{6}\eta^3\right)L_{1,3}^{-4}A_n\\ &+\tfrac{1}{D}\left(\tfrac{1+2\beta}{12}\eta^3-\tfrac{\beta}{6}\eta-\tfrac{\eta^2}{12}\right)\left(\beta L_{1,4}^{-2}A_n+L_{1,4}^{-3}A_n\right)\\ &-L_1^{-4}A_n,\ n\geq 0, \end{aligned}$$

where

$$\begin{aligned} A_0&=-S\eta f_0'''(\eta)-(3S+M^2)f_0''(\eta)+2Sf_0(\eta)f_0'''(\eta),\\ A_n&=-S\eta f_n'''(\eta)-(3S+M^2)f_n''(\eta)\\ &\quad+2S\textstyle\sum_{k=0}^{n}f_k(\eta)f_{n-k}'''(\eta),\ n\geq 1. \end{aligned}$$

From the recursion scheme, we obtain the solution components and solution approximants.

We take $S=M=1$ and $\beta=0.1$ to compute the solution approximants. The solution approximants $\phi_n(\eta)$, $n=2,3,4$, are plotted in Fig. 7, where the three curves overlap. In Fig. 8, we plot the error remainder functions $ER_n(\eta)$ for $n=2,3,4$. The maximal error remainder parameters MER_n, $n=2$ through 7, are listed in Table 2. The logarithmic plots of these values are displayed in Fig. 9, where the points are distributed almost on a straight line thus indicating an approximate exponential rate of convergence.

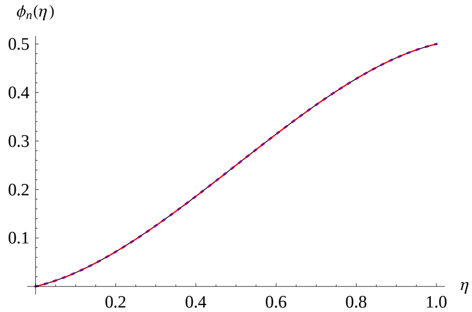

Fig. 7: Curves of $\phi_2(\eta)$ (solid line), $\phi_3(\eta)$ (dash line) and $\phi_4(\eta)$ (dot line) for $S=M=1$ and $\beta=0.1$.

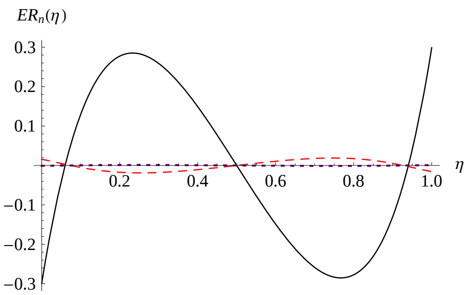

Fig. 8: Curves of $ER_2(\eta)$ (solid line), $ER_3(\eta)$ (dash line) and $ER_4(\eta)$ (dot line) for $S=M=1$ and $\beta=0.1$.

Table 2: The maximal error remainder parameters MER_n in Example 3.

n	2	3	4
MER_n	0.298549	0.0188325	0.00124835
n	5	6	7
MER_n	0.0000845968	5.85126×10^{-6}	4.10099×10^{-7}

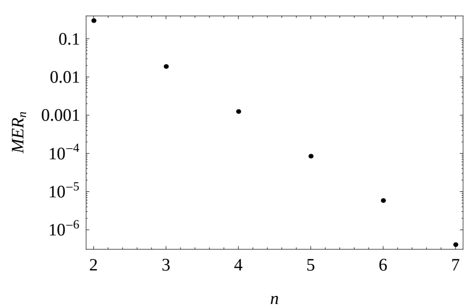

Fig. 9: Logarithmic plots of MER_n for $n=2$ through 7.

4 Conclusions

We have presented a new modification of the Adomian decomposition method to conveniently solve a wide class of higher-order, multipoint, nonlinear boundary value problems with higher-order Robin-type boundary conditions. Our new approach yields a rapidly convergent sequence of analytic functions for the solution without any incorporating undetermined coefficients within the modified recursion scheme for computing successive solution. Thus we avoid the complications resulting from the necessity of evaluating such undetermined coefficients at each stage of approximation. As this resulting sequence of nonlinear algebraic or transcendental equations produces more than one root for the possible value of each undetermined coefficient at each stage, we are then confronted with the ambiguity of multiple roots. Usually this difficulty is handled by discarding unphysical roots, which does not lend itself to automation. Furthermore we can also parameterize the new modified recursion scheme by inserting a predetermined parameter in order to achieve simple-to-integrate series without regard to the possible complexity of the original initial solution component, which is comprised of the boundary values and the input function. The value of the predetermined parameter can favorably increase the rate of convergence and extend the interval of convergence of the resulting parameterized decomposition series solution. We have demonstrated the practicality and efficiency of our new modification of the Adomian decomposition method by several numerical examples. In point of fact, the convergence was already sufficiently rapid, where we demonstrated an approximately exponential rate of convergence of our new modified decomposition series. The overall efficiency of our new modification of the ADM is further enhanced by the new algorithms and subroutines crafted by Duan for generating the Adomian polynomials quickly and to high orders at will. Furthermore the rigorous mathematical convergence of the Adomian decomposition series to the exact solution has already been established by many researchers. In many physical models, an exact closed-form solution is not always possible; however our expository examples have once again demonstrated that only a few terms of the truncated decomposition series provides an excellent approximation even in the complex mathematical models.

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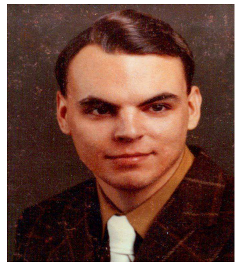
References

[1] J.S. Duan, Z. Wang, S.Z. Fu, T. Chaolu, Parametrized temperature distribution and efficiency of convective straight fins with temperature-dependent thermal conductivity by a new modified decomposition method, Int. J. Heat Mass Transfer **59**, 137-143 (2013).
[2] R. Rach, A.M. Wazwaz, J.S. Duan, A reliable analysis of oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics, Int. J. Biomath. **7**, ID 1450020 (2014).
[3] L. Lu, J.S. Duan, L.Z. Fan, Solution of the magnetohydrodynamics Jeffery-Hamel flow equations by the modified Adomian decomposition method, Adv. Appl. Math. Mech. (2015), doi: 10.4208/aamm.2014.m543.
[4] J.S. Duan, R. Rach, A.M. Wazwaz, Solution of the model of beam-type micro- and nano-scale electrostatic actuators by a new modified Adomian decomposition method for nonlinear boundary value problems, Int. J. Nonlinear Mech. **49**, 159-169 (2013).
[5] A.M. Wazwaz, R. Rach, J.S. Duan, Adomian decomposition method for solving the Volterra integral form of the Lane-Emden equations with initial values and boundary conditions, Appl. Math. Comput. **219**, 5004-5019 (2013).
[6] J.S. Duan, R. Rach, A.M. Wazwaz, T. Chaolu, Z. Wang, A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions, Appl. Math. Model. **37**, 8687-8708 (2013).
[7] Q. Mao, Application of Adomian modified decomposition method to free vibration analysis of rotating beams, Math. Probl. Eng. **2013**, ID 284720 (2013).
[8] R. Rach, J.S. Duan, A.M. Wazwaz, Solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method, J. Math. Chem. **52**, 255-267 (2014).
[9] J.S. Duan, R. Rach, A.M. Wazwaz, A reliable algorithm for positive solutions of nonlinear boundary value problems by the multistage Adomian decomposition method, Open Engineering **5**, 59-74 (2015).
[10] G. Adomian, Stochastic Systems, Academic, New York, 1983.
[11] G. Adomian, Nonlinear Stochastic Operator Equations, Academic, Orlando, FL, 1986.
[12] G. Adomian, Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer Academic, Dordrecht, 1989.
[13] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic, Dordrecht, 1994.
[14] E. Momoniat, C. Harley, An implicit series solution for a boundary value problem modelling a thermal explosion, Math. Comput. Modelling **53**, 249–260 (2011).
[15] X. Shang, P. Wu, X. Shao, An efficient method for solving Emden–Fowler equations, J. Franklin Institute, **346**, 889–897 (2009).
[16] A.M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Higher Education Press, Beijing, and Springer-Verlag, Berlin, 2009.
[17] A.M. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications, Higher Education Press, Beijing, and Springer-Verlag, Berlin, 2011.

[18] S.E. Serrano, Engineering Uncertainty and Risk Analysis: A Balanced Approach to Probability, Statistics, Stochastic Modeling, and Stochastic Differential Equations, Second Revised Edition, HydroScience, Ambler, PA, 2011.
[19] J.S. Duan, R. Rach, D. Baleanu, A.M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, Commun. Frac. Calc. **3**, 73-99 (2012).
[20] R. Rach, A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011, Kybernetes **41**, 1087-1148 (2012).
[21] G. Adomian, R. Rach, Inversion of nonlinear stochastic operators, J. Math. Anal. Appl. **91**, 39-46 (1983).
[22] G. Adomian, R. Rach, On composite nonlinearities and the decomposition method, J. Math. Anal. Appl. **113**, 504-509 (1986).
[23] R. Rach, A convenient computational form for the Adomian polynomials, J. Math. Anal. Appl. **102**, 415-419 (1984).
[24] R. Rach, A new definition of the Adomian polynomials, Kybernetes **37**, 910-955 (2008).
[25] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. **111**, 53-69 (2000).
[26] F. Abdelwahid, A mathematical model of Adomian polynomials, Appl. Math. Comput. **141**, 447-453 (2003).
[27] K. Abbaoui, Y. Cherruault, V. Seng, Practical formulae for the calculus of multivariable Adomian polynomials, Math. Comput. Modelling **22**, 89-93 (1995).
[28] Y. Zhu, Q. Chang, S. Wu, A new algorithm for calculating Adomian polynomials, Appl. Math. Comput. **169**, 402-416 (2005).
[29] J. Biazar, M. Ilie, A. Khoshkenar, An improvement to an alternate algorithm for computing Adomian polynomials in special cases, Appl. Math. Comput. **173**, 582-592 (2006).
[30] M. Azreg-Aïnou, A developed new algorithm for evaluating Adomian polynomials, CMES-Comput. Model. Eng. Sci. **42**, 1-18 (2009).
[31] J.S. Duan, Recurrence triangle for Adomian polynomials, Appl. Math. Comput. **216**, 1235-1241 (2010).
[32] J.S. Duan, An efficient algorithm for the multivariable Adomian polynomials, Appl. Math. Comput. **217**, 2456-2467 (2010).
[33] J.S. Duan, Convenient analytic recurrence algorithms for the Adomian polynomials, Appl. Math. Comput. **217**, 6337-6348 (2011).
[34] J.S. Duan, New recurrence algorithms for the nonclassic Adomian polynomials, Comput. Math. Appl. **62**, 2961-2977 (2011).
[35] J.S. Duan, R. Rach, A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations, Appl. Math. Comput. **218**, 4090-4118 (2011).
[36] J.S. Duan, R. Rach, Z. Wang, On the effective region of convergence of the decomposition series solution, J. Algorithms Comput. Tech. **7**, 227-247 (2013).
[37] L. Lu, J.S. Duan, How to select the value of the convergence parameter in the Adomian decomposition method, CMES-Comput. Model. Eng. Sci. **97**, 35-52 (2014).
[38] J.S. Duan, R. Rach, A.M. Wazwaz, A new modified Adomian decomposition method for higher-order nonlinear dynamical systems, CMES-Comput. Model. Eng. Sci. **94**, 77-118 (2013).
[39] M. Usman, A. Nazir, Z. Naheed, S.T. Mohyud-Din, Adomian's decomposition method to squeezing flow and heat transfer between two parallel disks with velocity slip and temperature jump, Adv. Diab. Metab. **1**, 37-44 (2013).



Randolph Rach is retired and formerly a Senior Engineer at Microwave Laboratories Inc. His experience includes research and development in microwave electronics and travelling-wave tube technology, and his research interests span nonlinear system analysis, nonlinear ordinary and partial differential equations, nonlinear integral equations and nonlinear boundary value problems. He has both authored and co-authored more than 100 papers in Applied Mathematics and is an early contributor to the Adomian decomposition method.

Jun-Sheng Duan is a Professor of Mathematics at Shanghai Institute of Technology in Shanghai, PR China. He has both authored and co-authored more than 80 papers in applied mathematics and mathematical physics. He has contributed notably to theoretical advances in the Adomian decomposition method, including new efficient algorithms leading to fast computer subroutines.

Abdul-Majid Wazwaz is a Professor of Mathematics at Saint Xavier University in Chicago, Illinois, USA. He has both authored and co-authored more than 400 papers in applied mathematics and mathematical physics. He is the author of five books on the subjects of discrete mathematics, integral equations and partial differential equations. Furthermore, he has contributed extensively to theoretical advances in solitary waves theory, the Adomian decomposition method and the variational iteration method.