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A New Method to Modelling the Additive Functional Equations

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Abstract: This paper elucidates a possible method to model additive type of functional equations using eigenvalues and eigenvectors of matrices with suitable numerical examples. In order to model a functional equation the author has defined a new type of Consecutive Column Matrix(CCM) and discussed its eigenvalues and eigenvectors. The author has also modelled a famous additive Cauchy functional equation using identity matrix.

Keywords: Additive functional equation, eigenvalues and eigenvectors of matrices

1 Introduction

The study of functional equations is a contemporary area of mathematics that provides a powerful approach to working with important concepts and relationships in analysis and algebra such as symmetry, linearity and equivalence. Although the systematic study of such equations is a relatively recent area of mathematical study, they have been considered earlier in various forms by mathematicians such as Euler in the 18^{th} century and Cauchy in the 19^{th} century.

The theory of functional equations[6,7] is a growing branch of mathematics which has contributed a lot to the development of the strong tools in today's mathematics. Many new applied problems and theories have motivated functional equations to develop new approaches and methods. D'Alembert, Euler, Gauss, Cauchy, Abel, Weierstrass, Darboux and Hilbert are among the great mathematicians who have been concerned with functional equations and methods of solving them.

Functional equations represent an alternative way of modelling problems in Physics. The interest of modelling physical problems by functional equations is that we do not have to assume the differentiability of the function f. Consequently, the functional equations lead often to other solutions than those given by partial differential equations, and these other solutions can be of interest to physicists. The most appealing characteristic of functional equation is its capacity to design mathematical models.

Their have been so many researchers study the solution and stability of different types of functional equations like additve, quadratic, cubic, quartic and mixed type of additive-quadratic, quadratic-cubic and so on. How ever, studies over the origin and formation of such functional equations are not convincing as they are only structured based trial and error method. This fact has been influential for the author to under go a formal study on modelling a functional equations.

This paper is organised as follows: In section 2, author discusses the preliminaries and definitions of functional equations. In section 3, the author introduces and discusses the new type of Consecutive Column Matrices(CCM) and models the additive functional equations using CCM along with its eigenvalues and eigenvectors. In section 4, author models the most famous additive Cauchy functional equation and gives the conclusion in section 5.

2 Preliminaries and Definitions

A Hungarian Mathematician J. Aczel [1], an excellent specialist in functional equations, defines the functional equation as follows:

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Definition 1.*Functional Equation:*
Functional Equations are equations in which both sides are terms constructed from the finite number of unknown functions and a finite number of independent variables.

Example 1.(i) $f(x+y) = f(x) + f(y)$
(Cauchy Additive Functional Equation)
(ii) $f(x+y) + f(x-y) = 2f(x) + 2f(y)$
(Lee-An-Park Quadratic Functional Equation)
(iii) $f(2x+y) + f(2x-y) = 8f(x) + 2f(y)$
(Abbas Najati Cubic Functional Equation)

Definition 2.*Solutions of Functional Equation:*
A solution of a functional equation is a function which satisfies the equation.

*Example 2.*Cauchy Functional Equations

$$\begin{aligned} f(x+y) &= f(x) + f(y), \\ f(x+y) &= f(x)\ f(y), \\ f(xy) &= f(x) + f(y) \end{aligned}$$

have solutions $f(x) = kx$, $f(x) = e^x$, $f(x) = lnx$, respectively.

The functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

is the most famous among the functional equations. It is often called the additive Cauchy functional equation in honor of A. L. Cauchy. The properties of the additive Cauchy equation are powerful tools in almost every field of natural and social sciences.

Consider the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{2}$$

The function $f(x) = x^2$ is the solution of the functional equation (2). Hence it is called the quadratic functional equation (or) Euler - Lagrange functional equation [8] and every solution of the quadratic functional equation (2) is called quadratic function [2,3,4,5].

In 2001, J. M. Rassias [9] introduced the pioneering cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) = 6f(y) \tag{3}$$

and established the solution of the Ulam stability problem for these cubic mappings. It is easy to show that the function $f(x) = x^3$ satisfies the functional equation(3), which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

Very recently, using the fixed point method, the following authors: Tian Zhou Xu, John Michael Rassias, Wan Xin Xu [10] are investigated the generalized Hyers-Ulam stability of the general mixed additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} & f(x+ny) + f(x-ny) \\ &= n^2 f(x+y) + n^2 f(x-y) + 2(1-n^2) f(x) \\ &+ \frac{n^4 - n^2}{12}[f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned} \tag{4}$$

for fixed integers n with $n \neq 0, 1$ in multi-Banach spaces. It is easy to show that the function $f(x) = x^4 + x^3 + x^2 + x$ satisfies the functional equation(4), which is called a mixed type functional equation.

3 Main Results

Definition 3.*Consecutive Column Matrix(CCM):*
A square matrix is said to be consecutive column matrix if it has consecutive numbers in each column entries further the sum of each row entries are equal. A consecutive column matrix of order 2 is denoted by CCM-2.

*Example 3.*A general CCM-2 matrix is given by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where $a_{11}, a_{21}; a_{12}, a_{22}$ are consecutive numbers with $a_{11} + a_{12} = a_{21} + a_{22}$.

Numerical example:

$$(i) \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad (ii) \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

*Note 1.*The identity matrix of order 2 is the best example of CCM-2 matrix.

Lemma 1. *If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then the eigenvalues of A are*
(i) $(a_{11} - a_{21})$ which is always ± 1.
(ii) $a_{11} + a_{12}$ is another eigenvalue.

*Proof.*The characteristic equation of the matrix A is $\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$, which gives

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0. \tag{5}$$

If λ_1 and λ_2 are the two eigenvalues then by solving equation (5), we get

$$\begin{aligned} &\big((\lambda - (a_{11} - a_{21}))(\lambda - (a_{11} + a_{12}))\big) = 0 \ (or) \\ &\big((\lambda - (a_{22} - a_{12}))(\lambda - (a_{11} + a_{12}))\big) = 0. \end{aligned} \tag{6}$$

From the equation (6), we arrive
$\lambda_1 = a_{11} - a_{21}$ and $\lambda_2 = a_{11} + a_{12}$

Numerical example:
The CCM-2 matrices and its eigen values are

(i) $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}; \lambda_1 = 1 \, and \, \lambda_2 = 5.$

(ii) $\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}; \lambda_1 = -1 \, and \, \lambda_2 = 1.$

Remark 1. If λ_1 and λ_2 be the eigenvalues of CCM-2 matrix, then the difference between eigenvalues is equal to the sum of second diagonal elements in a matrix. That is $\lambda_2 - \lambda_1 = a_{12} + a_{21}$.

Numerical example:

For the CCM-2 matrix $\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}; \lambda_1 = -1, \lambda_2 = 1$, we note that $\lambda_2 - \lambda_1 = a_{12} + a_{21} = 2$.

The following remark gives the eigenvectors X_1 and X_2 corresponding to the eigenvalues λ_1 and λ_2 of the CCM-2 matrix.

Remark 2. For the CCM-2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, its eigenvalues are $\lambda_1 = a_{11} - a_{21}$, $\lambda_2 = a_{11} + a_{12}$. Let X_1 and X_2 are eigenvectors of the CCM-2 matrix A corresponding to the eigenvalues λ_1 and λ_2 respectively, then

$$X_1 = \begin{pmatrix} a_{12} \\ -a_{21} \end{pmatrix}, \; X_2 = \begin{pmatrix} a_{11} + a_{12} \\ a_{11} + a_{12} \end{pmatrix}.$$

Numerical example:

When $\lambda_1 = -1$ is the eigen value of CCM-2 matrix $\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$, the eigenvector X_1 corresponding to λ_1 is $X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. When $\lambda_2 = 1$ is the eigenvalue of CCM-2 matrix, the eigenvector X_2 corresponding to λ_2 is $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Definition 4. ***Additive Matrix:***
A matrix is said to be additive matrix, if it gives the additive functional equation.

Possible Model Matrices for CCM-2:
The possible model matrices for CCM-2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ are $M_1 = [\alpha X_1, \beta X_2]$ and $M_2 = [\beta X_2, \alpha X_1]$, where $\alpha, \beta \in \mathbb{R}$.

Now, using A and M_1 or M_2 we may model the following functional equation. Let X and Y be the Vector spaces. Define a mapping $f : X \to Y$, then

$$f(a_{11}x + m_{11}y) + f(a_{12}x + m_{12}y) = f(a_{21}x + m_{21}y) + f(a_{22}x + m_{22}y) + \alpha f((\lambda_2 - \lambda_1)y) \tag{7}$$

for all $x, y \in X$, $\alpha \in \mathbb{R}$, λ_1 and λ_2 are eigenvalues of A.

It is easy to check that the functional equation (7) is additive for $\alpha, \beta \in \mathbb{R}$.

Hence, the CCM-2 matrices are additive matrices, since it gives additive functional equations.

Numerical example: Let $B = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ be the CCM-2 matrix.

Here, the author selects some of the following possible model matrices of B and models the functional equations using (7).

$$N_1 = \begin{pmatrix} -2 & 5 \\ 4 & 5 \end{pmatrix} \text{ for } \alpha = -1, \; \beta = 1,$$

$$N_2 = \begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix} \text{ for } \alpha = -1, \; \beta = \tfrac{1}{5},$$

$$N_3 = \begin{pmatrix} 2 & 5 \\ -4 & 5 \end{pmatrix} \text{ for } \alpha = 1, \beta = 1,$$

$$N_4 = \begin{pmatrix} -1 & 5 \\ 2 & 5 \end{pmatrix} \text{ for } \alpha = -\tfrac{1}{2}, \; \beta = 1,$$

$$N_5 = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \text{ for } \alpha = -\tfrac{1}{2}, \; \beta = \tfrac{1}{5},$$

$$N_6 = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \text{ for } \alpha = \tfrac{1}{2}, \; \beta = \tfrac{1}{5}.$$

Now, using B and N_1 through (7), we may model the following functional equation

$$f(3x - 2y) + f(2x + 5y) = f(4x + 4y) + f(x + 5y) - f(6y) \tag{8}$$

for all $x, y \in X$. Using B and N_2 through (7), we may model the following functional equation

$$f(3x - 2y) + f(2x + y) = f(4x + 4y) + f(x + y) - f(6y) \tag{9}$$

for all $x, y \in X$. Using B and N_3 through (7), we may model the following functional equation

$$f(3x + 2y) + f(2x + 5y) = f(4x - 4y) + f(x + 5y) + f(6y) \tag{10}$$

for all $x, y \in X$. Using B and N_4 through (7), we may model the following functional equation

$$f(3x - y) + f(2x + 5y) = f(4x + 2y) + f(x + 5y) - \frac{1}{2} f(6y) \tag{11}$$

for all $x, y \in X$. Using B and N_5 through (7), we may model the following functional equation

$$f(3x - y) + f(2x + y) = f(4x + 2y) + f(x + y) - \frac{1}{2} f(6y) \tag{12}$$

for all $x, y \in X$. Using B and N_6 through (7), we may model the following functional equation

$$\begin{aligned} & f(3x+y)+f(2x+y) \\ &\quad = f(4x+y)+f(x-2y)+\frac{1}{2}f(6y) \end{aligned} \tag{13}$$

for all $x, y \in X$.

Clearlly the functional equations from (8) to (13) are additive, since $f(x) = x$ is the solution.

Hence, the CCM-2 matrices are additive matrices.

4 Modelling the Additive Cauchy Functional Equation

In this section, the author models additive Cauchy functional equation (1) which is one of the most famous functional equations.

Consider CCM-2 matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is a well known Identity matrix. Based on the possible model matrices M_1 and M_2, we get the model matrices of I are

$$N_7 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \alpha = 1, \ \beta = 1 \text{ and}$$

$$N_8 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad \alpha = 1, \ \beta = 1.$$

Now, using I and N_7 or N_8 through (7) with $f(0) = 0$, we get the following additive Cauchy functional equation

$$f(x)+f(y) = f(x+y).$$

5 Conclusion

Thus, a new type of matrix CCM-2 is introduced and its eigenvalues and eigenvectors have been discussed and explained. This is the first attempt to model additive functional equations using eigenvalues and eigenvectors of matrices.

A famous Cauchy functional equation (1) has also been modelled using Identity matrix of order 2 with the proposed method.

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