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Some Fixed Point Theorems in Multiplicative Cone b-Metric Spaces

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Abstract: In this paper, we introduces the notion of multiplicative cone b-metric space and prove some fixed point theorems in setting of complete multiplicative cone b-metric space. An application of our main result is also provided.

Keywords: Multiplicative cone b-metric space, fixed point, contractive mapping.
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1 Introduction

The study of fixed point theory become a subject of great interest due to its applications in Mathematics as well as in other areas of research. There are many researchers who have worked in fixed point theory of contractive mapping see, [1,5]. In [1], Banach presented a most outstanding result concerning to contraction mapping, this famous result is known as Banach contraction principle. The idea of b-metric space was presented by [2] as a generalized form of metric spaces. In [2], Bakhtin proves the contraction mapping principle in b-metric spaces that generalized the Banach contraction principle in metric spaces. Moreover the idea of cone metric space was presented by Haung and Zhang in [7]. Their work includes some fixed point results for contractive type mappings in cone metric spaces. In 1993, Cezerwik [4] proved some fixed point theorems in b-metric spaces. In [8], Hussain and Shah introduced cone b-metric spaces, they established some topological properties in cone b-metric spaces. Moreover Huaping and Shaoyuan in [9], proves some fixed point theorems of contractive mappings with some new examples and applications in cone b-metric spaces, and this results improves some fixed point results in metric spaces and b-metric spaces, as well as they expanded other concern work and results in cone metric spaces. In 2015, George et al. [6] generalized cone metric, cone b-metric and rectangular metric space and introduced the new concept of generalized cone b-metric space. They also proved a well known contraction principle and presented some fixed point results in their work. Ozavsar and Cevikel [11] introduced the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings on a complete multiplicative metric space. They also gave some topological properties of the relevant multiplicative metric space. Hxiaoju et al. [10] studied common fixed points for weak commutative mappings on a multiplicative metric space. For further details about multiplicative metric space and related concepts, we refer the reader to [3,11].
In this paper we, introduces the concept of multiplicative cone b-metric space and proved some fixed point results in multiplicative cone b-metric space. We also presented an example to hold up the main result and show the uniqueness and existence of a solution of first order ordinary differential equation as an application point of view.

2 Preliminaries

Consistent with Haung and Zhang [7] & Hussain and Shah [8], we will be needed the following definitions in this paper.
Suppose E be a real Banach space and P be a subset of E. Then P is called cone if and only if:
(i) P is closed, nonempty and $P \neq \{0\}$,
(ii) $cx + dy \in P$ for all $x, y \in P$ and non-negative real

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numbers c,d.
(iii) $P \cap (-P) = \{0\}$.

Definition 2.1[7] *Suppose P be a cone in real Banach space E, define a partial ordering $\leq$ with respect to P by $a \leq b \Longleftrightarrow b - a \in P$. We can write $a < b$ to show that $a \leq b$ but $a \neq b$, while $a \ll b$ will stand for $b - a \in \operatorname{int} P$, where $\operatorname{int} P$ denotes interior of P.*

Through this paper we always suppose that E is a Banach space, P is a cone in E with int $P \neq \emptyset$ and $\leq$ is partial ordering w.r.t cone.

Definition 2.2[7] *Let Y be a non-empty set. Let the mapping $d : Y \times Y \to E$ satisfies:*
(i) $0 \leq d(x,y)$ *for all* $x,y \in Y$ *with* $x \neq y$;
(ii) $d(x,y) = 0$ *if and only if* $x = y$;
(iii) $d(x,y) = d(y,x)$ *for all* $x,y \in Y$;
(iv) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x,y,z \in Y$.
Then the function d is said to be a cone metric on Y and (Y,d) is called a cone metric space.

Definition 2.3[8] *Suppose that Y be a non-empty set and $s \geq 1$ be a given positive real number. A function $d : Y \times Y \to E$ is said to be cone b-metric if and only if, $\forall\, x,y,z \in Y$, the following assertions are satisfied:*
(i) $0 \leq d(x,y)$ *for all* $x,y \in Y$ *with* $x \neq y$;
(ii) $d(x,y) = 0$ *if and only if* $x = y$;
(iii) $d(x,y) = d(y,x)$ *for all* $,y \in Y$;
(iv) $d(x,y) \leq s[d(x,z) + d(z,y)]$ *for all* $x,y,z \in Y$.
Then the function d is said be a cone b-metric on Y and (Y,d) is called a cone b-metric space.

It is to be noted that any cone metric space is cone b-metric space but generally the converse is not true.
In this work we modified cone b-metric space to multiplicative cone b-metric space.

Definition 2.4*Let Y be a non-empty set and $s \geq 1$ be a given positive real number. A mapping $d : Y \times Y \to E$ satisfies:*
(i) $1 \leq d(x,y)$ *for all* $x,y \in Y$ *with* $x \neq y$;
(ii) $d(x,y) = 1$ *if and only if* $x = y$;
(iii) $d(x,y) = d(y,x)$ *for all* $x,y \in Y$;
(iv) $d(x,y) \leq [d(x,z).d(z,y)]^s$ *for all* $x,y,z \in Y$.
Then the function d is said to be a multiplicative cone b-metric on Y and (Y,d) is said to be a multiplicative cone b-metric space.

Example 2.5*Let* $d(x,y) = a^{(\sum_{i=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}}}$ *for all* $x,y \in X$; *P>1*
we show that $d(x,y)$ is multiplicative cone b-metric space i.e (m_{cb}, d), but not multiplicative cone metric space i.e (m_c, d).
***Solution**: Since*

$$(a+b)^p \leq (2\,max(a,b))^p \leq 2^p(|a|^p + |b|^p) \text{ for all } a,b \geq 0. \tag{2.1}$$

Let $u = x_i - z_i$ *and* $x_i - y_i = u + v$
By inequality (2.1),

$$\begin{aligned}
a^{|x_i-y_i|^p} &\leq a^{(|u+v|^p)} \\
&\leq a^{(|u|+|v|)^p} \\
&\leq a^{2^p(\Sigma|u|^p+\Sigma|v|^p)} \\
&\leq a^{2^p(\Sigma|u|^p+\Sigma|v|^p)^{\frac{1}{p}}} \\
&\leq (a^{2^p\Sigma|u|^p}.a^{2^p\Sigma|v|^p})^{\frac{1}{p}} \\
&\leq (a^{(\Sigma|x_i-z_i|^p)^{\frac{1}{p}}}.a^{(\Sigma|z_i-y_i|^p)^{\frac{1}{p}}})^{2^p} \\
&\leq (d(x,z).d(z,y))^{2^p} \\
\Rightarrow d(x,y) &\leq (d(x,z).d(z,y))^{2^p}.
\end{aligned}$$

Which shows that $d(x,y)$ is multiplicative cone b-metric space.
Conversely: Let

$$\begin{aligned}
(a+b)^p &> a^{(a^p+b^p)} \\
|u+v|^p &= |x-y|^p \\
a^{|x_i-y_i|^p} &= a^{(u+v)^p} > a^{(u^p+v^p)} = a^{((x_i-z_i)^p+(z_i-y_i)^p)} \\
a^{(\Sigma|x_i-y_i|^p)^{\frac{1}{p}}} &> a^{(\Sigma(x_i-z_i)^p)^{\frac{1}{p}}}.a^{(\Sigma(z_i-y_i)^p)^{\frac{1}{p}}} \\
\Rightarrow d(x,y) &> d(x,z).d(z,y).
\end{aligned}$$

Thus the Triangle inequality of (m_c, d) is not satisfying. That is, (Y,d) is not multiplicative cone metric space.

We state the following lemmas without proof.

Lemma 1.*Suppose that $\{g_m\}$ be a sequence in real Banach space E and P be a cone. If $c \in \operatorname{int} P$ and $g_m \to 1$ as $(m \to \infty)$, then there exists N such that for all $m > N$, we have $g_m \ll c$.*

Lemma 2.*Suppose P be a cone and $g \ll c$ for all $c \in \operatorname{int} P$, then $g = 1$.*

Lemma 3.*Suppose P be a cone. If $g \in P$ and $g \leq g^k$ for some $k \in [0,1)$, then $g = 1$.*

3 Main Results

In this section, we proved some fixed point theorems in the setting of multiplicative cone b-metric spaces. Our main result is follow as:

Theorem 3.1*Let (Y,d) be a complete multiplicative cone b-metric space with power $s \geq 1$. Suppose the mapping $T : Y \to Y$ satisfies the following contractive condition,*

$$d(Ta, Tb) \leq d(a,b)^{\lambda} \text{ for all } a,b \in Y$$

where $0 \leq \lambda < 1$ is a constant. Then T has a unique fixed point in Y.

Proof. Choose $a_0 \in Y$. We construct the iterative sequence $\{a_n\}$, where $a_n = Ta_{n-1}$, $n \geq 1$, i.e., $a_{n+1} = Ta_n = T^{n+1}a_0$. We have

$$d(a_{n+1},a_n) = d(Ta_n,Ta_{n-1}) \leq d(a_n,a_{n-1})^{\lambda} \leq ... \leq d(a_1,a_0)^{\lambda^n}.$$

For any $m \geq 1$, $p \geq 1$, it follows that

$$\begin{aligned} d(a_m,a_{m+p}) &\leq [d(a_m,a_{m+1}).d(a_{m+1},a_{m+p})]^s \\ &\leq [d(a_m,a_{m+1})]^s.[d(a_{m+1},a_{m+2}).d(a_{m+2},a_{m+p})]^{s^2} \\ &\leq d(a_m,a_{m+1})^s.[d(a_{m+1},a_{m+2})]^{s^2}.[d(a_{m+2},a_{m+3})]^{s^3} \\ &. \quad \ldots.d(a_{m+p-2},a_{m+p-1})^{s^{p-1}}.d(a_{m+p-1},a_{m+p})^{s^{p-1}} \\ &\leq d(a_1,a_0)^{s\lambda^m}.d(a_1,a_0)^{s^2\lambda^{m+1}}.d(a_1,a_0)^{s^3\lambda^{m+2}} \\ &. \quad \ldots.d(a_1,a_0)^{s^{p-1}\lambda^{m+p-2}}.d(a_1,a_0)^{s^p\lambda^{m+p-1}} \\ &= d(a_1,a_0)^{(s\lambda^m+s^2\lambda^{m+1}+s^3\lambda^{m+2}+\ldots+s^p\lambda^{m+p-1})} \\ &\leq d(a_1,a_o)^{\frac{s\lambda^m}{1-s\lambda}} \end{aligned}$$

Let $1 \ll c$ be given. Notice that $d(a_1,a_o)^{\frac{s\lambda^m}{1-s\lambda}} \to 1 \,\text{as}\, m \to \infty$ for any p. By Lemma 1, we find $m_0 \in N$ such that

$$d(a_1,a_o)^{\frac{s\lambda^m}{1-s\lambda}} \ll c,$$

for each $m > m_0$. thus,

$$d(a_m,a_{m+p}) \leq d(a_1,a_o)^{\frac{s\lambda^m}{1-s\lambda}} \ll c,$$

for all $m > m_0$ and any p. So, a_n is Cauchy sequence in (Y,d).
Since (Y,d) is a complete multiplicative cone b-metric space, there exist $a^* inY$ such that $a_n \to a^*$. Take $n_0 \in \mathbb{N}$ such that $d(a_n,a^*) \ll \frac{c}{s(\lambda+1)}$ for all $n > n_0$. Hence

$$d(Ta^*,a^*) \leq [d(Ta^*,Ta_n).d(Ta_n,a^*)]^s \leq [\lambda d(a^*,a_n).d(a_{n+1},a^*)]^s \ll c,$$

for each $m > m_0$. By use of Lemma 2, we deduce that $d(Ta^*,a^*) = 1$, i.e., $Ta^* = a^*$. That is, a^* is a fixed point of T.
For proving uniqueness, if there is another fixed point b^*, then by the given assertion,

$$d(a^*,b^*) = d(Ta^*,Tb^*) \leq d(a^*,b^*)^{\lambda}.$$

By Lemma 3, $a^* = b^*$. This complete the proof.

Theorem 3.2 Let (Y,d) be a complete multiplicative cone b-metric space with power $s \geq 1$. suppose the mapping $T : Y \to Y$ holds the contractive condition:*

$$d(Ta,Tb) \leq d(a,Ta)^{\lambda_1}.d(b,Tb)^{\lambda_2}.d(a,Tb)^{\lambda_3}.d(b,Ta)^{\lambda_4} \text{ for } a,\, b \in Y,$$

where $0 \leq \lambda_i < 1/s$ is a constant. Then T has a unique fixed point in Y.

Proof. Choose $a_0 \in Y$ and $a_1 = Ta_0$ and $a_{n+1} = Ta_n = T^{n+1}a_0$.
First we see

$$\begin{aligned} d(a_{n+1},a_n) &= d(Ta_n,Ta_{n-1}) \\ &\leq d(a_n,Ta_n)^{\lambda_1}.d(a_{n-1},Ta_{n-1})^{\lambda_2}.d(a_n,Ta_{n-1})^{\lambda_3}.d(a_{n-1},Ta_n)^{\lambda_4} \\ &\leq d(a_n,a_{n+1})^{\lambda_1}.d(a_{n-1},a_n)^{\lambda_2}.[d(a_{n-1},a_n).d(x_n,a_{n+1})]^{s\lambda_4} \\ &\leq d(a_n,a_{n+1})^{(\lambda_1+s\lambda 4)}.d(a_n,a_{n-1})^{(\lambda_2+s\lambda 4)}. \end{aligned}$$

It follows that

$$d(a_{n+1},a_n)^{(1-\lambda_1-s\lambda 4)} \leq d(a_n,a_{n-1})^{(\lambda_2+s\lambda 4)}. \tag{3.1}$$

Secondly,

$$\begin{aligned} d(a_{n+1},a_n) &= d(Ta_n,Ta_{n-1}) = d(Ta_{n-1},Ta_n) \\ &\leq d(a_{n-1},Ta_{n-1})^{\lambda_1}.d(a_n,Ta_n)^{\lambda_2}.d(a_{n-1},Ta_n)^{\lambda_3}.d(a_n,Ta_{n-1})^{\lambda_4} \\ &\leq d(a_{n-1},a_n)^{\lambda_1}.d(a_n,a_{n+1})^{\lambda_2}.[d(a_{n-1},a_n).d(x_n,a_{n+1})]^{s\lambda_3} \\ &\leq d(a_n,a_{n+1})^{(\lambda_2+s\lambda_3)}.d(a_n,a_{n-1})^{(\lambda_1+s\lambda_3)}. \end{aligned}$$

This establishes that

$$d(a_{n+1},a_n)^{(1-\lambda_2-s\lambda_3)} \leq d(a_n,a_{n-1})^{(\lambda_1+s\lambda_3)}. \tag{3.2}$$

Adding (3.1) and (3.2) gives
$$d(a_{n+1},a_n) = d(a_n,a_{n-1})^{\frac{\lambda_1+\lambda_2+s(\lambda_3+\lambda_4)}{2-\lambda_1-\lambda_2-s(\lambda_3+\lambda_4)}}.$$

Put $\lambda = \frac{\lambda_1+\lambda_2+s(\lambda_3+\lambda_4)}{2-\lambda_1-\lambda_2-s(\lambda_3+\lambda_4)}$, it is easy to see that $\lambda \in [0,1)$. Thus,
$d(a_{n+1},a_n) \leq d(a_n,a_{n-1})^{\lambda} \ldots d(a_1,a_0)^{\lambda^n}$.
We can follow the same argument that is given in Theorem 3.1, there exist $a^* \in Y$ such that $a_n \to a^*$. Let $c \gg 1$ be arbitrary. Since $a_n \to a^*$, there exist N such that

$$d(a_n,a^*) \ll \frac{2-s\lambda_1-s\lambda_2-s^2\lambda_3-s^2\lambda_4}{2s^2+2s}c$$

Next we want to show that a^* is a fixed point of T. Actually, on the one hand,

$$\begin{aligned} d(Ta^*,a^*) &\leq [d(Ta^*,Ta_n).d(Ta_n,a^*)]^s = d(Ta^*,Ta_n)^s.d(a_{n+1},a^*)^s \\ &\leq [d(a^*,Ta^*)^{\lambda_1}.d(a_n,Ta_n)^{\lambda_2}.d(a^*,Ta_n)^{\lambda_3}.d(a_n,Ta^*)^{\lambda_4}]^s.d(a_{n+1},a^*)^s \\ &= [d(a^*,Ta^*)^{\lambda_1}.d(a_n,a_{n+1})^{\lambda_2}.d(a^*,a_{n+1})^{\lambda_3}.d(a_n,Ta^*)^{\lambda_4}]^s.d(a_{n+1},a^*)^s \\ &\leq d(a^*,Ta^*)^{s\lambda_1}.d(a_n,a^*)^{s^2\lambda_2}.d(a^*,a_{n+1})^{s^2\lambda_2}.d(a^*,a_{n+1})^{s\lambda_3} \\ &.d(a_n,a^*)^{s^2\lambda_4}.d(a^*,Ta^*)^{s^2\lambda_4}.d(a_{n+1},a^*)^s \\ &= d(a^*,Ta^*)^{(s\lambda_1+s^2\lambda_4)}.d(a_n,a^*)^{(s^2\lambda_2+s^2\lambda_4)} \\ &.d(a^*,a_{n+1})^{(s^2\lambda_2+s\lambda_3+s)}. \end{aligned}$$

Implies that

$$d(a^*,Ta^*)^{(1-s\lambda_1-s^2\lambda_4)} \leq d(a_n,a^*)^{(s^2\lambda_2+s^2\lambda_4)}.d(a^*,a_{n+1})^{(s^2\lambda_2+s\lambda_3+s)}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} d(a^*,Ta^*) &\leq [d(a^*,Ta_n).d(Ta_n,Ta^*)]^s = d(a^*,x_{n+1})^s.d(Tx_n,Tx^*)^s \\ &\leq d(a^*,a_{n+1})^s.[d(a_n,Ta_n)^{\lambda_1}.d(a^*,Ta^*)^{\lambda_2}.d(a_n,Ta^*)^{\lambda_3}.d(a^*,Ta_n)^{\lambda_4}]^s \\ &= d(a^*,a_{n+1})^s.[d(a_n,a_{n+1})^{\lambda_1}.d(a^*,Ta^*)^{\lambda_2}.d(a_n,Ta^*)^{\lambda_3}.d(a^*,a_{n+1})^{\lambda_4}]^s \\ &\leq d(a^*,a_{n+1})^s.d(a_n,a^*)^{s^2\lambda_1}.d(a^*,a_{n+1})^{s^2\lambda_1}.d(a^*,Ta^*)^{s\lambda_2}.d(a_n,a^*)^{s\lambda_3} \\ &.d(a^*,Ta^*)^{s^2\lambda_3}.d(a^*,a_{n+1})^{s\lambda_4} \\ &= d(a^*,Ta^*)^{(s\lambda_2+s^2\lambda_3)}.d(a_n,a^*)^{(s^2\lambda_1+s^2\lambda_3)} \\ &.d(a^*,a_{n+1})^{(s^2\lambda_1+s\lambda_4+s)}. \end{aligned}$$

Which means that

$$d(a^*,Ta^*)^{(1-s\lambda_2-s^2\lambda_3)} \leq d(a_n,a^*)^{(s^2\lambda_1+s^2\lambda_3)}.d(a^*,a_{n+1})^{(s^2\lambda_1+s\lambda_4+s)}. \tag{3.4}$$

Adding (3.3) and (3.4) gives

$$\begin{aligned}&(2-s\lambda_1-s\lambda_2-s^2\lambda_3-s^2\lambda_4)d(a^*,Ta^*)\\ \leq\ & d(a_n,a^*)^{s^2(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}.d(a^*,a_{n+1})^{(s^2\lambda_1+s^2\lambda_2+s\lambda_3+s\lambda_4)}\\ \leq\ & d(a_n,a^*)^{s^2}.d(a^*,a_{n+1})^{(s^2+2s)}.\end{aligned}$$

By simple calculations,

$$d(a^*,Ta^*)\leq \frac{s^2d(a_n,a^*)+(s^2+2s)d(a^*,a_{n+1})}{2-s\lambda_1-s\lambda_2-s^2\lambda_3-s^2\lambda_4}\ll c.$$

we can see easily from Lemma 2 that $d(a^*,Ta^*)=1$, i.e., the mapping T has a fixed point a^*. At last, for uniqueness, if there is another fixed point b^*, then

$$\begin{aligned}d(a^*,b^*)&=d(Ta^*,Tb^*)\\ &\leq d(a^*,Ta^*)^{\lambda_1}.d(b^*,Tb^*)^{\lambda_2}.d(a^*,Tb^*)^{\lambda_3}.d(b^*,Ta^*)^{\lambda_4}\\ &\leq [d(a^*,b^*).d(b^*,Tb^*)]^{s\lambda_3}.[d(b^*,a^*).d(a^*,Ta^*)]^{s\lambda_4}\\ &= d(a^*,b^*)^{s(\lambda_3+\lambda_4)}.\end{aligned}$$

Owing to $0\leq s(\lambda_3+\lambda_4)<1$, we deduce from Lemma 3 that $a^*=b^*$. This compete the proof.

4 Application

We can make use of Theorem (3.1) in this section to the given first order boundary problem of periodic type

$$\begin{aligned}b^*&=F(t,b(t))\\ b(1)&=b_0\end{aligned}\tag{4.1}$$

We can write (4.1) as, $b(t)=b_0\int_1^t F(t,b(t))^{dt}$

Example 4.1*Consider the boundary problem* (4.1) *with the continuous function F and let $F(a,b)$ satisfies the local multiplicative Lipschitz condition, that is,*

$$\left|\frac{G(a,b)}{G(a,c)}\right|\leq L^{\left|\frac{b}{c}\right|}$$

Set $N=\max_{[-h,h]\times[a,b]}\Big|F(a,b)\Big|$, then it must have a unique solution of (4.1).

Solution *Let $Y=E=C^*([-h,h])$ and $P=\{a\in E: a\geq 0\}$.*
Put $d^*: Y\times Y\to E$ as $d^*(a,b)=\max_{-h\leq t\leq h}\left|\frac{a(t)}{b(t)}\right|$ with $f=C^*[-h,h]\to\mathbb{R}$.
Clearly (Y,d^*) is a complete multiplicative cone b-metric space. We can write (4.1) *as,*

$$x(t)=y_0\int_1^t F(s,x(s))^{ds}$$

A mapping is defined as $T: C([-h,h])\to\mathbb{R}$ by $Ta(t)=b_0\int_1^t F(s,a(s))^{ds}$.

If $a(t),b(t)\in B(\xi,\delta)=\{\phi(t)\in C^*[-h,h]: d(\phi,\xi)\leq\delta\}$ then from

$$\begin{aligned}d^*(Ta,Tb)&=\max_{-h\leq t\leq h}\left|\frac{Ta}{Tb}\right|\\ &\leq \max_{-h\leq t\leq h}\left(\int_1^t\left|\frac{F(s,s(a))}{F(s,b(s))}\right|\right)^{ds}\\ &\leq \max_{-h\leq t\leq h}\left(\int_1^t L^{\left|\frac{a(s)}{b(s)}\right|}\right)^{ds}\\ &\leq \max_{-h\leq t\leq h}\left(\int_1^t 1^{ds}\right)^{L^{d(a,b)}}\\ &=\max_{-h\leq t\leq h}\Big(|t-1|\Big)^{L^{d(a,b)}}\leq\Big(K^L\Big)^{d(a,b)}\\ &\leq d(a,b)^{\lambda}\end{aligned}$$

$$\Rightarrow d(Ta,Tb)\leq d(a,b)^{\lambda}$$

and

$$\begin{aligned}d^*(Ta,b_0)=d^*(Ta,b_0)&=\max_{-h\leq t\leq h}\left|\frac{Ta}{b_0}\right|=\max_{-h\leq t\leq h}\left|\int_1^t F(s,a(s))^{ds}\right|\\ &\leq \max_{-h\leq t\leq h}\Big|F(s,a(s))\Big|\leq\delta\end{aligned}$$

we calculate $T: B(\xi,\delta)\to B(\xi,\delta)$ is a contractive mapping. At last, we will show that $(B(\xi,\delta),d)$ is complete, because $\{a_n\}$ is Cauchy sequence in $B(\xi,\delta)$, for all $a\in Y$ such that $d^*(a_n,a)\ll C$.
Thus

$$d^*(\xi,a)\leq d^*(a_n,\xi).d^*(a_n,a)\leq\delta.C$$

Hence $d(\xi,a)\leq\delta$, which means that $a\in B(\xi,\delta)$, that is $(B(\xi,\delta),d)$ is complete. Thus the fixed point of T is unique i.e., $a(t)\in B(\xi,\delta)$.
Thus, we conclude that, there exist a unique solution of 4.1.

5 Conclusion

. In this manuscript, we introduces for the first time the notion of multiplicative cone b-metric space and presented some fixed point theorems in setting of multiplicative cone b-metric space. We also give an application to support our main result.

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